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DERANDOMIZATION IN GAME-THEORETIC PROBABILITY

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Abstract. We give a general method for constructing a deterministic strategy of Reality from a randomized strategy in game-theoretic probability. The construction can be seen as derandomization in game-theoretic probability.

1. Introduction

1.1. **Reality's strategy in Game-theoretic probability.**

Game-theoretic probability [18] is a probability theory based on a betting game between two players, Skeptic and Reality. Sometimes we add the third player called Forecaster. In game-theoretic probability an almost sure event is (usually) formalized as an event such that Skeptic can increase his capital to infinity if the event does not happen. In this case we say that Skeptic has a winning strategy. In game-theoretic probability, in order to prove that an event happens almost surely, we construct a winning strategy of Skeptic. A number of such strategies have been constructed so far. Often these strategies of Skeptic correspond to well-known proofs in measure-theoretic probability that a certain event happens with probability one.

In this paper, when we just refer to a strategy, it is a deterministic strategy. We explicitly say "randomized strategy", when a strategy utilizes random variables in the sense of measure-theoretic probability.

There is no counterpart of Reality's strategy in measure-theoretic probability, because in measure-theoretic probability Reality is simply generating random variables under a given probability distribution without any specific strategy. Hence it is more difficult to derive results on Reality's strategies. Reality's strategies correspond to the notion of derandomization, since if Reality is following a strategy she is not random in the measure-theoretic sense.

If a two-player game (without Forecaster) is with perfect information, then at least one of the players has a winning strategy in the game by Martin's Theorem (Theorem 5.8). If Skeptic does not have a winning strategy, Reality should have. For instance, consider the game-theoretic version of Kolmogorov's strong law of large numbers [18, Proposition 4.1]. Shafer and Vovk proved the existence of Reality's winning strategy in Section 4.3 in their book [18]. The proof is, however, nonconstructive. The two main tools of the proof are

(i) the randomized strategy for Reality that was devised by Kolmogorov,
(ii) Martin's theorem.

It is unnatural that we need to use such a big theorem, Martin's theorem, to answer such a simple question. It was a long-standing question to give a concrete deterministic strategy of Reality.

Vovk [20] finally gave such a strategy. The proof is simple, which is a desired property. The proof seems to be based on the randomized strategy by Kolmogorov, but it is not clear how these two strategies are related. Thus it is difficult to know

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from his proof how to modify the randomized strategy to answer a similar question in different games. On the other hand, Miyabe and Takemura [12] derived a rather strong result on strategies of Reality. The essential idea is that Reality uses a "fictional" strategy of Skeptic. In fact, Theorem 4.12 of Miyabe and Takemura [12] showed the existence of Reality's strategy. However, they did not give a concrete strategy. Thus we did not know how the strategy looks like.

In this paper we construct a concrete strategy of Reality based on the idea above. The construction goes as follows.

(i) Take a randomized strategy of Reality.
(ii) Construct a strategy of Skeptic that forces the random event.
(iii) Construct a strategy of Reality using it.

Each step is straightforward and does not require coming up with a new strategy. Since we construct a deterministic strategy from a randomized one, we call this *derandomization in game-theoretic probability.*

1.2. **Derandomization.** Randomized algorithm [13, 11] has been frequently used in complexity theory [7, 1]. One of the reasons is that there are some problems such that it seems difficult to prove that they are polynomial-time computable but they are polynomial-time computable with an random oracle with high probability.

The class **BPP** (Bounded-error Probabilistic Polynomial-time) is, roughly speaking, the set of problems that are polynomial-time computable with a randomized algorithm. It has been conjectured that every problem in **BPP** is actually polynomial-time computable, that is, **BPP** = **P**. In other words, the conjecture is asking whether we can always derandomize in this setting.

An analogous question in computability theory [3, 15, 16] has been solved. On Cantor space $2^{\mathbb{N}}$ with the uniform measure μ, if $A \in 2^{\mathbb{N}}$ is not computable, then the set of all sequences that compute A has measure 0 [4, 17]. Thus, if a sequence is computable by a randomized strategy, then the sequence should be computable. In other words, we can always derandomize if we do not care about computational resource.

Derandomization asks the question how we can deterministically construct a sequence random enough. Construction of such a sequence has been studied in the theory of algorithmic randomness [5, 14] to separate some randomness notions. The essential idea is diagonalization. One recent interesting application is the construction of an absolutely normal number in polynomial time [10, 6, 2].

The same technique can be applied to construct a strategy of Reality that complies with an event in game-theoretic probability. Derandomization itself is much easier in this case because we do not care about computability at all. In contrast, we need to consider a sequence of reals in our case while derandomization in complexity theory and the theory of algorithmic randomness usually considers an infinite binary sequence.

1.3. **Overview of this paper.** The main theme of this paper is the construction of Reality's strategy. In Section 3 we study Borel-Cantelli lemmas in game-theoretic probability. This is a simple case and illustrates how the construction goes. Then, the result will be used in the next section. In Section 4 we give a deterministic strategy of Reality that complies with the success and the failure of the strong law of large numbers. In Section 5 we give a general theory of the notion of compliance and look at some examples.

2. Preliminaries

In this paper we mainly consider the unbounded forecasting game defined in Chapter 4 of Shafer and Vovk [18].

Unbounded Forecasting Game (UFG)
Players: Forecaster, Skeptic, Reality
Protocol:
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots$:
Forecaster announces $m_n \in \mathbb{R}$ and $v_n \geq 0$.
Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.
Reality announces $x_n \in \mathbb{R}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n) + V_n((x_n - m_n)^2 - v_n)$.
Collateral Duties: Skeptic must keep $\mathcal{K}_n$ non-negative. Reality must keep $\mathcal{K}_n$ from tending to infinity.

An infinite sequence $\xi = (m_1, v_1, x_1, m_2, v_2, x_2, \cdots)$ of moves of Forecaster and Reality is called a *path*. Define the sample space

$$\Omega = \{\xi = (m_1, v_1, x_1, m_2, v_2, x_2, \cdots) \ : \ m_n \in \mathbb{R},\ v_n \geq 0,\ x_n \in \mathbb{R}\}$$

as the set of paths. Any subset $E \subseteq \Omega$ is called an *event*. We say that a strategy P of Skeptic *forces* an event E if the capital $\mathcal{K}_n^P(\xi)$ of Skeptic with P is non-negative for all $\xi \in \Omega$ and for all $n \geq 0$, and $\xi \notin E$ implies $\limsup_n \mathcal{K}_n^P(\xi) = \infty$. Skeptic *can force* an event if there is a strategy P of Skeptic that forces the event. Note that we are not distinguishing "weak forcing" and "forcing", since they are equivalent ([18, Lemma 3.1]).

Definition 2.1 (Miyabe and Takemura [12])**.** By a strategy R, Reality *complies* with an event $E \subseteq \Omega$ if

(i) $\xi \in E$, irrespective of the moves of Forecaster and Skeptic, with Skeptic observing his collateral duty,
(ii) $\sup_n \mathcal{K}_n < \infty$.

Reality *strongly complies* with E if (ii) is replaced with $\mathcal{K}_n \leq \mathcal{K}_0$ for all n.

Theorem 2.2 (Shafer and Vovk [18, Proposition 4.1])**.** *In the unbounded forecasting game,*

(i) *Skeptic can force*

$$\sum_{n=1}^{\infty} \frac{v_n}{n^2} < \infty \Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (x_i - m_i) = 0.$$

(ii) *Reality can comply with*

$$\sum_{n=1}^{\infty} \frac{v_n}{n^2} = \infty \Rightarrow \left(\frac{1}{n} \sum_{i=1}^{n} (x_i - m_i) \text{ does not converge to } 0 \right).$$

We call the event of (i) the *success of SLLN* (Strong Law of Large Numbers) and the event of (ii) the *failure of SLLN*. In the proof of (ii), Shafer and Vovk [18, Proposition 4.1] use a randomized strategy of Reality and Martin's theorem, but did not give a concrete strategy. Vovk [20] gave a concrete strategy. The result also follows from Theorem 4.12 of Miyabe and Takemura [12].

The following is the key fact to give a strategy of Reality.

Theorem 2.3 (Miyabe and Takemura [12, Proposition 4.10])**.** *In the unbounded forecasting game, if Skeptic can force an event E, then Reality strongly complies with E.*

In the proof of this theorem, a strategy of Reality was constructed using the strategy of Skeptic that forces the event.

3. Borel-Cantelli lemmas

In this section we focus on game-theoretic versions of Borel-Cantelli lemmas, which will play an important role to give the strategy of Reality that complies with the success and the failure of SLLN in the next section.

Coin-Tossing Game
Players: Forecaster, Skeptic, Reality
Protocol:
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots$:
Forecaster announces $0 < p_n < 1$.
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in \{0, 1\}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - p_n)$.
Collateral Duties: Skeptic must keep $\mathcal{K}_n$ non-negative. Reality must keep $\mathcal{K}_n$ from tending to infinity.

The following result is a game-theoretic version of Lévy's Extension of the Borel-Cantelli Lemma.

Proposition 3.1 (Miyabe-Takemura [12, Example 2.3])**.** *In the coin-tossing game Skeptic can force*

$$\sum_n p_n < \infty \iff \sum_n x_n < \infty. \tag{1}$$

By essentially the same argument in the proof of Theorem 2.3, we can show that Reality can comply with the event (1). Here we give a concrete strategy. In order to do that, we need a concrete strategy of Skeptic that forces (1) and the simpler the better. Proposition 3.1 follows from the following two lemmas.

Lemma 3.2. *In the coin-tossing game, Skeptic can force*

$$\sum_n p_n = \infty \Rightarrow \sum_n x_n = \infty.$$

Proof. Let

$$H_n = \{k < n \ : \ x_k = 1\}, \ T_n = \{k < n \ : \ x_k = 0\} \tag{2}$$

be the sets of time indices of heads and tails before the round n and let

$$b_n = \# H_n \tag{3}$$

denote the number of heads before the round n. Note that for any distinct $k, j \in H_n$, we have $k < j \Rightarrow b_k < b_j$. Hence

$$\sum_{k \in H_n} 2^{-b_k - 1} \le 1. \tag{4}$$

Consider the following strategy of Skeptic:

$$M_n = -2^{-b_n - 1}.$$

We claim that this strategy forces the event.

First, we show that this strategy keeps $\mathcal{K}_n$ non-negative. Note that

$$\mathcal{K}_n = 1 - \sum_{k \in H_n} 2^{-b_k - 1}(1 - p_n) - \sum_{k \in T_n} 2^{-b_k - 1}(-p_n) > 1 - \sum_{k \in H_n} 2^{-b_k - 1} \ge 0.$$

Next, suppose that $\sum_n p_n = \infty$ and $\sum_n x_n < \infty$. Then, there exists N such that $n \ge N \Rightarrow x_n = 0$. Thus, for every $n \ge N$,

$$\mathcal{K}_n = \mathcal{K}_{N-1} + 2^{-b_N - 1} \sum_{k=N}^{n} p_k \to \infty.$$

□

Lemma 3.3. *In the coin-tossing game, Skeptic can force*

$$\sum_n p_n < \infty \Rightarrow \sum_n x_n < \infty.$$

Proof. Let c_n be the natural number satisfying

$$c_n - 1 \le \sum_{k=1}^{n} p_k < c_n. \tag{5}$$

Note that

$$\sum_{k:c_k=i} p_k = \sum_{k:c_k\le i} p_k - \sum_{k:c_k\le i-1} p_k < i-(i-2) = 2.$$

Consider the following strategy of Skeptic:

$$M_n = 2^{-c_n-1}.$$

We claim that this strategy forces the event.

First we show that this strategy keeps $\mathcal{K}_n$ non-negative. Note that

$$\sum_{k\in T_n} 2^{-c_k-1}p_k \le \sum_{k=1}^{\infty} 2^{-c_k-1}p_k \le \sum_{i=1}^{\infty}\sum_{k:c_k=i} 2^{-i-1}p_k \le \sum_{i=1}^{\infty} 2^{-i} \le 1, \tag{6}$$

where T_n is defined in (2). Then

$$\mathcal{K}_n \ge 1 + \sum_{k\in T_n} M_k(x_k - p_k) > 1 - \sum_{k\in T_n} 2^{-c_k-1}p_k \ge 0.$$

Next we show that this strategy forces the event. Assume that $\sum_n p_n < \infty$ and $\sum_n x_n = \infty$. Let c be such that $c-1 < \sum_n p_n \le c$. Then there exists N_0 such that $c-1 < \sum_{k=1}^{N_0} p_k$. Since $\sum_n p_n < \infty$, there exists N_1 such that $n \ge N_1 \Rightarrow p_n < 1/2$. Let $N = \max\{N_0, N_1\}$. For $n \ge N$ such that $x_n = 1$, we have

$$\mathcal{K}_n - \mathcal{K}_{n-1} \ge 2^{-c-1}(1-p_n) \ge 2^{-c-2}.$$

For $n \ge N$ such that $x_n = 0$, we have

$$\mathcal{K}_n - \mathcal{K}_{n-1} \ge -2^{-c-1}p_n.$$

Since $\sum_n p_n < \infty$, we have $\limsup_n \mathcal{K}_n = \infty$. □

Using these strategies, Reality can strongly comply with the same event.

Theorem 3.4. *In the coin-tossing game, Reality strongly complies with*

$$\sum_n p_n < \infty \iff \sum_n x_n < \infty.$$

Proof. Let b_n, c_n be defined by (3) and (5). We claim that the following strategy of Reality strongly complies with the event:

> Reality waits for the first round n_0 such that Skeptic announces $M_n \ne 0$. If such a round does not exist, let $n_0 = \infty$.
>
> For $n < n_0$ including the case $n_0 = \infty$, Reality announces x_n as
>
> $$x_n = \begin{cases} 1 & \text{if } c_n \ne c_{n-1} \\ 0 & \text{if } c_n = c_{n-1}. \end{cases}$$
>
> For $n = n_0$, Reality announces x_n as
>
> $$x_n = \begin{cases} 1 & \text{if } M_n < 0 \\ 0 & \text{if } M_n > 0. \end{cases}$$

If $\mathcal{K}_{n_0} = 0$, then for $n > n_0$ Reality announces x_n as

$$x_n = \begin{cases} 1 & \text{if } c_n \neq c_{n-1} \\ 0 & \text{if } c_n = c_{n-1}. \end{cases}$$

If $\mathcal{K}_{n_0} > 0$, then let

$$\epsilon = 1 - \frac{\mathcal{K}_{n_0}}{\mathcal{K}_0}$$

For $n > n_0$, Reality announces x_n as

$$x_n = \begin{cases} 1 & \text{if } M_n \leq d_n \\ 0 & \text{if } M_n > d_n \end{cases}$$

where

$$d_n = \frac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon}(2^{-b_n-2} - 2^{-c_n-2}).$$

We show that this strategy strongly complies with the event. If $n_0 = \infty$, then clearly $\sum_n p_n < \infty$ if and only if $\sum_n x_n < \infty$ and $\mathcal{K}_n = \mathcal{K}_0$. Then, we can assume that $n_0 < \infty$.

Since $\mathcal{K}_{n_0} - \mathcal{K}_{n_0-1} < 0$, we have $\mathcal{K}_{n_0} < \mathcal{K}_0$. If $\mathcal{K}_{n_0} = 0$, then Skeptic should announce $M_n = 0$ for every $n > n_0$ in order to keep $\mathcal{K}_n$ non-negative. Thus, $\mathcal{K}_n = 0$ for every $n > n_0$. In this case clearly $\sum_n p_n < \infty$ if and only if $\sum_n x_n < \infty$.

In what follows, we assume $\mathcal{K}_{n_0} > 0$. Then $0 < \epsilon < 1$. Suppose that $\sum_n p_n = \infty$ and $\sum_n x_n < \infty$. Since b_n is bounded and c_n goes to infinity, there exists δ such that $d_n > \delta$ and $x_n = 0$ for all sufficiently large $n > N$. Then for such an n, we have

$$\begin{aligned} \mathcal{K}_n =& \mathcal{K}_N - \sum_{k=N+1}^{n} M_k p_k < \mathcal{K}_N - \sum_{k=N+1}^{n} d_k p_k \\ <& \mathcal{K}_N - \sum_{k=N+1}^{n} \delta p_k \to -\infty \end{aligned}$$

as $n \to \infty$. Thus, such a strategy of Skeptic is not allowed.

Suppose that $\sum_n p_n < \infty$ and $\sum_n x_n = \infty$. Since c_n is bounded, b_n goes to infinity and p_n goes to 0, there exists $\delta > 0$ such that $d_n < -\delta$ and $p_n \leq 1/2$ for all sufficiently large $n > N$. For $n > N$ such that $x_n = 1$,

$$\mathcal{K}_n - \mathcal{K}_{n-1} = M_n(1-p_n) \leq -\delta(1-p_n) \leq -\frac{\delta}{2}.$$

For $n > N$ such that $x_n = 0$,

$$\mathcal{K}_n - \mathcal{K}_{n-1} = -M_n p_n < \delta p_n.$$

Thus, $\mathcal{K}_n \to -\infty$ and such a strategy of Skeptic is not allowed.

Finally we show that $\sup_n \mathcal{K}_n \leq 1$. Since we have $\mathcal{K}_0 = (1-\epsilon)\mathcal{K}_{n_0}$, it suffices to show that

$$\mathcal{K}_n \leq \frac{\mathcal{K}_{n_0}}{1-\epsilon} = \mathcal{K}_{n_0} + \frac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon}.$$

For $n \geq n_0$ such that $x_n = 1$, we have

$$\mathcal{K}_n - \mathcal{K}_{n-1} \leq M_n(1-p_n) < d_n \leq \frac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon} 2^{-b_n-2}.$$

For $n \geq n_0$ such that $x_n = 0$, we have

$$\mathcal{K}_n - \mathcal{K}_{n-1} = -M_n p_n < \frac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon} 2^{-c_n-2} p_n.$$

By (4) and (6), we have $\mathcal{K}_n - \mathcal{K}_{n_0} \leq \dfrac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon}$. □

From now on we explain how we derived the above strategy. The goal is to construct a strategy of Reality that complies with the event E of $\sum_n p_n < \infty \iff \sum_n x_n < \infty$. It suffices to give a strategy with which Reality's move is "random" in the following two senses:

(a) The capital is finite.
(b) The path satisfies the almost-sure property E.

The meaning of randomness in measure-theoretic probability is not clear. In the theory of algorithmic randomness, one formulation of randomness is finiteness of the capital for all betting strategies that are effective in some sense. In game-theoretic probability, randomness of a path means the finiteness of the capital in the game. For instance, Vovk and Shen [21] have used the terminology of "game-random". With this view, we express the property (b) by the finiteness of the capital with respect to the strategy with which Skeptic can force the event E.

By the proof of Lemma 3.2 and Lemma 3.3, the following strategy F of Skeptic forces the event E:

$$M_n = 2^{-c_n-2} - 2^{-b_n-2}.$$

Reality uses this strategy F as a fictional strategy. We denote by S the real strategy of Skeptic. In order to comply with the event E, all Reality has to do is to make the capital with the strategy $(S+F)/2$ finite. Note that the finiteness of the capital with $(S+F)/2$ implies the finiteness of the capital with S and the capital with F. Furthermore, the finiteness of the capital with F implies the event E because F forces the event E. The strategy $O = (S+F)/2$ announces

$$M_n^O = \frac{M_n^S + 2^{-c_n-2} - 2^{-b_n-2}}{2}.$$

Reality can make $\mathcal{K}_n^O \leq \mathcal{K}_{n-1}^O$ by announcing

$$x_n = \begin{cases} 1 & \text{if } M_n^O \leq 0 \\ 0 & \text{if } M_n^O > 0. \end{cases}$$

Note that

$$M_n^O \leq 0 \iff M_n^S \leq 2^{-b_n-2} - 2^{-c_n-2}.$$

Then this strategy complies with the event E. Notice that this strategy gives the essential part of the strategy in the proof of Theorem 3.4.

To give a strategy that "strongly" complies with the event, we need a little trick. The idea is taken from the proof of Proposition 4.10 in [12]. A rough idea is as follows. Wait until the round n_0 satisfying $M_{n_0} \neq 0$ so that Reality can make the capital strictly less than the initial capital. Let $1-\epsilon$ be the ratio of the capital at n_0 and the initial capital. After the round n_0, Reality only has to make the capital with $(1-\epsilon)S + \epsilon F$ non-increasing. The derived strategy is the strategy in the proof of Theorem 3.4.

4. Derandomization in the Unbounded Forecasting Game and its generalization

In this section we give Reality's strategies complying with the success and the failure of the strong law of large numbers at the same time in the Unbounded Forecasting Game and its generalization.

4.1. A strategy of Reality for the Unbounded Forecasting Game.

Theorem 4.1. *In the unbounded forecasting game, Reality strongly complies with*

$$\sum_n \frac{v_n}{n^2} < \infty \iff \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n (x_i - m_i) = 0.$$

Note that this theorem implies (ii) of Theorem 2.2. We show this by giving a concrete strategy of Reality.

In measure-theoretic probability, the failure of SLLN was shown by the following randomized strategy devised by Kolmogorov [8]: if $v_n < n^2$,

$$x_n := \begin{pmatrix} n \\ -n \\ 0 \end{pmatrix} \text{ with probability } \begin{pmatrix} v_n/(2n^2) \\ v_n/(2n^2) \\ 1 - v_n/n^2 \end{pmatrix},$$

respectively; if $v_n \geq n^2$,

$$x_n := \begin{pmatrix} \sqrt{v_n} \\ -\sqrt{v_n} \end{pmatrix} \text{ with probability } \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Here, to show the non-convergence, the second part of the Borel-Cantelli lemmas is used. In contrast, if $\sum_n v_n/n^2 < \infty$, then by the first part of the Borel-Cantelli lemma, SLLN holds almost surely.

Motivated by this strategy we restrict $x_n \in \{0, \pm n\}$ if $v_n < n^2$, and $x_n \in \{\pm\sqrt{v_n}\}$ if $v_n \geq n^2$. Then, the strategy we need to construct is the one of Reality that complies with

$$\sum_n \frac{v_n}{n^2} < \infty \iff x_n = 0 \text{ for all but finitely many } n.$$

Recall that we can construct a strategy of Reality if we have a strategy of Skeptic that forces the event. Such a strategy of Skeptic can be constructed by modifying the strategy constructed in the previous section.

In what follows, without loss of generality, we assume that $m_n = 0$ for every n. Furthermore, we can forget the round n such that $v_n = 0$ by letting $x_n = 0$. Thus, we assume that $v_n > 0$ for every n.

Proof. We claim that the following strategy of Reality strongly complies with the event:

Let

$$b_n = \#\{k < n \ : \ x_k \neq 0\}$$

and c_n be the natural number satisfying

$$c_n - 1 \leq \sum_{k=1}^{n} \frac{v_k}{k^2} < c_n.$$

Reality waits for the first round n_0 such that Skeptic announces $(M_n, V_n) \neq (0, 0)$. If such a round does not exist, let $n_0 = \infty$.

For $n < n_0$ including the case $n_0 = \infty$, Reality announces x_n as

$$x_n = \begin{cases} n & \text{if } c_n \neq c_{n-1} \\ 0 & \text{if } c_n = c_{n-1}. \end{cases}$$

For $n = n_0$, Reality announces x_n as

$$x_n = \begin{cases} 1 & \text{if } V_n = 0 \text{ and } M_n < 0 \\ -1 & \text{if } V_n = 0 \text{ and } M_n > 0 \\ 0 & \text{if } V_n > 0. \end{cases}$$

If $\mathcal{K}_{n_0} = 0$, then for $n > n_0$ Reality announces x_n as

$$x_n = \begin{cases} n & \text{if } c_n \neq c_{n-1} \\ 0 & \text{if } c_n = c_{n-1}. \end{cases}$$

If $\mathcal{K}_{n_0} > 0$, let

$$\epsilon = 1 - \frac{\mathcal{K}_{n_0}}{\mathcal{K}_0}.$$

For $n > n_0$ Reality announces x_n as

$$x_n = \begin{cases} n & \text{if } v_n < n^2,\ V_n \le d_n \text{ and } M_n < 0 \\ -n & \text{if } v_n < n^2,\ V_n \le d_n \text{ and } M_n \ge 0 \\ 0 & \text{if } v_n < n^2,\ V_n > d_n \\ \sqrt{v_n} & \text{if } v_n \ge n^2 \text{ and } M_n < 0 \\ -\sqrt{v_n} & \text{if } v_n \ge n^2 \text{ and } M_n \ge 0, \end{cases}$$

where

$$d_n = \frac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon} \frac{2^{-b_n-2} - 2^{-c_n-2}}{n^2}.$$

We show that this strategy strongly complies with the event. If $n_0 = \infty$, then clearly $\sum_n v_n/n^2 < \infty$ if and only if $x_n = 0$ for all but finitely many n, and $\mathcal{K}_n = \mathcal{K}_0$ for every n. Then, we can assume that $n_0 < \infty$.

Consider the round $n = n_0$. If $V_n = 0$, then

$$\mathcal{K}_{n_0} - \mathcal{K}_{n_0-1} = -|M_n| < 0.$$

If $V_n > 0$, then

$$\mathcal{K}_{n_0} - \mathcal{K}_{n_0-1} = -V_n v_n < 0.$$

Thus, $\mathcal{K}_{n_0} < \mathcal{K}_0$.

If $\mathcal{K}_{n_0} = 0$, then Skeptic should announce $(M_n, V_n) = (0,0)$ for every $n > n_0$ in order to keep $\mathcal{K}_n$ non-negative. Thus, $\mathcal{K}_n = 0$ for every $n > n_0$. In this case clearly $\sum_n v_n/n^2 < \infty$ if and only if $x_n = 0$ for all but finitely many n.

In what follows we assume $\mathcal{K}_{n_0} > 0$. Then $0 < \epsilon < 1$.

Suppose that $\sum_n v_n/n^2 = \infty$ and $x_n = 0$ for all but finitely many n. Since b_n is bounded and c_n goes to infinity, there exists δ such that $d_n > \delta/n^2$ for all sufficiently large $n > N_0$. Then, for such an n,

$$\begin{aligned} \mathcal{K}_n =& \mathcal{K}_{N_0} - \sum_{k=N_0+1}^{n} V_k v_k < \mathcal{K}_{N_0} - \sum_{k=N_0+1}^{n} d_k v_k \\ <& \mathcal{K}_{N_0} - \sum_{k=N_0+1}^{n} \delta \cdot \frac{v_k}{k^2} \to -\infty \end{aligned}$$

as $n \to \infty$. Thus, such a strategy of Skeptic is not allowed.

Suppose that $\sum_n v_n/n^2 < \infty$ and $x_n \ne 0$ for infinitely many n. Since c_n is bounded and b_n goes to infinity, d_n is negative for all sufficiently large n. Since $\sum_n v_n/n^2 < \infty$, we have $v_n < n^2$ for all sufficiently large n. Thus, x_n should be 0 for all sufficiently large n. This is a contradiction.

Finally we show that $\sup_n \mathcal{K}_n \le 1$. It suffices to show that $\mathcal{K}_n \le \mathcal{K}_{n_0} + \frac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon}$. For $n \ge n_0$ such that $x_n \ne 0$, we have

$$\mathcal{K}_n - \mathcal{K}_{n-1} \le V_n n^2 \le \frac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon} 2^{-b_n-2}.$$

For $n \ge n_0$ such that $x_n = 0$, we have

$$\mathcal{K}_n - \mathcal{K}_{n-1} = -V_n v_n \le 0.$$

Thus, $\mathcal{K}_n - \mathcal{K}_{n_0} \le \dfrac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon}$. □

4.2. **A strategy of Reality in a generalization of the Unbounded Forecasting Game.** There are some possible ways in which we generalize our result for the Unbounded Forecasting Game. Kumon, Takemura and Takeuchi [9] have obtained similar results in a game which generalizes the Unbounded Forecasting Game. Miyabe and Takemura [12] have shown the existence of the strategy that strongly complies with the failure of SLLN in a rather general setting. Here, we give a stronger result in the general setting in the following senses.

(i) We give a concrete deterministic strategy.
(ii) The strategy strongly complies with the success and the failure of SLLN at the same time.
(iii) We use weaker assumptions.
(iv) The strategy is much simpler.

The following protocol is from Section 5 in Miyabe and Takemura [12].

UNBOUNDED FORECASTING GAME WITH GENERAL HEDGE (UFGH)
Parameters: A single hedge $h : \mathbb{R} \to \mathbb{R}$
Players: Forecaster, Skeptic, Reality
Protocol:
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \dots$:
Forecaster announces $m_n \in \mathbb{R}$ and $v_n \geq 0$.
Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.
Reality announces $x_n \in \mathbb{R}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n) + V_n(h(x_n - m_n) - v_n)$.
Collateral Duties: Skeptic must keep $\mathcal{K}_n$ non-negative. Reality must keep $\mathcal{K}_n$ from tending to infinity.

Assumption

(A0) $h(x) = h(|x|) \geq 0$.
(A1) $h(x)/x$ is monotone increasing for $x > 0$.
(A2) $h(x)/x^2$ is monotone decreasing for $x > 0$.
(A3) $h(x) = x^2$ for $|x| \leq 1$.

Here, we are taking into account Remark 5.3 of [12].

Theorem 4.2 (Theorem 5.9 in [12]). *Suppose that h satisfies (A0)-(A3) and that g is a positive increasing function. Then in UFGH, Reality strongly complies with*

$$\sum_n \frac{v_n}{g(A_n)} = \infty \Rightarrow \sum_{k=1}^{n} \frac{x_k - m_k}{h^{-1} \circ g(A_k)} \text{ does not converge.}$$

Theorem 4.3 (Theorem 5.10 in [12]). *Let $h(x) = x^r$ where $1 \leq r \leq 2$ and g be a positive increasing function. Then in UFGH, Reality strongly complies with*

$$\sum_n \frac{v_n}{g(A_n)} = \infty \Rightarrow \frac{\sum_{k=1}^{n}(x_k - m_k)}{h^{-1} \circ g(A_n)} \text{ does not converge.}$$

We consider a slightly weaker condition, but the following assumption is cricital to show the strong compliance. See Remark 4.7 for details.

Assumption

(A4) $h(0) = 0$.

From now on we show the following theorem.

Theorem 4.4. *Suppose that h satisfies (A0)-(A2), (A4) and that g is a positive increasing function. Then in UFGH, Reality strongly complies with*

$$\sum_n \frac{v_n}{g(A_n)} < \infty \iff \frac{\sum_{k=1}^{n}(x_k - m_k)}{h^{-1} \circ g(A_n)} \text{ converges.} \tag{7}$$

Note that Theorem 4.4 implies the two theorems above. Theorem 4.4 also implies Proposition 2.1 in [9] by letting $m_n = 0$, $v_n = v$, $h(x) = x^r$ and $g(x) = x/v$. Furthermore, Theorem 4.4 also implies Proposition 3.1 in [9] by letting $m_n = 0$, $v_n = v$ and $g(x) = h(vx)$.

In what follows, without loss of generality, we assume that $m_n = 0$ for every n. Furthermore, we can forget the round n such that $v_n = 0$ by letting $x_n = 0$. Thus, we assume that $v_n > 0$ for every n.

Before giving the strategy, we recall the following lemma.

Lemma 4.5 (Miyabe and Takemura [12, Lemma 4.15]). *Let $\{a_n\}$ be a sequence of positive reals. Then there exists a sequence $\{\epsilon_n\}$ of positive reals such that*

(i) *ϵ_n is determined only by $a_1, \cdots, a_n$,*
(ii) *$\epsilon_n a_n \le 1$,*
(iii) *$\sum_n a_n = \infty$ implies $\sum_n \epsilon_n a_n = \infty$ and $\epsilon_n \to 0$.*

Furthermore, by the proof, we can assume that

(iv) $\sum_n a_n < \infty$ implies that $\{\epsilon_n\}$ converges to a positive real.

Now we are ready to give the strategy.

In UFGH, we consider the following strategy of Reality:

Let

$$b_n = \#\{k < n \ : \ x_k \neq 0\}$$

and c_n be the natural number satisfying

$$c_n - 1 \le \sum_{k=1}^{n} \frac{\epsilon_k v_k}{g(A_k)} < c_n$$

where $\{\epsilon_n\}$ is the sequence determined by Lemma 4.5 for $\{v_n/g(A_n)\}$. Reality waits for the first round n_0 such that Skeptic announces $(M_n, V_n) \neq (0, 0)$. If such a round does not exist, $n_0 = \infty$.

For $n < n_0$ including the case $n_0 = \infty$, Reality announces x_n as

$$x_n = \begin{cases} e_n & \text{if } c_n \neq c_{n-1} \\ 0 & \text{if } c_n = c_{n-1}. \end{cases}$$

where

$$e_n = h^{-1}(g(A_n) \cdot \epsilon_n^{-1}).$$

For $n = n_0$, Reality announces x_n as

$$x_n = \begin{cases} 1 & \text{if } V_n = 0 \text{ and } M_n < 0 \\ -1 & \text{if } V_n = 0 \text{ and } M_n > 0 \ . \\ 0 & \text{if } V_n > 0. \end{cases}$$

If $\mathcal{K}_{n_0} = 0$, then for $n > n_0$ Reality announces x_n as

$$x_n = \begin{cases} e_n & \text{if } c_n \neq c_{n-1} \\ 0 & \text{if } c_n = c_{n-1}. \end{cases}$$

If $\mathcal{K}_{n_0} > 0$, let

$$\epsilon = 1 - \frac{\mathcal{K}_{n_0}}{\mathcal{K}_0}.$$

For $n > n_0$ Reality announces x_n as

$$x_n = \begin{cases} e_n & \text{if } \epsilon_n v_n < g(A_n),\ V_n \le d_n \text{ and } M_n < 0 \\ -e_n & \text{if } \epsilon_n v_n < g(A_n),\ V_n \le d_n \text{ and } M_n \ge 0 \\ 0 & \text{if } \epsilon_n v_n < g(A_n),\ V_n > d_n \\ h^{-1}(v_n) & \text{if } \epsilon_n v_n \ge g(A_n) \text{ and } M_n < 0 \\ -h^{-1}(v_n) & \text{if } \epsilon_n v_n \ge g(A_n) \text{ and } M_n \ge 0. \end{cases}$$

where

$$d_n = \frac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon} \frac{2^{-b_n-2} - 2^{-c_n-2}}{g(A_n) \cdot \epsilon_n^{-1}}.$$

We show that this strategy strongly complies with the success and the failure of SLLN at the same time. In the proof we use the following lemma.

Lemma 4.6 (Miyabe and Takemura [12, Lemma 4.14])**.** *Let $\{y_n\}$ be a sequence of reals and let $\{g_n\}$ be a non-decreasing sequence of positive reals. If $(\sum_{k\le n} y_k)/g_n$ converges to d, then $|y_n/g_n| \le |d| + 1$ for all but finitely many n.*

Proof of Theorem 4.4. By a similar argument as the proof of Theorem 4.1, we can show that Reality can comply with

$$\sum_n \frac{v_n}{g(A_n)} < \infty \iff \sum_n \frac{\epsilon_n v_n}{g(A_n)} < \infty \iff x_n = 0 \text{ for all but finitely many } n$$

except the following two places. (1) The capital of the case such that $\sum_n v_n/g(A_n) = \infty$ and $x_n = 0$ for all but finitely many n is as follows:

$$\begin{aligned} \mathcal{K}_n =& \mathcal{K}_{N_0} - \sum_{k=N_0+1}^{n} V_k v_k < \mathcal{K}_{N_0} - \sum_{k=N_0+1}^{n} d_k v_k \\ <& \mathcal{K}_{N_0} - \sum_{k=N_0+1}^{n} \delta \cdot \frac{v_k}{g(A_k)} \to -\infty. \end{aligned}$$

(2) For $n \ge n_0$ such that $x_n \ne 0$, we have

$$\begin{aligned} \mathcal{K}_n - \mathcal{K}_{n-1} \le& V_n(h(e_n) - v_n) \le d_n \cdot h(e_n) \\ \le& \frac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon} \cdot \frac{2^{-b_n-2}}{g(A_n)\cdot \epsilon_n^{-1}} \cdot g(A_n) \cdot \epsilon_n^{-1} = \frac{\epsilon \mathcal{K}_{n_0}}{1-\epsilon} \cdot 2^{-b_n-2} \end{aligned}$$

if $\epsilon_n v_n < g(A_n)$, and $\mathcal{K}_n - \mathcal{K}_{n-1} \le 0$ if $\epsilon_n v_n \ge g(A_n)$.

From now on, we show that $x_n = 0$ for all but finitely many n if and only if $\sum_{k=1}^n x_k / h^{-1} \circ g(A_n)$ converges.

Suppose that $x_n = 0$ for all but finitely many n and $\lim_n A_n < \infty$. Then, $\sum_n x_n$ converges and $h^{-1} \circ g(A_n)$ converges. Thus, $\sum_{k=1}^n x_k / h^{-1} \circ g(A_n)$ converges.

Suppose that $x_n = 0$ for all but finitely many n and $\lim_n A_n = \infty$. Then, $\sum_n x_n$ converges and $\lim_n h^{-1} \circ g(A_n) = \infty$. Thus, $\sum_{k=1}^n x_k / h^{-1} \circ g(A_n)$ converges to 0.

Suppose that $x_n \ne 0$ for infinitely many n. This means that

$$|x_n| = \begin{cases} e_n & \text{if } \epsilon_n v_n < g(A_n) \\ h^{-1}(v_n) & \text{if } \epsilon_n v_n \ge g(A_n) \end{cases}$$

for infinitely many n. Note that, if $\epsilon_n v_n \ge g(A_n)$, then $h^{-1}(v_n) \ge h^{-1}(g(A_n) \cdot \epsilon_n^{-1})$ by the monotonicity of h. Thus,

$$\frac{|x_n|}{h^{-1} \circ g(A_n)} \ge \frac{h^{-1}(g(A_n) \cdot \epsilon_n^{-1})}{h^{-1} \circ g(A_n)}$$

for infinitely many n. We claim that the right-hand side goes to infinity. Then, by Lemma 4.6, $\sum_{k=1}^n x_k / h^{-1} \circ g(A_n)$ does not converge.

Since $\epsilon_n \to 0$ as $n \to \infty$, we have

$$g(A_n) \le g(A_n) \cdot \epsilon_n^{-1}$$

for all sufficiently large n. Since h is non-decreasing, so is h^{-1} and

$$h^{-1}(g(A_n)) \le h^{-1}(g(A_n) \cdot \epsilon_n^{-1}).$$

By Assumption (A2), we have

$$\frac{g(A_n)}{(h^{-1}(g(A_n)))^2} \ge \frac{g(A_n) \cdot \epsilon_n^{-1}}{(h^{-1}(g(A_n) \cdot \epsilon_n^{-1}))^2},$$

which implies that

$$\frac{h^{-1}(g(A_n) \cdot \epsilon_n^{-1})}{h^{-1} \circ g(A_n)} \ge \epsilon_n^{-1/2} \to \infty.$$

□

Remark 4.7. Notice that Assumption (A4) is used to show that $\mathcal{K}_{n_0} < \mathcal{K}_0$. At the round $n = n_0$, if $V_n > 0$, then $x_n = 0$ and

$$\mathcal{K}_{n_0} - \mathcal{K}_{n_0-1} = V_n(h(0) - v_n),$$

which is negative because $h(0) = 0$.

5. On the notion of Compliance

In this section we give a general theory on compliance. We consider the unbounded forecasting game or the coin-tossing game, but most theorems can be applied to a similar game.

5.1. **The strength of compliance.** Recall Theorem 2.3, which says that Reality strongly complies with the event that Skeptic can force. An interpretation of this fact in the usual notion of probability is like this:

> For each event with probability 1, one can deterministically take a path in the event.

Clearly, the probability of the event is closely related to the supremum of the capital of Skeptic. The strongness of the compliance seems to be due to probability 1 of the event. With this motivation we study the relation between the notion of compliance and the upper and lower probability.

We denote strategies of Forecaster, Skeptic and Reality by F, S and R respectively.

Definition 5.1 (Shafer and Vovk [18, Chapter 8.3])**.** The upper probability of an event E is defined as

$$\overline{P}(E) = \inf\{a \mid (\exists S)(\forall F)(\forall R)\mathcal{K}_0 = a \ \& \ E \Rightarrow \sup_n \mathcal{K}_n \ge 1\}$$

where S needs to keep the capital non-negative. The lower probability is defined as

$$\underline{P}(E) = 1 - \overline{P}(E^c).$$

The following are some properties of $\overline{P}$ and $\underline{P}$.

Proposition 5.2 (see [18, Proposition 8.12])**.** *Skeptic can force an event E if and only if $\underline{P}(E) = 1$.*

The "only if" direction holds because the convex combination of strategies of Skeptic is possible in the game.

Proposition 5.3 (see [18, Proposition 8.10])**.** *The upper probability $\overline{P}$ is an outer measure and the lower probability $\underline{P}$ is an inner measure.*

Proposition 5.4 (see [19, Lemma 1])**.** *For every event E, we have*

$$0 \le \underline{P}(E) \le \overline{P}(E) \le 1.$$

We define a similar function based on the notion of compliance.

Definition 5.5. For an event E, let

$$\overline{Q}(E) = \sup\{a \mid (\exists R)(\forall F)(\forall S)\mathcal{K}_0 = a \Rightarrow E \wedge \sup_n \mathcal{K}_n \le 1\}.$$

Let

$$\underline{Q}(E) = 1 - \overline{Q}(E^c).$$

The following are immediate by definition.

Proposition 5.6. *If Reality strongly complies with E, then $\overline{Q}(E) = 1$.*

Proposition 5.7. *If $\overline{Q}(E) > 0$, then Reality complies with E.*

A two-player game is called *determined* if one of the player has a winning strategy.

Theorem 5.8 (Martin's theorem; see [18, Chapter 4.6])**.** *If the winning condition is Borel, then the game is determined.*

Remark 5.9. In fact, Martin's theorem says that quasi-Borel is enough.

If Forecaster does not exist, then the game is with perfect information. Then, by Martin's theorem, we can show that Q is equal to P in this case.

Theorem 5.10. *Suppose that Forecaster's strategy is fixed in advance and Borel. If E is Borel, then $\overline{P}(E) = \overline{Q}(E)$ and $\underline{P}(E) = \underline{Q}(E)$.*

Proof. Suppose the game whose winning condition C_a of Skeptic is

$$\mathcal{K}_0 = a \wedge E \Rightarrow \sup_n \mathcal{K}_n \ge 1.$$

This condition can be written as

$$A \wedge \left(E^c \vee \bigwedge_m \bigvee_n B_{n,m}\right)$$

where A is $\mathcal{K}_0 = a$ and $B_{n,m}$ is $\mathcal{K}_n \ge 1 - 2^{-m}$. Note that A and $B_{n,m}$ are Borel. Recall that the class of Borel sets is closed under countable union, countable intersection and complement. Thus, the condition is Borel for each a.

Suppose $\overline{P}(E) = x > 0$, and let ϵ be a positive real small enough. If $\mathcal{K}_0 = x - \epsilon$, then no strategy S can guarantee to win. Then Reality has a winning strategy by Martin's theorem. It follows that $\overline{Q}(E) \ge x = \overline{P}(E)$.

Suppose $\overline{P}(E) = 0$. Then for each $x > 0$ there exists a strategy S such that $\mathcal{K}_0 < x/2$ and $E \Rightarrow \sup_n \mathcal{K}_n \ge 1$. Thus, the strategy $2S$ forces $\mathcal{K}_0 < x$ and $E \Rightarrow \sup_n \mathcal{K}_n > 1$. This means that no Reality's strategy complies with $\mathcal{K}_0 < x$ and $E \Rightarrow \sup_n \mathcal{K}_n \le 1$. Hence, $\overline{Q}(E) \le x$. Since x is arbitrary, we have $\overline{Q}(E) = 0$.

Since E is Borel, then so is E^c. Note that $\overline{P}(E^c) = \overline{Q}(E^c)$. Then $\underline{P}(E) = \underline{Q}(E)$. □

Theorem 2.3 says that, in our terminology,

$$\underline{P}(E) = 1 \Rightarrow \overline{Q}(E) = 1.$$

By a little modification, we can show the following.

Proposition 5.11.

$$\underline{P}(E) \le \overline{Q}(E)$$

and

$$\underline{Q}(E) \le \overline{P}(E).$$

Proof. We can assume $\underline{P}(E) = x > 0$. Then $\overline{P}(E^c) = 1 - x$. Hence there exists S such that $\mathcal{K}_0^S = 1 - x + \epsilon$, and $E^c \Rightarrow \sup_n \mathcal{K}_n^S \geq 1$ for small enough $\epsilon > 0$.

Now consider a strategy T of Skeptic such that $\mathcal{K}_0^T = x - 2\epsilon$. Then $S + T$ is a strategy such that $\mathcal{K}_0^{S+T} = 1 - \epsilon$. Hence Reality has the strategy satisfying $\sup_n \mathcal{K}_n^{S+T} \leq 1 - \epsilon$. It follows that $\sup_n \mathcal{K}_n^S \leq 1 - \epsilon$ and $\sup_n \mathcal{K}_n^T \leq 1 - \epsilon$. Since $E^c \Rightarrow \sup_n \mathcal{K}_n^S \geq 1$, we have E. Then this strategy is the witness of $\overline{Q}(E) \geq x$. □

Corollary 5.12. *If $\underline{P}(E) > 0$, then Reality can comply with E.*

5.2. Examples.

5.2.1. *Coin-tossing game.* In the following we look at some examples. Some examples show the difference between P and Q.

Example 5.13. In the coin-tossing game, we have $\overline{Q}(x_1 = 0) = \overline{Q}(x_1 = 1) = 0$.

Proof. We show that $\overline{Q}(x_1 = 1) = 0$.

Consider the following strategy of Forecaster and Skeptic;

$$\mathcal{K}_0 = \epsilon > 0, \ p_1 = \epsilon/2, \ M_1 = 2.$$

If $x_1 = 0$, then

$$\mathcal{K}_1 = \epsilon + 2\left(0 - \frac{\epsilon}{2}\right) \geq 0.$$

If $x_1 = 1$, then

$$\mathcal{K}_1 = \epsilon + 2\left(1 - \frac{\epsilon}{2}\right) = 2 > 1.$$

Then this strategy satisfies the collateral duty. This fact means that no strategy of Reality complies with $\mathcal{K}_0 = \epsilon > 0 \Rightarrow x_1 = 1 \wedge \sup_n \mathcal{K}_n \leq 1$. This shows that $\overline{Q}(x_1 = 1) = 0$.

We can show that $\overline{Q}(x_1 = 0) = 0$ in a similar way. □

Corollary 5.14. *The function $\overline{Q}$ is not an outer measure in general and $\underline{Q}$ is not an inner measure in general.*

Proof. In the coin-tossing game, let $A = \{x_1 = 0\}$ and $B = \{x_1 = 1\}$. Then, $\overline{Q}(A) = \overline{Q}(B) = 0$ and $\overline{Q}(A \cup B) = 1$. Thus, the inequality $\overline{Q}(A \cup B) \leq \overline{Q}(A) + \overline{Q}(B)$ does not hold even if A and B are disjoint. □

Recall that, in the coin-tossing game, we can assume that Skeptic announces $N_n \in \mathbb{R}$ and $V_n \geq 0$ and $\mathcal{K}_n := \mathcal{K}_{n-1} + N_n(x_n - p_n) + V_n((x_n - p_n)^2 - p_n(1 - p_n))$ by letting $M_n = N_n + V_n(1 - 2p_n)$. See Miyabe and Takemura [12, before Proposition 4.8] for details.

Recall the game-theoretic version of Lévy's extension of the Borel-Cantelli lemmas (Proposition 3.1). In the following, we focus on the case $\sum_n p_n < \infty$ and study the distribution of $\sum_n x_n$.

Example 5.15. In the coin-tossing game, consider the event $E_{c,k}$ of

$$\sum_n p_n(1 - p_n) \leq c^2 \text{ and } |\sum_n (x_n - p_n)| \geq ck.$$

Then

$$\overline{P}(E_{c,k}) \leq \frac{1}{k^2}.$$

Remark 5.16. This is Chebyshev's inequality.

Proof. Let

$$\mathcal{S}_n = \sum_{i=1}^{n}(x_i - p_i),\ \mathcal{T}_n = \mathcal{S}_n^2 - \sum_{i=1}^{n} p_i(1-p_i).$$

Then $\mathcal{T}_n$ is a capital process because

$$\begin{aligned}\mathcal{T}_n =& \mathcal{S}_n^2 - \sum_{i=1}^{n} p_i(1-p_i)\\ =& \mathcal{T}_{n-1} + 2\mathcal{S}_{n-1}(x_n - p_n) + ((x_n-p_n)^2 - p_n(1-p_n))\end{aligned}$$

Consider the strategy with

$$\mathcal{K}_0 = \frac{1}{k^2},\ N_n = \frac{2\mathcal{S}_n}{c^2k^2},\ V_n = \frac{1}{c^2k^2}.$$

Then,

$$\mathcal{K}_n = \frac{1}{k^2} + \frac{\mathcal{T}_n}{c^2k^2} \geq \frac{1}{k^2} - \frac{1}{k^2} \geq 0.$$

Hence, $\mathcal{K}_n$ keeps nonnegative.

Suppose $|\mathcal{S}_\infty| \geq ck$. Then $\mathcal{T}_\infty \geq c^2k^2 - c^2$ and

$$\sup_n \mathcal{K}_n \geq \frac{1}{k^2} + \frac{c^2k^2 - c^2}{c^2k^2} = 1.$$

□

If $k > 1$, then $\underline{P}(E^c_{c,k}) \geq 1 - \frac{1}{k^2} > 0$ and Reality can comply with $E^c_{c,k}$.

If $\overline{Q}(E)$ is positive, then Reality complies with the event E. The converse does not hold. We give a counterexample.

Example 5.17. In the coin-tossing game, Reality can comply with the event E of

$$\sum_n p_n < \infty \Rightarrow x_n = 0 \text{ for all } n.$$

but $\overline{Q}(E) = 0$.

Proof. Reality's strategy is just to announce $x_n = 0$ for every n. Consider a strategy of Skeptic. Since Skeptic should keep $\mathcal{K}_n$ non-negative,

$$\mathcal{K}_{n-1} + M_n(1-p_n) \geq 0,$$

and

$$M_n \geq -\frac{\mathcal{K}_{n-1}}{1-p_n}.$$

Then,

$$\mathcal{K}_n \leq \mathcal{K}_{n-1} - \frac{\mathcal{K}_{n-1}}{1-p_n} \cdot (-p_n) = \mathcal{K}_{n-1} \cdot \frac{1}{1-p_n}.$$

Thus,

$$\sup_n \mathcal{K}_n \leq \mathcal{K}_0 \cdot \frac{1}{\prod_n (1-p_n)}.$$

If $\sum_n p_n < \infty$, then $\prod_n (1-p_n)$ converges to a positive real. Thus, $\sup_n \mathcal{K}_n < \infty$. Hence, Reality can comply with this event.

Suppose $\overline{Q}(E) > 0$ for a contradiction. Reality should announce $x_n = 0$ for every n. Consider the strategy such that

$$0 < \mathcal{K}_0 < \overline{Q}(E) \text{ and } M_n = -\frac{\mathcal{K}_{n-1}}{1-p_n}.$$

Then, $\mathcal{K}_n$ keeps non-negative. Let $m \in \mathbb{N}$ be such that $\mathcal{K}_0 2^m > 1$. Forecaster announces $p_n = 1/2$ for each $n \leq m$ and $p_n = 2^{-n}$ for each $n > m$. Then $\sum_n p_n < \infty$. However,

$$\sup_n \mathcal{K}_n = \mathcal{K}_0 \cdot \frac{1}{\prod_n (1 - p_n)} > \mathcal{K}_0 \cdot 2^m > 1.$$

□

5.2.2. *Bounded forecasting game.*

Bounded Forecasting Game
Players: Forecaster, Skeptic, Reality
Protocol:
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots$:
Forecaster announces $p_n \in [0, 1]$.
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in [0, 1]$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - p_n)$.
Collateral Duties: Skeptic must keep $\mathcal{K}_n$ non-negative. Reality must keep $\mathcal{K}_n$ from tending to infinity.

Example 5.18. In the bounded forecasting game with $p_n = 1/2$ for all n, $\overline{Q}(x_1 \in [0, 1/2)) = \overline{Q}(x_1 \in [1/2, 1]) = 1$.

Proof. First we show that $\overline{Q}(x_1 \in [1/2, 1]) = 1$. Reality announces x_1 as follows:

$$x_1 = \begin{cases} 1 & \text{if } M_1 < 0 \\ 1/2 & \text{if } M_1 \geq 0 \end{cases}.$$

If $M_1 < 0$, then

$$\mathcal{K}_1 - \mathcal{K}_0 = M_1(1 - 1/2) < 0.$$

If $M_1 \geq 0$, then

$$\mathcal{K}_1 - \mathcal{K}_0 = M_1(1/2 - 1/2) = 0.$$

Reality announces $x_n = 1/2$ for all $n \geq 2$. Then, this strategy of Reality strongly complies with this event.

Next we show that $\overline{Q}(x_1 \in [0, 1/2)) = 1$. Let $\epsilon > 0$. It suffices to give a strategy of Reality that satisfies the following condition: if $\mathcal{K}_0 = 1 - \epsilon$ then $x_1 \in [0, 1/2)$ and $\sup_n \mathcal{K}_n \leq 1$.

We claim that the following strategy works: Reality announces

$$x_1 = \begin{cases} \frac{1-\epsilon}{2} & \text{if } M_1 < 0, \\ 0 & \text{if } M_1 \geq 0, \end{cases}$$

and $x_n = 1/2$ for all $n \geq 2$.

First notice that, if $\mathcal{K}_0 = 1 - \epsilon$, then $(1 - \epsilon)/2 \in [0, 1/2)$.

We claim that $\sup_n \leq 1$. It suffices to show that $\mathcal{K}_1 \leq 1$. If $M_1 \geq 0$, then

$$\mathcal{K}_1 = 1 - \epsilon + M_1(0 - 1/2) < 1.$$

Suppose that $M_1 < 0$. Since Skeptic needs to keep $\mathcal{K}_1$ non-negative, we have

$$\mathcal{K}_1 = 1 - \epsilon + M_1\left(1 - \frac{1}{2}\right) \geq 0,$$

which implies $M_1 \geq -2(1-\epsilon) > -2$. Then,

$$\begin{aligned}\mathcal{K}_1 =& 1-\epsilon + M_1\left(\frac{1-\epsilon}{2} - \frac{1}{2}\right)\\ =& 1-\epsilon + \frac{-M_1}{2}\cdot\epsilon < 1-\epsilon+\epsilon = 1.\end{aligned}$$

□

Corollary 5.19. *The function $\overline{Q}$ is not an inner measure in general and $\underline{Q}$ is not an outer measure in general.*

Corollary 5.20. *There is a game and an event E in it such that $\overline{Q}(E) = 1$ but Reality can not strongly comply with E.*

Proof. Consider the event E of $x_1 \in [0, 1/2)$ in the bounded forecasting game with $p_n = 1/2$ for all n. Then $\overline{Q}(E) = 1$ as is shown above. Suppose that Skeptic announces $M_1 = -1$ with $\mathcal{K}_0 = 1$. This move keeps $\mathcal{K}_1$ non-negative. Since $x_1 \in [0, 1/2)$,

$$\mathcal{K}_1 = 1 - 1(x_1 - 1/2) > 1.$$

Hence, Reality can not strongly comply with this event. □

Corollary 5.21. *We do not have $\underline{Q}(E) \leq \overline{Q}(E)$ or $\underline{Q}(E) \geq \overline{Q}(E)$ in general.*

Proof. In the bounded forecasting game with $p_n \in (0, 1)$ for all n, consider the event E that $x_n = p_n$ for infinitely many n. Reality can strongly comply with this event E by announcing $x_n = p_n$ for every n, thus $\overline{Q}(E) = 1$. On the other hand Reality can also strongly complies with E^c by announcing

$$x_n = \begin{cases} 1 & \text{if } M_n \leq 0 \\ 0 & \text{if } M_n > 0, \end{cases}$$

thus $\overline{Q}(E^c) = 1$ and $\underline{Q}(E) = 0$.

In the coin-tossing game, consider the event F that $x_n = 1$ for at most finitely many n. When $\sum_n p_n = \infty$, Skeptic can force the event F^c. Thus, $\overline{Q}(F) = 0$. When $\sum_n p_n < \infty$, Skeptic can force the event F. Thus, $\overline{Q}(F^c) = 0$ and $\underline{Q}(F) = 1$. □

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