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Abstract

We explore the networks that yield the largest mean consensus time of voter models under different update rules. By analytical and numerical means, we show that the so-called lollipop graph, barbell graph, and double star graph maximise the mean consensus time under the update rules called the link dynamics, voter model, and invasion process, respectively. For each update rule, the largest mean consensus time scales as $O(N^3)$, where N is the number of nodes in the network.

1 Introduction

Collective opinion formation in society seems to be an emerging phenomenon that occurs in networks of agents that interact and exchange opinions. The voter model is an individual-based stochastic model for collective opinion formation that has been studied in, above all, statistical physics and probability theory communities for decades [1–4].

In the voter model, a node assumes one of the different states (i.e., opinions) that can stochastically change over time. The configuration in which all nodes take the same state, i.e., consensus, is the only type of absorbing configuration of the voter model. The mean time to consensus, which we denote by $\langle T \rangle$, is a fundamental property of the voter model. For a given size of the population, $\langle T \rangle$ depends on the network structure, which suggests that

opinions spread faster on some networks than others. Not surprisingly, $\langle T \rangle$ is small for well-connected networks such as the complete graph for which $\langle T \rangle = O(N)$, where N is the number of nodes in the network [5]. More complex networks with small mean path lengths also yield linear or sublinear dependence of $\langle T \rangle$ on N [6–15]. In contrast, $\langle T \rangle$ can be large for networks in which communication between nodes is difficult for topological reasons. For example, the one-dimensional chain yields $\langle T \rangle = O(N^2)$ [2, 4, 16]. Networks with community structure also yield $\langle T \rangle = O(N^2)$ when intercommunity links are rare [17] because coordination between the different communities is a bottleneck of the entire process.

In this study, we explore network structure that maximises the consensus time. In practice, answering this question may help us understand why consensus is difficult in real society [18, 19]. In theory, we expect that there exist networks for which $\langle T \rangle$ is larger than $O(N^2)$ for the following reason. The mean consensus time of the voter model is often analysed through the random walk. In fact, the so-called coalescence random walk is the dual process of the voter model, implying that the mean time before random walkers coalesce into one gives $\langle T \rangle$ for the voter model [1, 16, 20, 21]. In addition, in the voter model on the chain, positions of active links, i.e., boundaries between adjacent nodes possessing the opposite states, perform a random walk until different active links meet and annihilate. This relationship helps us to calculate $\langle T \rangle$ for the chain (i.e., $\langle T \rangle = O(N^2)$) through the hitting or cover time of the random walk. Because the hitting time [22–25] and cover time [26] of the random walk is known to scale as $O(N^3)$ for some networks, $\langle T \rangle$ of the voter model may also scale as $O(N^3)$ for these and other networks.

In networks in which the degree (i.e., number of neighbouring nodes for a node) is heterogeneous, the consensus time depends on the specific rule with which we update the state of nodes [8,13,14]. Therefore, we explore the networks that maximise $\langle T \rangle$ separately for different update rules. For each update rule, we determine such networks by combining exact numerical calculations of $\langle T \rangle$ for small networks (section 3), coarse analytical arguments to evaluate lower and upper bounds of $\langle T \rangle$ (sections 4.1 and 4.2), analysis of the coalescing random walk (section 4.3), and direct numerical simulations for large networks (section 5). The results are summarised in Table 1.

2   Model

We consider undirected networks possessing N nodes and M links. Each node possesses either of the two states **0** and **1** at any time t. In each update event, the state of a node is updated, and time $1/N$ is consumed. We repeat this procedure until either the consensus of state **0** or that of state **1** is reached. It should be noted that a node is updated once per unit time on average.

We examine three update rules according to Refs. [13, 14]. Under the link dynamics (LD) [8,11,13,14], we first select a link with probability $1/M$. Denote by i and j the two endpoints of the selected link. With probability $1/2$, i copies j's state. With the remaining probability $1/2$, j copies i's state. In this manner, local consensus is obtained in a single update event. If i and j have the same state beforehand, the state of neither node changes. Under the voter model (VM) update rule [6,7,9–15], we first select a node i to be updated with probability $1/N$. Then, a neighbour of i, denoted by j, is selected out of the neighbours of i with the equal probability, i.e., the inverse of i's degree. Then, i copies j's state. Under the invasion process (IP) [8,13,14], we first select a parent node i with probability $1/N$. Then, a neighbour of i, again denoted by j, is selected for updating with the probability equal to the inverse of i's degree. Then, j copies i's state. In the following, we collectively refer to the dynamics under LD, VM, and IP as opinion dynamics.

The three update rules coincide for regular networks (i.e., networks in which all nodes have the same degree). In heterogeneous networks, opinion dynamics including $\langle T \rangle$ depends on the update rule [8,13,14]. In the present study, we allow heterogeneous networks.

3 Small networks maximising the mean consensus time

3.1 Methods

In this section, we consider small networks and exactly calculate $\langle T \rangle$ for any given network and numerically maximise $\langle T \rangle$ by gradually morphing network structure with N (i.e., number of nodes) and M (i.e., number of links) fixed.

We refer to the collection of the states of the N nodes as configuration. There are 2^N possible configurations because each node takes either state **0** or **1**. Two configurations correspond to consensus. The mean consensus time depends on the starting configuration. We denote a configuration by $[s_1 \cdots s_N]$, where $s_i \in \{0, 1\}$ is the state of node i ($1 \leq i \leq N$), and the mean consensus time starting from configuration $[s_1 \cdots s_N]$ by $\langle T \rangle_{[s_1 \cdots s_N]}$. It should be noted that $\langle T \rangle_{[0 \cdots 0]} = \langle T \rangle_{[1 \cdots 1]} = 0$.

In a single update, the configuration may switch to a neighbouring configuration, where the neighbourhood of a configuration is defined as the configurations that differ from the original configuration just in a single s_i. Each configuration has N neighbours, as shown in Fig. 1(a).

To explain the method for exactly calculating $\langle T \rangle$, we refer to the network with $N = 4$ nodes shown in Fig. 1(b) and consider configuration [0101] (i.e., nodes 1 and 3 possess state **0**, and nodes 2 and 4 possess state **1**).

Under LD, node 1 imitates node 2 in a single update with probability $1/8$, and node 1 imitates node 4 with probability $1/8$. If either of these events occurs, the configuration transits from [0101] to [1101]. If the link connecting nodes 1 and 3 is selected for the update, the configuration does not change. This event occurs with probability $1/4$. By taking into account all possible transitions from [0101] in a similar manner, we obtain

$$\langle T \rangle_{[0101]} = \frac{1}{4} \langle T \rangle_{[0101]} + \frac{1}{4} \langle T \rangle_{[0001]} + \frac{1}{8} \langle T \rangle_{[0100]} + \frac{1}{8} \langle T \rangle_{[0111]} + \frac{1}{4} \langle T \rangle_{[1101]} + \frac{1}{N}. \tag{1}$$

The last term on the right-hand side of Eq. (1) results from the fact that an update consumes time $1/N$ by definition.

We can similarly derive the recursive equations for $2^N - 2$ starting configurations in total, i.e., those except [0000] and [1111]. By solving the set of the $2^N - 2$ linear equations, we obtain the mean consensus time for all initial configurations. Finally, we define $\langle T \rangle$ as the mean consensus time averaged over the initial configurations that possess the equal number of state **0** and **1** nodes. When N is even, there are $N!/[(N/2)!]^2$ such configurations. When N is odd, the average is taken over the $N!/\{[(N+1)/2]![(N-1)/2]!\}$ configurations in which the number of nodes in state **0** is smaller than that in state **1** just by one.

Using this exact method for calculating $\langle T \rangle$, we numerically explored the network structure with the largest $\langle T \rangle$ value for a given N, M, and update rule as follows.

1. Generate an initial network G using one of the two following methods. In the first method, we prepare the star with N nodes and $N - 1$ links. Then, we add the remaining $M - N + 1$ links between pairs of leaf (i.e., non-hub) nodes with the uniform density. In the second method, we prepare the chain with N nodes and $N - 1$ links. Then, we similarly add the remaining $M - N + 1$ links between randomly selected nonadjacent pairs of nodes. In either method, we prohibit multiple links when adding the $M - N + 1$ links. Networks generated by the two methods represent two extreme initial conditions.

2. Calculate $\langle T \rangle$ for G under the given update rule.

3. Select a link with the uniform probability $1/M$. Denote the endpoints of the selected link by i and j. If the degrees of i and j are at least two, delete the link, select a pair of nonadjacent nodes with the uniform probability, and create a link between the two nodes. If the degree of i is equal to one and that of j is at least two, we disconnect the link from j but keep i as an endpoint of the new link. We select the other endpoint of the new link with the uniform probability from the nodes that are not adjacent to i and connect it to i. It should be noted that either the degree of i or that of j is at least two because the network is

in fact connected throughout the procedure. If the generated network is disconnected, we repeat the procedure until a connected network is generated. We refer to the new network as G'.

4. Calculate $\langle T \rangle$ for G'.

5. If $\langle T \rangle$ is larger for G' than G, we replace G by G'.

6. We repeat steps 3, 4, and 5 until the local maximum of $\langle T \rangle$ is reached. In practice, if the rewiring does not occur in more than 5000 steps, we stop repeating steps 3, 4, and 5, calculate $\langle T \rangle$, and record the network structure.

We run simulations with five initial networks generated by each of the two methods (i.e., ten initial conditions in total). When the final network depends on the initial network, we select the one yielding the largest $\langle T \rangle$ value. It should be noted that the obtained network realises a local, but not global, maximum of $\langle T \rangle$.

3.2 Results

First, we set $N = 10$ and apply the optimisation procedure described in section 3.1.

The networks maximising $\langle T \rangle$ under LD are shown for various M values in Fig. 2(a). The generated networks are close to the lollipop graph [23, 24, 26], which is defined as a network composed of a clique and a one-dimensional chain grafted to the clique. By definition, the lollipop graph can be only formed for specific values of M given a value of N. Figure 2(a) shows that when M are these values (i.e., $M = 10$, 12, 15, 19), the network that maximises $\langle T \rangle$ is the lollipop graph. For other values of M, the obtained networks are close to the lollipop graph. The maximised $\langle T \rangle$ value is plotted against M in Fig. 3(a). The peaks are located at the values of M that enable the lollipop graph. Therefore, the lollipop graph is suggested to maximise $\langle T \rangle$ under LD.

The networks maximising $\langle T \rangle$ under VM are shown in Fig. 2(b). When $M = 11$, 15, and 21, the generated networks are the barbell graph [22, 25], which is defined as the network possessing two cliques of the equal size connected by a chain. For the other M values, $\langle T \rangle$ is the largest for networks close to the barbell graph. The maximised $\langle T \rangle$ value is plotted against M in Fig. 3(b). The peaks of $\langle T \rangle$ are located at $M = 11$, 13, and 15. When $M = 13$, the generated network is an asymmetric variant of the barbell graph in which one clique has three nodes and the other clique has four nodes [Fig. 2(b)]. Taken together with the results for $M = 11$ and 15, the barbell graph is suggested to maximise $\langle T \rangle$ under VM.

The networks maximising $\langle T \rangle$ under IP are shown in Fig. 2(c). When $M = 9$, which is the minimum possible value to keep the network connected,

the generated network is the so-called double star graph (Fig. 4), in which two stars are connected by an additional link between the two hubs [27]. When M is slightly larger than $N - 1$, the generated networks are close to the double star graph [Fig. 2(c)]. For larger M values, the generated networks are not similar to the double star graph. The maximised $\langle T \rangle$ value is plotted against M in Fig. 3(c). The $\langle T \rangle$ value monotonically decreases with M, suggesting that the double star graph maximises $\langle T \rangle$ under IP.

Next, for LD, we confine ourselves to the lollipop graph and search for the lollipop graph with the largest value of $\langle T \rangle$. To this end, we set $N = 15$ and $N = 20$, and vary the size of the clique. We are allowed to use larger values of N as compared to the numerical simulations described in Figs. 2 and 3, for which $N = 10$, for two reasons. First, in the current set of numerical simulations, we do not have to maximise $\langle T \rangle$ by gradually changing networks. Second, because of the symmetry inherent in the lollipop graph, we can considerably reduce the number of the linear equations to solve [e.g., Eq. (1)]. Concretely, we have to only maintain the number of nodes in the **1** state in the clique and the configuration of the chain, because all nodes in the clique are structurally equivalent.

In Fig. 5(a), $\langle T \rangle$ is plotted as a function of the size of the clique in the lollipop graph when N is fixed. The figure indicates that $\langle T \rangle$ is the largest for the lollipop graph having approximately half the nodes in the clique. The corresponding results for the barbell graph under VM are shown in Fig. 5(b). The figure indicates that $\langle T \rangle$ is the largest for the barbell graph that has approximately $N/3$ nodes in each clique and the chain.

4 Analytical estimation of the mean consensus time

In this section, we analytically assess the dependence of $\langle T \rangle$ on N for the lollipop, barbell, and double star graphs under the three update rules. Our analysis is based on the probability and mean time for transitions among typical coarse-grained configurations.

4.1 Lollipop graph

Because we are interested in the asymptotic dependence of $\langle T \rangle$ on N, we assume in this section that the lollipop graph contains $2N$ nodes; the clique and chain in the lollipop graph contain N nodes each. For all three update rules, consensus within the clique is reached in $O(N)$ time if the nodes in the chain do not affect the opinion dynamics within the clique.

In the following, we estimate the mean consensus time by assessing approximate lower and upper bounds of $\langle T \rangle$. Typical configurations of the opinion dynamics on the lollipop graph are schematically shown in Fig. 6.

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An open and filled circle represents a node in the **0** and **1** states, respectively. The initial configuration is totally random and depicted as configuration I in Fig. 6.

4.1.1 LD

We call a consecutive segment on the chain occupied by the same state (i.e., **0** or **1**) the domain. Under LD, a node in the chain has degree two (except for the one at the end of the chain, whose degree is one) and is updated once in $N(N-1)/2 + N = O(N^2)$ steps on average. Because an update event consumes time $1/2N$ by definition, the mean time before a node on the chain is updated is $O(N)$. In the ordinary voter model on the one-dimensional chain, the domain size grows to an $O(N)$ size in $O(N^2)$ time [28, 29]. Therefore, under LD, the time needed for domains to grow to an $O(N)$ size is equal to $O(N^3)$. By this time, consensus would be reached within the clique because consensus of the clique if it were isolated occurs in $O(N)$ time. To summarise, the transition from configuration I to configuration II occurs in $O(N^3)$ time. It should be noted that configuration II shown in Fig. 6(a) include configurations in which the chain contains two or more domains of the opposite states with characteristic length $O(N)$. We obtain

$$\langle T \rangle_{\mathrm{I}} = O(N^3) + \langle T \rangle_{\mathrm{II}}, \tag{2}$$

where $\langle T \rangle_{\mathrm{I}}$ and $\langle T \rangle_{\mathrm{II}}$ are the mean consensus time starting from configurations I and II, respectively. Likewise we define $\langle T \rangle_{\mathrm{III}}$, $\langle T \rangle_{\mathrm{IV}}$, and $\langle T \rangle_{\mathrm{V}}$, where III, IV, and V correspond to the three configurations shown in Fig. 6(a).

There are two types of configurations that can be reached from configuration II. With a $O(1)$ probability p^{LD}, which is in fact equal to the fraction of nodes on the chain that takes the same state as that of the clique [14], configuration II transits to the consensus of the entire network (Fig. 6(a)). This event requires $O(N^3)$ time because each node on the chain is updated once per $O(N)$ time and consensus of the chain requires $O(N^2)$ update events per node [16]. Otherwise, starting from configuration II, the chain is eventually occupied by the state opposite to that taken by the clique (configuration IV shown in Fig. 6(a)) with probability $1 - p^{\mathrm{LD}} = O(1)$. This event also requires $O(N^3)$ time. Therefore, we obtain

$$\langle T \rangle_{\mathrm{II}} = p^{\mathrm{LD}} O(N^3) + \left(1 - p^{\mathrm{LD}}\right) \left[O(N^3) + \langle T \rangle_{\mathrm{IV}}\right]. \tag{3}$$

There are two possible types of events that occur in configuration IV. First, the state taken by the chain may invade a node in the clique (configuration III). Second, the state taken by the clique may invade the node on the chain adjacent to the clique (configuration V). Either event occurs with probability $1/2$ because the link connecting the clique and chain must be taken, and the direction of opinion transmission on the selected link determines the type of the event. The occurrence of either event requires $O(N)$

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time because the mentioned single link in the chain must be selected for either event to occur. Therefore, we obtain

$$\langle T \rangle_{\mathrm{IV}} = \frac{1}{2}\left[O(N) + \langle T \rangle_{\mathrm{III}}\right] + \frac{1}{2}\left[O(N) + \langle T \rangle_{\mathrm{V}}\right]. \tag{4}$$

From configuration V, the consensus of the entire lollipop graph is reached in $O(N^3)$ time with probability $1/N$ because, under LD, the consensus probability of a state is equal to the number of nodes possessing the state in an arbitrary undirected network (chain in the present case) [13,14]. With probability $(N-1)/N$, configuration IV is revisited. This event requires $O(N)$ updates per node, which corresponds to $O(N^2)$ time. These probabilities and times can be calculated as the hitting probability and time of the random walk on the chain with two absorbing boundaries [3,30]. Therefore, we obtain

$$\langle T \rangle_{\mathrm{V}} = \frac{1}{N}O(N^3) + \frac{N-1}{N}\left[O(N^2) + \langle T \rangle_{\mathrm{IV}}\right]. \tag{5}$$

From configuration III, the single invader in the clique spreads its opinion in the clique to lead to consensus of the entire network after $O(N)$ time with probability approximately equal to $1/N$. This probability may be an overestimate because the clique and chain in fact interact possibly to prevent consensus from being reached in $O(N)$ time. Otherwise, configuration IV or a more complicated configuration as represented by configuration I is revisited. The transition to the latter configuration may occur because, starting from configuration III, the clique may experience a mixture of the **0** and **1** states for some time during which the clique transforms the chain to alternating small domains of the opposite states. However, to assess a lower bound of $\langle T \rangle$, we pretend that configuration IV is reached from configuration III with probability $(N-1)/N$ after short time $O(\ln N)$, which is true if the clique were disconnected from the chain [5]. Then, we obtain

$$\langle T \rangle_{\mathrm{III}} = \frac{1}{N}O(N) + \frac{N-1}{N}\left[O(\ln N) + \langle T \rangle_{\mathrm{IV}}\right]. \tag{6}$$

Because consensus would require longer time if we start from configuration I than IV, we assume that Eq. (6) gives a lower bound of $\langle T \rangle$. By solving Eqs. (2), (3), (4), (5), and (6), we obtain $\langle T \rangle_{\mathrm{I}}$, $\langle T \rangle_{\mathrm{II}}$, $\langle T \rangle_{\mathrm{III}}$, $\langle T \rangle_{\mathrm{IV}}$, $\langle T \rangle_{\mathrm{V}} = O(N^3)$ as a rough lower bound of $\langle T \rangle$.

To estimate an upper bound of $\langle T \rangle$, we consider the transitions between typical configurations shown in Fig. 6(b), which differs from Fig. 6(a) only in the transitions from configuration III. Now, with probability $q^{\mathrm{LD}} = 1/N$, we presume that the configuration returns to a random one (configuration I), which is adversary to consensus. With probability $1 - q^{\mathrm{LD}}$, we safely assume that configuration IV is revisited in time $O(\ln N)$. It should be noted that in time $O(\ln N)$, a link on the chain has rarely been selected for large N, such

that the chain would not be divided into multiple domains of the opposite states in time $O(\ln N)$. By combining Eqs. (2), (3), (4), (5), and

$$\langle T \rangle_{\text{III}} = q^{\text{LD}} \left[\langle T \rangle_{\text{I}} + O(N) \right] + (1 - q^{\text{LD}}) \left[\langle T \rangle_{\text{IV}} + O(\ln N) \right], \qquad (7)$$

we obtain $\langle T \rangle_{\text{I}}, \langle T \rangle_{\text{II}}, \langle T \rangle_{\text{III}}, \langle T \rangle_{\text{IV}}, \langle T \rangle_{\text{V}} = O(N^3)$ as a rough upper bound of $\langle T \rangle$. It should be noted that the same conclusion holds true as long as $q^{\text{LD}} = O(1/N)$.

In fact, $\langle T \rangle$ is exactly upper-bounded by $O(N^3)$ for arbitrary networks on the basis of the following alternative arguments. We denote the number of nodes in the **1** state by N_1. In an update event, N_1 increases or decreases by one with the same probability [13, 14]. With the remaining probability, N_1 does not change. Therefore, N_1 performs an unbiased random walk on interval $[0, N]$. The probability that N_1 changes in an update event is at least $O(1/N^2)$ because there are at most $N(N-1)/2$ links in the network. Because a single update event consumes time $1/N$ and the random walk hits either boundary (i.e., 0 or N) after $O(N^2)$ actual hops, we obtain $[(1/N)/O(1/N^2)] \times O(N^2) = O(N^3)$ as an upper bound of $\langle T \rangle$.

On the basis of the rough lower and upper bounds, we conclude $\langle T \rangle = O(N^3)$ for the combination of the lollipop graph and LD.

4.1.2   VM

We evaluate approximate lower and upper bounds of $\langle T \rangle$ under VM and IP in a similar manner. It should be noted that a node in the chain is updated once per $O(1)$ time under VM and IP, which is in contrast to the case of LD.

To derive an approximate lower bound under VM, we consider the same transitions among typical configurations as those under LD [Fig. 6(a)]. We obtain

$$\langle T \rangle_{\text{I}} = O(N^2) + \langle T \rangle_{\text{II}}, \qquad (8)$$

$$\langle T \rangle_{\text{II}} = p^{\text{VM}} O(N^2) + \left(1 - p^{\text{VM}}\right) \left[O(N^2) + \langle T \rangle_{\text{IV}} \right], \qquad (9)$$

$$\langle T \rangle_{\text{III}} = \frac{1}{N} O(N) + \frac{N-1}{N} \left[O(\ln N) + \langle T \rangle_{\text{IV}} \right], \qquad (10)$$

$$\langle T \rangle_{\text{IV}} = \frac{2}{N+2} \left[O(N) + \langle T \rangle_{\text{III}} \right] + \frac{N}{N+2} \left[O(1) + \langle T \rangle_{\text{V}} \right], \qquad (11)$$

$$\langle T \rangle_{\text{V}} = \frac{1}{2(N-1)} O(N^2) + \frac{2N-3}{2(N-1)} \left[O(N) + \langle T \rangle_{\text{IV}} \right], \qquad (12)$$

where $p^{\text{VM}}, 1 - p^{\text{VM}} = O(1)$. To derive Eq. (11), we used the fact that, in a single update event, the transition from configuration IV to III occurs with probability $(1/2N) \times (1/N)$ and that from configuration IV to V occurs with probability $(1/2N) \times (1/2)$. To derive Eq. (12), we used the fact that the fixation probability for a node i (i.e., probability that the consensus of

the state taken by i is reached when all the other $N - 1$ nodes possess the opposite state) under VM is proportional to i's degree [13,14]. Consider an isolated chain of length N in which the leftmost node is in state **1** and all the other $N-1$ nodes are in state **0**, as schematically shown in configuration V in Fig. 6(a). Then, **0** and **1** fixate with probability $(2N - 3)/[2(N - 1)]$ and $1/[2(N - 1)]$, respectively; $2(N - 1)$ is equal to the sum of the degree of all nodes. Equations (8), (9), (10), (11), and (12) yield $\langle T \rangle_\mathrm{I}$, $\langle T \rangle_\mathrm{II}$, $\langle T \rangle_\mathrm{III}$, $\langle T \rangle_\mathrm{IV}$, $\langle T \rangle_\mathrm{V} = O(N^2)$.

To estimate an upper bound of $\langle T \rangle$, we consider transitions among typical configurations shown in Fig. 6(c). It is different from the diagram for LD [Fig. 6(b)] in that the transition from configuration III to configuration IV is not present. We modified the diagram because, under VM, the node that belongs to the chain and is adjacent to the clique imitates the state of a node in the clique every $O(1)$ time. Therefore, even within time $O(\ln N)$, nodes in the chain may flip the state many times such that configuration IV is rarely visited directly from configuration III. Given this situation, we replace Eq. (10) by

$$\langle T \rangle_\mathrm{III} = O(N) + \langle T \rangle_\mathrm{I}. \tag{13}$$

In other words, once configuration III is reached, we assume that configuration I is always reached after $O(N)$ time. Because configuration I is considered to be adversary to consensus as compared to configuration IV, we assume that Eq. (13) gives an upper bound of $\langle T \rangle$. The $O(N)$ time on the right-hand side of Eq. (13) comes from the fact that the opinion dynamics in an isolated clique of size N ends up with consensus in $O(N)$ time. Using Eqs. (8), (9), (11), (12), and (13), we obtain $\langle T \rangle_\mathrm{I}$, $\langle T \rangle_\mathrm{II}$, $\langle T \rangle_\mathrm{III}$, $\langle T \rangle_\mathrm{IV}$, $\langle T \rangle_\mathrm{V} = O(N^2)$.

Therefore, we conclude $\langle T \rangle = O(N^2)$ for the combination of the lollipop graph and VM.

4.1.3 IP

For an approximate lower bound under IP, we obtain

$$\langle T \rangle_\mathrm{I} = O(N^2) + \langle T \rangle_\mathrm{II}, \tag{14}$$

$$\langle T \rangle_\mathrm{II} = p^\mathrm{IP} O(N^2) + \left(1 - p^\mathrm{IP}\right) \left[O(N^2) + \langle T \rangle_\mathrm{IV}\right], \tag{15}$$

$$\langle T \rangle_\mathrm{III} = \frac{1}{N} O(N) + \frac{N - 1}{N} \left[O(\ln N) + \langle T \rangle_\mathrm{IV}\right], \tag{16}$$

$$\langle T \rangle_\mathrm{IV} = \frac{N}{N + 2} \left[O(1) + \langle T \rangle_\mathrm{III}\right] + \frac{2}{N + 2} \left[O(N) + \langle T \rangle_\mathrm{V}\right], \tag{17}$$

$$\langle T \rangle_\mathrm{V} = \frac{2}{N + 2} O(N^2) + \frac{N}{N + 2} \left[O(N) + \langle T \rangle_\mathrm{IV}\right], \tag{18}$$

where p^IP, $1 - p^\mathrm{IP} = O(1)$. To derive Eq. (17), we used the fact that, in a single update event, configuration IV transits to III with probability

$(1/2N) \times (1/2)$ and V with probability $(1/2N) \times (1/N)$. To derive Eq. (18), we used the fact that the fixation probability under IP is inversely proportional to the degree of the node [13, 14]. Similar to the case of VM, consider an isolated chain of length N in which the leftmost node is in state **1** and all the other $N - 1$ nodes are in state **0**. Then, **0** and **1** fixate with probability $\propto (N - 2) \times (1/2) + 1 \times (1/1)$ and $\propto 1 \times (1/1)$, respectively. Equations (14), (15), (16), (17), and (18) yield $\langle T \rangle_{\mathrm{I}}$, $\langle T \rangle_{\mathrm{II}} = O(N^2)$ and $\langle T \rangle_{\mathrm{III}}$, $\langle T \rangle_{\mathrm{IV}}$, $\langle T \rangle_{\mathrm{V}} = O(N \ln N)$. Because a random initial condition yields $\langle T \rangle_{\mathrm{I}} = O(N^2)$, we regard $O(N^2)$ as a rough lower bound of $\langle T \rangle$.

To estimate an upper bound of $\langle T \rangle$, we consider the same transitions among typical configurations as those for LD [Fig. 6(b)] and replace Eq. (16) by

$$\langle T \rangle_{\mathrm{III}} = q^{\mathrm{IP}} \left[\langle T \rangle_{\mathrm{I}} + O(N) \right] + (1 - q^{\mathrm{IP}}) \left[\langle T \rangle_{\mathrm{IV}} + O(\ln N) \right], \qquad (19)$$

where $q^{\mathrm{IP}} = 1/N$. To derive Eq. (19), we assumed that configuration IV is reached in $O(\ln N)$ time with probability $1 - q^{\mathrm{IP}} = (N - 1)/N$. Within this time, the state monopolising the clique would not invade the chain because such an event requires $O(N)$ time to occur. By combining Eqs. (14), (15), (17), (18), and (19), we obtain $\langle T \rangle_{\mathrm{I}}$, $\langle T \rangle_{\mathrm{II}}$, $\langle T \rangle_{\mathrm{III}}$, $\langle T \rangle_{\mathrm{IV}}$, $\langle T \rangle_{\mathrm{V}} = O(N^2)$ as a rough upper bound of $\langle T \rangle$.

Therefore, we conclude $\langle T \rangle = O(N^2)$ for the combination of the lollipop graph and IP.

4.2 Barbell graph

In the barbell graph, the two cliques may first reach the unanimity of the opposite states. This phenomenon may delay consensus, which is the case for the two-clique graph [14, 17]. For simplicity, we assume in this section that each clique and the chain have N nodes each such that the barbell graph is composed of $3N$ nodes. We evaluate approximate lower and upper bounds for each update rule in the manner similar to that for the lollipop graph. Typical configurations and the transitions among them for the barbell graph are schematically shown in Fig. 7.

4.2.1 LD

Under LD, a node belonging to the chain is updated once per $O(N)$ time. Therefore, the consensus of the chain would require $O(N^3)$ time if it were isolated. When the consensus of the chain is reached, each clique would have also reached consensus because the consensus within a clique occurs in $O(N)$ time. When two cliques end up with the same state after $O(N^3)$ time, the consensus of the entire barbell graph is realised. Otherwise, configuration III shown in Fig. 7(a) is reached. Either event occurs with probability $1/2$.

Therefore, we obtain

$$\langle T \rangle_{\mathrm{I}} = \frac{1}{2} O(N^3) + \frac{1}{2} \left[O(N^3) + \langle T \rangle_{\mathrm{III}} \right]. \tag{20}$$

Starting from configuration III, the next change in the state occurs at the boundary between the chain and one of the two cliques whose state is opposite to that of the chain. Similar to the transition from configuration IV in the case of the lollipop graph (section 4.1.1), we obtain

$$\langle T \rangle_{\mathrm{III}} = \frac{1}{2} \left[O(N) + \langle T \rangle_{\mathrm{II}} \right] + \frac{1}{2} \left[O(N) + \langle T \rangle_{\mathrm{IV}} \right]. \tag{21}$$

Starting from configuration IV [Fig. 7(a)], the state taken by just one node in the chain, which is adjacent to a clique, fixates in the chain in $O(N^3)$ time with probability $1/N$. Otherwise, with probability $(N-1)/N$, the state taken by $N-1$ nodes in the chain fixates in the chain in $O(N^2)$ time. In both cases, configuration III is recovered because of the symmetry. Therefore, we obtain

$$\langle T \rangle_{\mathrm{IV}} = O(N^2) + \langle T \rangle_{\mathrm{III}}. \tag{22}$$

We proceed similarly to the case of the lollipop graph to evaluate an approximate lower bound of $\langle T \rangle$. In other words, as shown in Fig. 7(a), starting from configuration II, we suppose that the consensus of the entire network is attained with probability $1/N$ in $O(N)$ time and that configuration III is revisited in $O(\ln N)$ time with the remaining probability. Then, we obtain

$$\langle T \rangle_{\mathrm{II}} = \frac{1}{N} O(N) + \frac{N-1}{N} \left[O(\ln N) + \langle T \rangle_{\mathrm{III}} \right]. \tag{23}$$

By combining Eqs. (20), (21), (22), and (23), we obtain $\langle T \rangle_{\mathrm{I}}, \langle T \rangle_{\mathrm{II}}, \langle T \rangle_{\mathrm{III}}, \langle T \rangle_{\mathrm{IV}} = O(N^3)$.

To evaluate an approximate upper bound, we only modify the transitions from configuration II, similar to the case of the lollipop graph. Without loss of generality, we assume that the chain and a first clique is occupied by the **1** state and that the second clique has a single node in state **1** and $N-1$ nodes in state **0**, as illustrated by configuration II in Fig. 7(a). With a small probability $r^{\mathrm{LD}} = O(1/N)$, the **1** state proliferates in the second clique to possibly occupy it in $O(N)$ time. Because a node in the chain is updated once per $O(N)$ time, the configuration of the chain may turn into a mixture of states **0** and **1** in the $O(N)$ time. Therefore, with probability r^{LD}, we assume that configuration II transits to configuration I. With probability $1 - r^{\mathrm{LD}}$, the single node in state **1** is extinguished in the second clique in $O(\ln N)$ time such that configuration III is revisited. It should be noted that a node in the chain is rarely updated in $O(\ln N)$ time such that the

unanimity on the chain is not perturbed in the $O(\ln N)$ time. By collecting these contributions, we obtain

$$\langle T \rangle_{\mathrm{II}} = r^{\mathrm{LD}} \left[O(N) + \langle T \rangle_{\mathrm{I}} \right] + \left(1 - r^{\mathrm{LD}} \right) \left[O(\ln N) + \langle T \rangle_{\mathrm{III}} \right]. \qquad (24)$$

By combining Eqs. (20), (21), (22), and (24), we obtain $\langle T \rangle_{\mathrm{I}}, \langle T \rangle_{\mathrm{II}}, \langle T \rangle_{\mathrm{III}}, \langle T \rangle_{\mathrm{IV}} = O(N^3)$. It should be noted that the arguments based on the unbiased random walk (section 4.1.1) also lead to same upper bound of $\langle T \rangle$.

Therefore, we conclude $\langle T \rangle = O(N^3)$ for the combination of the barbell graph and LD.

4.2.2   VM

For an approximate lower bound of $\langle T \rangle$ under VM, we assume the same types of transitions among configurations as those for LD [Fig. 7(a)] to obtain

$$\langle T \rangle_{\mathrm{I}} = \frac{1}{2} O(N^2) + \frac{1}{2} \left[O(N^2) + \langle T \rangle_{\mathrm{III}} \right], \qquad (25)$$

$$\langle T \rangle_{\mathrm{II}} = \frac{1}{N} O(N) + \frac{N-1}{N} \left[O(\ln N) + \langle T \rangle_{\mathrm{III}} \right], \qquad (26)$$

$$\langle T \rangle_{\mathrm{III}} = \frac{2}{N+2} \left[O(N) + \langle T \rangle_{\mathrm{II}} \right] + \frac{N}{N+2} \left[O(1) + \langle T \rangle_{\mathrm{IV}} \right], \qquad (27)$$

$$\langle T \rangle_{\mathrm{IV}} = O(N) + \langle T \rangle_{\mathrm{III}}, \qquad (28)$$

leading to $\langle T \rangle_{\mathrm{I}}, \langle T \rangle_{\mathrm{II}}, \langle T \rangle_{\mathrm{III}}, \langle T \rangle_{\mathrm{IV}} = O(N^3)$.

To estimate an upper bound, we assume the transitions among configurations shown in Fig. 7(c). It is different from those for LD [Fig. 7(b)] in the transitions starting from configuration II. Similar to the case of the LD, we assume without loss of generality that the chain, first clique, and a single node in the second clique are in state **1** and the other $N-1$ nodes in the second clique are in state **0**. The **1** state in the second clique fixates there with probability $r^{\mathrm{VM}} = O(1/N)$ in $O(N)$ time and is eradicated with probability $1 - r^{\mathrm{VM}}$ in $O(\ln N)$ time. Because each node in the chain is updated once per unit time, domains of characteristic length $O(\sqrt{\ln N})$ are formed within the $O(\ln N)$ time in the latter situation. The resulting configuration is shown as configuration V in Fig. 7(c). Given configuration V, the consensus of the chain occurs in $O(N^2)$ time, such that configuration III is revisited. By combining Eqs. (25), (27), (28), and

$$\langle T \rangle_{\mathrm{II}} = r^{\mathrm{VM}} \left[O(N) + \langle T \rangle_{\mathrm{I}} \right] + \left(1 - r^{\mathrm{VM}} \right) \left[O(\ln N) + O(N^2) + \langle T \rangle_{\mathrm{III}} \right], \quad (29)$$

we obtain $\langle T \rangle_{\mathrm{I}}, \langle T \rangle_{\mathrm{II}}, \langle T \rangle_{\mathrm{III}}, \langle T \rangle_{\mathrm{IV}} = O(N^3)$.

Therefore, we conclude $\langle T \rangle = O(N^3)$ for the combination of the barbell graph and VM.

4.2.3 IP

For an approximate lower bound of $\langle T \rangle$ under IP, we consider Fig. 7(a) to obtain

$$\langle T \rangle_{\mathrm{I}} = \frac{1}{2}O(N^2) + \frac{1}{2}\left[O(N^2) + \langle T \rangle_{\mathrm{III}}\right], \tag{30}$$

$$\langle T \rangle_{\mathrm{II}} = \frac{1}{N}O(N) + \frac{N-1}{N}\left[O(\ln N) + \langle T \rangle_{\mathrm{III}}\right], \tag{31}$$

$$\langle T \rangle_{\mathrm{III}} = \frac{N}{N+2}\left[O(1) + \langle T \rangle_{\mathrm{II}}\right] + \frac{2}{N+2}\left[O(N) + \langle T \rangle_{\mathrm{IV}}\right], \tag{32}$$

$$\langle T \rangle_{\mathrm{IV}} = O(N) + \langle T \rangle_{\mathrm{III}}, \tag{33}$$

leading to $\langle T \rangle_{\mathrm{I}} = O(N^2)$ and $\langle T \rangle_{\mathrm{II}}, \langle T \rangle_{\mathrm{III}}, \langle T \rangle_{\mathrm{IV}} = O(N \ln N)$. Because a random initial condition yields $\langle T \rangle_{\mathrm{I}} = O(N^2)$, we regard $O(N^2)$ as a rough lower bound of $\langle T \rangle$.

For an approximate upper bound, we consider the same diagram as that for LD [Fig. 7(b)]. The rational behind this choice is that the unanimity in the chain in configuration II is not disturbed in $O(\ln N)$ time. This holds true because it takes $O(N)$ time before the state taken by a clique may invade a node that belongs to the chain and is adjacent to the clique. Therefore, we replace Eq. (31) by

$$\langle T \rangle_{\mathrm{II}} = r^{\mathrm{IP}}\left[O(N) + \langle T \rangle_{\mathrm{I}}\right] + \left(1 - r^{\mathrm{IP}}\right)\left[O(\ln N) + \langle T \rangle_{\mathrm{III}}\right], \tag{34}$$

where $r^{\mathrm{IP}} = O(1/N)$. By combining Eqs. (30), (32), (33), and (34), we obtain $\langle T \rangle_{\mathrm{I}}, \langle T \rangle_{\mathrm{II}}, \langle T \rangle_{\mathrm{III}}, \langle T \rangle_{\mathrm{IV}} = O(N^2)$.

Therefore, we conclude $\langle T \rangle = O(N^2)$ for the combination of the barbell graph and IP.

4.3 Double star graph

We evaluate the mean consensus time for the double star graph under the three update rules using a different method from that for the lollipop and barbell graphs. The dual process of the opinion dynamics is the so-called coalescing random walk [1, 16, 20, 21]. Consider N simple random walkers, with one walker located at each node initially. Walkers that have arrived at the same node are assumed to coalesce into one. Then, all N walkers eventually coalesce into one in a finite network. The time at which the last two walkers coalesce is equal to the consensus time of the opinion dynamics. The time to the coalescence of the last two walkers is considered to dominate the entire coalescing process starting from the N walkers. Therefore, in this section, we assess $\langle T \rangle$ by measuring the mean time at which two walkers starting from different nodes in the double star graph meet. We used the same technique for a different network in a previous study [17].

Consider the double star graph with $2N$ nodes as shown in Fig. 4. We call two symmetric parts composed of N nodes the classes 1 and 2. Each class contains one hub node with degree N and $N-1$ leaf nodes with degree 1.

We define

$$\boldsymbol{p}(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \\ p_5(t) \end{pmatrix}, \tag{35}$$

where t denotes the time. In Eq. (35), $p_1(t)$ is the probability that the two walkers are located at different leaves in a single class (i.e., class 1 or 2) at time t (configuration 1 shown in Fig. 8). $p_2(t)$ is the probability that one walker stays in a class 1 leaf and the other walker stays in the class 1 hub, or one walker stays in a class 2 leaf and the other walker stays in the class 2 hub (configuration 2). $p_3(t)$ is the probability that one walker stays in a class 1 leaf and the other walker stays in the class 2 hub, or one walker stays in a class 2 leaf and the other walker stays in the class 1 hub (configuration 3). $p_4(t)$ is the probability that a walker stays in a class 1 leaf and the other walker stays in a class 2 leaf (configuration 4). Finally, $p_5(t)$ is the probability that a walker stays in the class 1 hub and the other walker stays in the class 2 hub (configuration 5). We denote by $q(t)$ the probability that the two walkers meet at time t.

4.3.1   LD

Under LD, a link with one of the two directions is selected with probability $1/[2(2N-1)]$ in an update event, which consumes time $1/2N$. Therefore, we obtain

$$\boldsymbol{p}\left(t + \frac{1}{2N}\right) = A^{\mathrm{LD}}\boldsymbol{p}(t), \tag{36}$$

$$q\left(t + \frac{1}{2N}\right) = \frac{1}{2(2N-1)}\begin{pmatrix} 0 & 2 & 0 & 0 & 2 \end{pmatrix}\boldsymbol{p}(t), \tag{37}$$

where

$$A^{\mathrm{LD}} = \frac{1}{2(2N-1)}\begin{pmatrix} 4(N-1) & N-2 & 0 & 0 & 0 \\ 2 & 3(N-1) & 1 & 0 & 0 \\ 0 & 1 & 3(N-1) & 2 & 2(N-1) \\ 0 & 0 & N-1 & 4(N-1) & 0 \\ 0 & 0 & 1 & 0 & 2(N-1) \end{pmatrix}. \tag{38}$$

Equation (38) leads to

$$(I - A^{\mathrm{LD}})^{-1}$$

$$=\frac{2N-1}{2N+3}\begin{pmatrix} N^2+N+1 & (N+1)(N-2) & N(N-2) & N(N-2) & (N-1)(N-2) \\ 2(N+1) & 2(N+1) & 2N & 2N & 2(N-1) \\ 2N & 2N & 6N & 6N & 6(N-1) \\ N(N-1) & N(N-1) & 3N(N-1) & 3N^2-N+3 & 3(N-1)^2 \\ 1 & 1 & 3 & 3 & 5 \end{pmatrix}$$

(39)

and

$$\frac{1}{2(2N-1)}\begin{pmatrix} 0 & 2 & 0 & 0 & 2 \end{pmatrix}(I - A^{\mathrm{LD}})^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

(40)

By using Eqs. (39) and (40), we obtain

$$\langle T \rangle \approx \sum_{t'=1}^{\infty} \frac{t'}{2N} q\left(\frac{t'}{2N}\right)$$

$$= \frac{1}{2N}\frac{1}{2(2N-1)}\begin{pmatrix} 0 & 2 & 0 & 0 & 2 \end{pmatrix}(I - A^{\mathrm{LD}})^{-2}\boldsymbol{p}(0)$$

$$= \frac{1}{2N}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}(I - A^{\mathrm{LD}})^{-1}\boldsymbol{p}(0)$$

$$= \frac{(2N-1)}{N(2N+3)}\begin{pmatrix} N^2+2N+2 & \frac{2N^2+2N+1}{2} & \frac{4N^2+3N+3}{2} & \frac{4N^2+5N+6}{2} & \frac{4N^2-N+2}{2} \end{pmatrix}\boldsymbol{p}(0).$$

(41)

Because

$$\langle T \rangle = \begin{pmatrix} O(N) & O(N) & O(N) & O(N) & O(N) \end{pmatrix}\boldsymbol{p}(0),$$

(42)

we conclude $\langle T \rangle = O(N)$ under LD.

4.3.2 VM

Consider a single update event under VM. There are two types of events. First, a leaf node imitates the state of the hub node in the same class with probability $1/2N$. In the random walk interpretation, this event corresponds to the movement of a walker located at this leaf (if so) to the hub. Second, a hub imitates the opinion of a neighbour, which is either a leaf in the same class or the hub in the opposite class, with probability $1/2N^2$. In the random walk, this event corresponds to the movement of a walker located at the hub to a neighbour. Neither walker moves if a walker is not located at the selected starting node. Therefore, we obtain

$$\boldsymbol{p}\left(t + \frac{1}{2N}\right) = A^{\mathrm{VM}}\boldsymbol{p}(t),$$

(43)

$$q\left(t + \frac{1}{2N}\right) = \frac{1}{2N^2}\begin{pmatrix} 0 & N+1 & 0 & 0 & 2 \end{pmatrix}\boldsymbol{p}(t),$$

(44)

where

$$A^{\mathrm{VM}} = \frac{1}{2N^2} \begin{pmatrix} 2N(N-1) & N-2 & 0 & 0 & 0 \\ 2N & 2N(N-1) & 1 & 0 & 0 \\ 0 & 1 & 2N(N-1) & 2N & 2(N-1) \\ 0 & 0 & N-1 & 2N(N-1) & 0 \\ 0 & 0 & N & 0 & 2N(N-1) \end{pmatrix}. \tag{45}$$

Equation (45) leads to

$$(I - A^{\mathrm{VM}})^{-1} = \frac{N^2}{2N+3} \begin{pmatrix} \frac{4N-1}{N} & \frac{2(N-2)}{N} & \frac{N-2}{N} & \frac{N-2}{N} & \frac{(N-1)(N-2)}{N^2} \\ 4 & 4 & 2 & 2 & \frac{2(N-1)}{N} \\ 2 & 2 & 2(N+2) & 2(N+2) & \frac{2(N-1)(N+2)}{N} \\ \frac{N-1}{N} & \frac{N-1}{N} & \frac{(N-1)(N+2)}{N} & \frac{N^2+3N+1}{N} & \frac{(N-1)^2(N+2)}{N^2} \\ 1 & 1 & N+2 & N+2 & \frac{N^2+3N+1}{N} \end{pmatrix} \tag{46}$$

and

$$\frac{1}{2N^2} \begin{pmatrix} 0 & N+1 & 0 & 0 & 2 \end{pmatrix} (I - A^{\mathrm{VM}})^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{47}$$

By using Eqs. (46) and (47), we obtain

$$\langle T \rangle \approx \sum_{t'=1}^{\infty} \frac{t'}{2N} q\left(\frac{t'}{2N}\right)$$
$$= \frac{1}{2N(2N+3)} \begin{pmatrix} a^{\mathrm{VM}} & b^{\mathrm{VM}} & c^{\mathrm{VM}} & d^{\mathrm{VM}} & e^{\mathrm{VM}} \end{pmatrix} \boldsymbol{p}(0), \tag{48}$$

where

$$a^{\mathrm{VM}} = 2N(6N-1), \tag{49}$$
$$b^{\mathrm{VM}} = 5N(2N-1), \tag{50}$$
$$c^{\mathrm{VM}} = 2N(2N^2+5N-2), \tag{51}$$
$$d^{\mathrm{VM}} = N(4N^2+12N-1), \tag{52}$$
$$e^{\mathrm{VM}} = 4N^3+8N^2-11N+4. \tag{53}$$

Therefore, we obtain

$$\langle T \rangle = \begin{pmatrix} O(1) & O(1) & O(N) & O(N) & O(N) \end{pmatrix} \boldsymbol{p}(0). \tag{54}$$

We conclude $\langle T \rangle = O(N)$ under VM because $\langle T \rangle = O(N)$ holds true for generic initial conditions corresponding to $p_3(0)$, $p_4(0)$, and $p_5(0)$.

4.3.3 IP

Under IP, a walker at a leaf moves to the hub in the same class with probability $1/2N^2$ in an update event. A walker at a hub moves to a leaf in the same class with probability $1/2N$ and to the hub in the opposite class with probability $1/2N^2$. With the remaining probability, neither walker moves. Therefore, we obtain

$$\boldsymbol{p}\left(t + \frac{1}{2N}\right) = A^{\mathrm{IP}}\boldsymbol{p}(t), \tag{55}$$

$$q\left(t + \frac{1}{2N}\right) = \frac{1}{2N^2}\begin{pmatrix} 0 & N+1 & 0 & 0 & 2 \end{pmatrix}\boldsymbol{p}(t), \tag{56}$$

where

$$A^{\mathrm{IP}} = \frac{1}{2N^2}\begin{pmatrix} 2(N+1)(N-1) & N(N-2) & 0 & 0 & 0 \\ 2 & (N+2)(N-1) & 1 & 0 & 0 \\ 0 & 1 & (N+2)(N-1) & 2 & 2N(N-1) \\ 0 & 0 & N(N-1) & 2(N+1)(N-1) & 0 \\ 0 & 0 & 1 & 0 & 2(N-1) \end{pmatrix}. \tag{57}$$

Equation (57) leads to

$$(I - A^{\mathrm{IP}})^{-1}$$
$$= \frac{2N^2}{N^3 + N + 3} \times$$
$$\begin{pmatrix} \frac{N^4 - 2N^3 + 4N^2 - 3N + 3}{2} & \frac{(C+1)N(N-2)}{2} & \frac{CN(N-2)}{2} & \frac{CN(N-2)}{2} & \frac{N^2(N-1)(N-2)}{2} \\ C+1 & C+1 & C & C & C-1 \\ C & C & C(N+2) & C(N+2) & N(N-1)(N+2) \\ \frac{CN(N-1)}{2} & \frac{CN(N-1)}{2} & \frac{CN(N-1)(N+2)}{2} & \frac{N^5 - N^3 + 3N^2 - N + 3}{2} & \frac{N^2(N-1)^2(N+2)}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{N+2}{2} & \frac{N+2}{2} & \frac{2N+3}{2} \end{pmatrix} \tag{58}$$

and

$$\frac{1}{2N^2}\begin{pmatrix} 0 & N+1 & 0 & 0 & 2 \end{pmatrix}(I - A^{\mathrm{IP}})^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{59}$$

where $C \equiv N^2 - N + 1$. By using Eqs. (58) and (59), we obtain

$$\langle T \rangle \approx \sum_{t'=1}^{\infty} \frac{t'}{2N} q\left(\frac{t'}{2N}\right)$$
$$= \frac{N}{N^3 + N + 3}\begin{pmatrix} a^{\mathrm{IP}} & b^{\mathrm{IP}} & c^{\mathrm{IP}} & d^{\mathrm{IP}} & e^{\mathrm{IP}} \end{pmatrix}\boldsymbol{p}(0), \tag{60}$$

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where

$$a^{\mathrm{IP}} = N^4 - 2N^3 + 5N^2 - 4N + 5, \tag{61}$$

$$b^{\mathrm{IP}} = \frac{2N^4 - 5N^3 + 10N^2 - 9N + 7}{2}, \tag{62}$$

$$c^{\mathrm{IP}} = \frac{N^5 + N^4 - 3N^3 + 10N^2 - 7N + 8}{2}, \tag{63}$$

$$d^{\mathrm{IP}} = \frac{N^5 + N^4 - 2N^3 + 10N^2 - 6N + 11}{2}, \tag{64}$$

$$e^{\mathrm{IP}} = \frac{N^5 + N^4 - 4N^3 + 8N^2 - 4N + 3}{2}. \tag{65}$$

Therefore, we obtain

$$T = \begin{pmatrix} O(N^2) & O(N^2) & O(N^3) & O(N^3) & O(N^3) \end{pmatrix} \boldsymbol{p}(0). \tag{66}$$

We conclude $\langle T \rangle = O(N^3)$ under IP because $\langle T \rangle = O(N^3)$ holds true for generic initial conditions corresponding to $p_3(0)$, $p_4(0)$, and $p_5(0)$.

5 Numerical simulations for large networks

To check the validity of the scaling between $\langle T \rangle$ and N derived in section 4, we carry out direct numerical simulations of the opinion dynamics for larger networks than those considered in section 3. As the lollipop graph, we consider those having $N/2$ nodes in the chain and clique. As the barbell graph, we consider those having $N/3$ nodes in the chain and each clique. In each run, $N/2$ randomly selected nodes initially possess state **0** and the other $N/2$ nodes state **1**. We calculate $\langle T \rangle$ as an average over 10^3 runs for each network and update rule.

The relationship between the numerically obtained $\langle T \rangle$ and N is shown in Fig. 9 for each combination of the network (i.e., lollipop, barbell, or double star) and update rule (i.e., LD, VM, or IP). The numerical results (symbols) agree well with the scaling law derived in section 4 (lines) in most cases. For the lollipop and barbell graphs combined with IP, the accuracy of the theoretical estimate improves as N increases. For the double star graph combined with VM and IP, $\langle T \rangle$ increases with N slightly more slowly than the theoretical prediction (i.e., $\langle T \rangle \propto N$ for VM and $\langle T \rangle \propto N^3$ for IP).

6 Discussion

We explored the networks that maximised the mean consensus time, $\langle T \rangle$, of the three variants of the voter model. The lollipop graph, barbell graph, and double star graph were suggested to maximise $\langle T \rangle$ under the LD, VM, and IP update rules, respectively. In addition, we evaluated $\langle T \rangle$ for the

three types of networks under each of the three update rules. The results are summarised in Table 1.

Our results are consistent with the previous results for random walks. The hitting and cover time for the random walk on the lollipop graph [23,24, 26] and the barbell graph [22,25] both scale as $O(N^3)$. For the lollipop graph, these previous results are consistent with ours for LD. For the barbell graph, the previous results are consistent with ours for LD and VM. However, different update rules yield $\langle T \rangle = O(N^2)$ for the lollipop and barbell graphs. The scaling between $\langle T \rangle$ and N depends on the update rule because a choice of the update rule corresponds to weighting of the links in the network. The link weight biases the probability that a particular link is used for state updating in opinion dynamics and the probability with which the random walk transits from one node to another [13, 14, 31]. It should be noted that more exact estimation of $\langle T \rangle$ for the lollipop and barbell graphs on the basis of the random walk, as we did for the double star graph, warrants future work.

The double star graph, which maximises $\langle T \rangle$ under IP, is far from the lollipop and barbell graphs in two aspects. First, it has a small diameter, i.e., three. Second, it has not been recognised as a network that slows down dynamics of the random walk. Previous theoretical results for opinion dynamics in heterogeneous random networks yielded $\langle T \rangle = O(N\mu_1\mu_{-1})$ for IP, where μ_1 and μ_{-1} are the mean degree and the mean of the inverse degree, respectively [13, 14]. For the double star with N nodes, where N is even, this theory predicts $\langle T \rangle = O(N)$ because $\mu_1 = 2(N-1)/N$ and $\mu_{-1} = (N^2 - 2N + 4)/N^2$. This estimate deviates from our results, i.e., $\langle T \rangle = O(N^3)$. In the theory developed in Refs. [13, 14], heterogeneous random networks are assumed such that the network does not have structure other than the degree distribution. In contrast, in the double star graph, leaf nodes are never adjacent to each other, and the two hubs are always adjacent. We consider this is the reason for the deviation. Similarly, the theory for heterogeneous random networks [13, 14] adapted to the degree distribution of the double star graph suggests $\langle T \rangle = N\mu_1^2/\mu_2 = O(1)$ under VM, where $\mu_2 = (N^2 + 2N - 4)/2N$ is the second moment of the degree. This estimate is also different from ours, i.e., $\langle T \rangle = O(N)$, presumably for the same reason.

Acknowledgments

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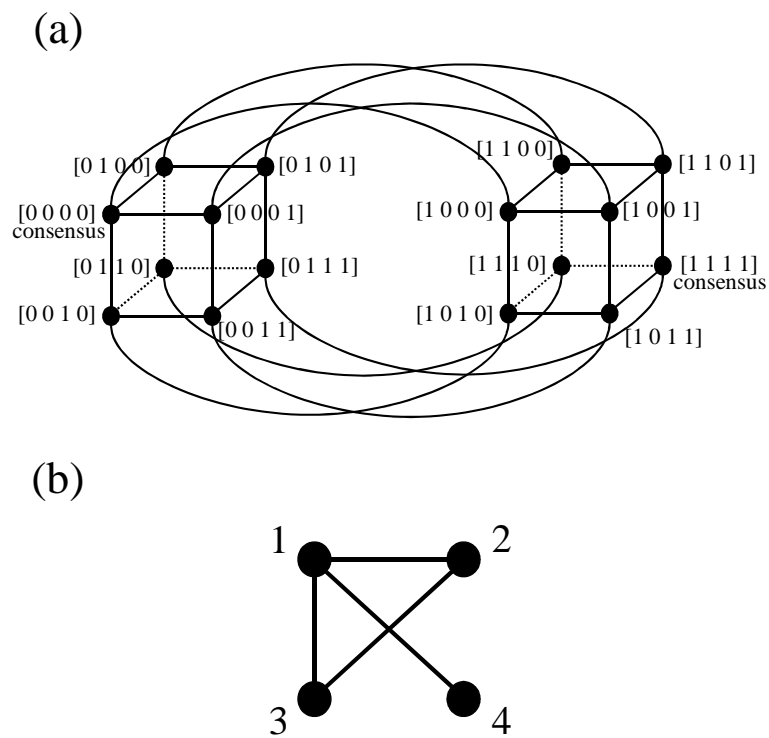
Figure 1: (a) Schematic of transitions between configurations in opinion dynamics on a network possessing $N = 4$ nodes. (b) A network with $N = 4$.
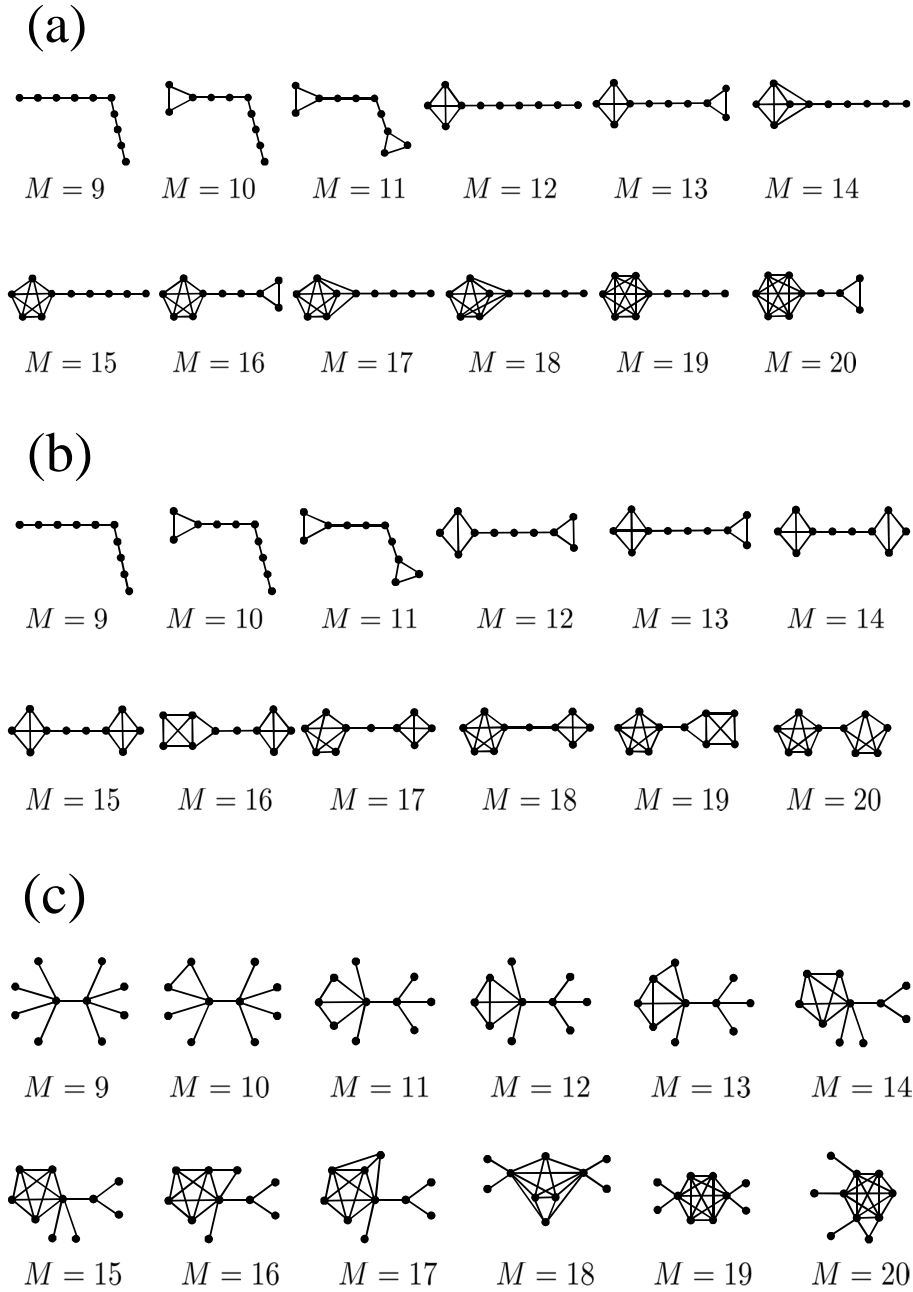
Figure 2: Networks that maximise the mean consensus time for different numbers of links M. We set $N = 10$. (a) LD. (b) VM. (c) IP.

(a)

(b)

(c)
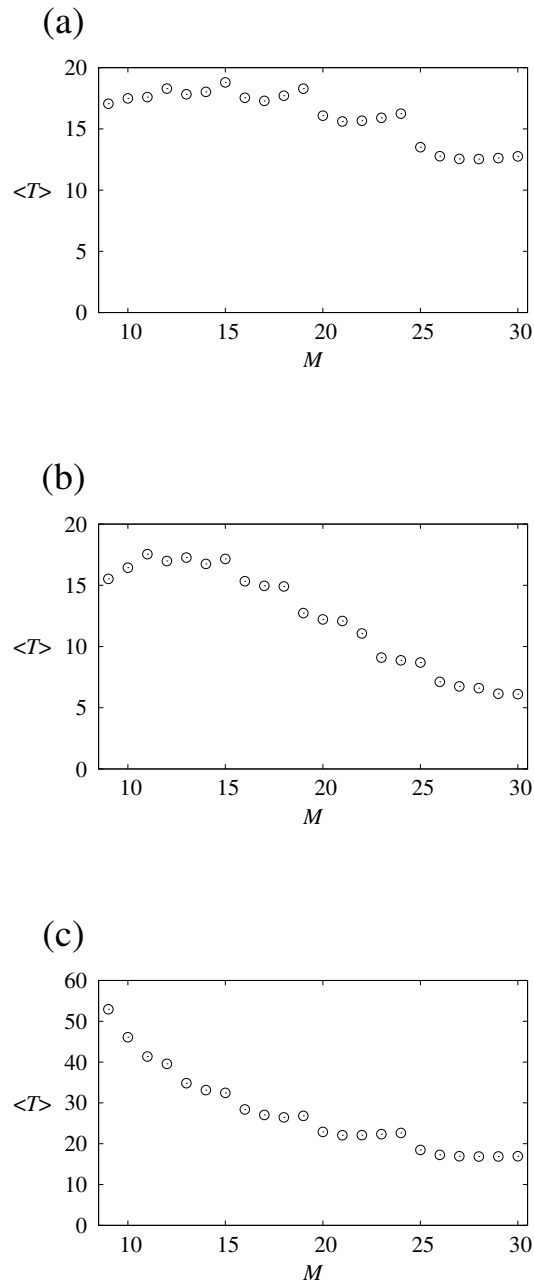
Figure 3: Largest mean consensus time, $\langle T \rangle$, with respect to network structure when we vary $M(\geq N - 1)$. We set $N = 10$. (a) LD. (b) VM. (c) IP.
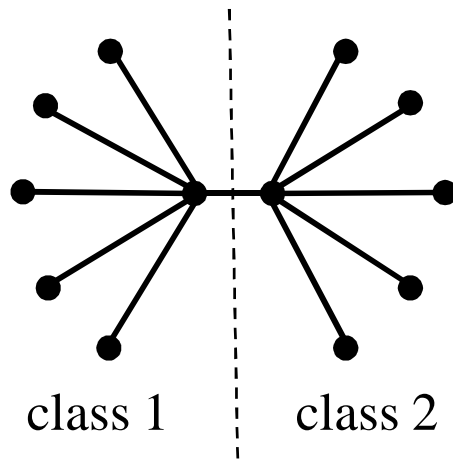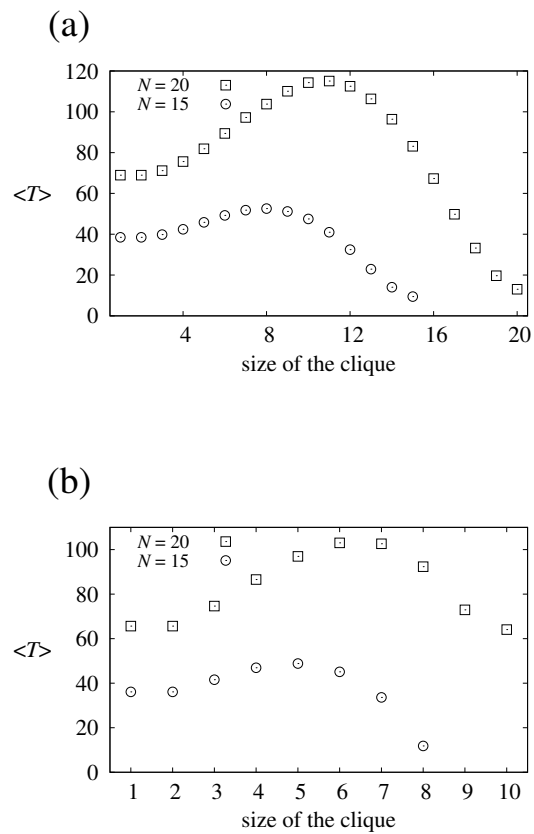
Figure 4: Double star graph.

(a)

(b)

Figure 5: (a) Mean consensus time, $\langle T \rangle$, for the lollipop graph with various M values under LD. (b) $\langle T \rangle$ for the barbell graph with various M values under VM. In (a) and (b), the horizontal axis represents the size of the clique in the lollipop or barbell graph. We set $N = 15$ and $N = 20$.
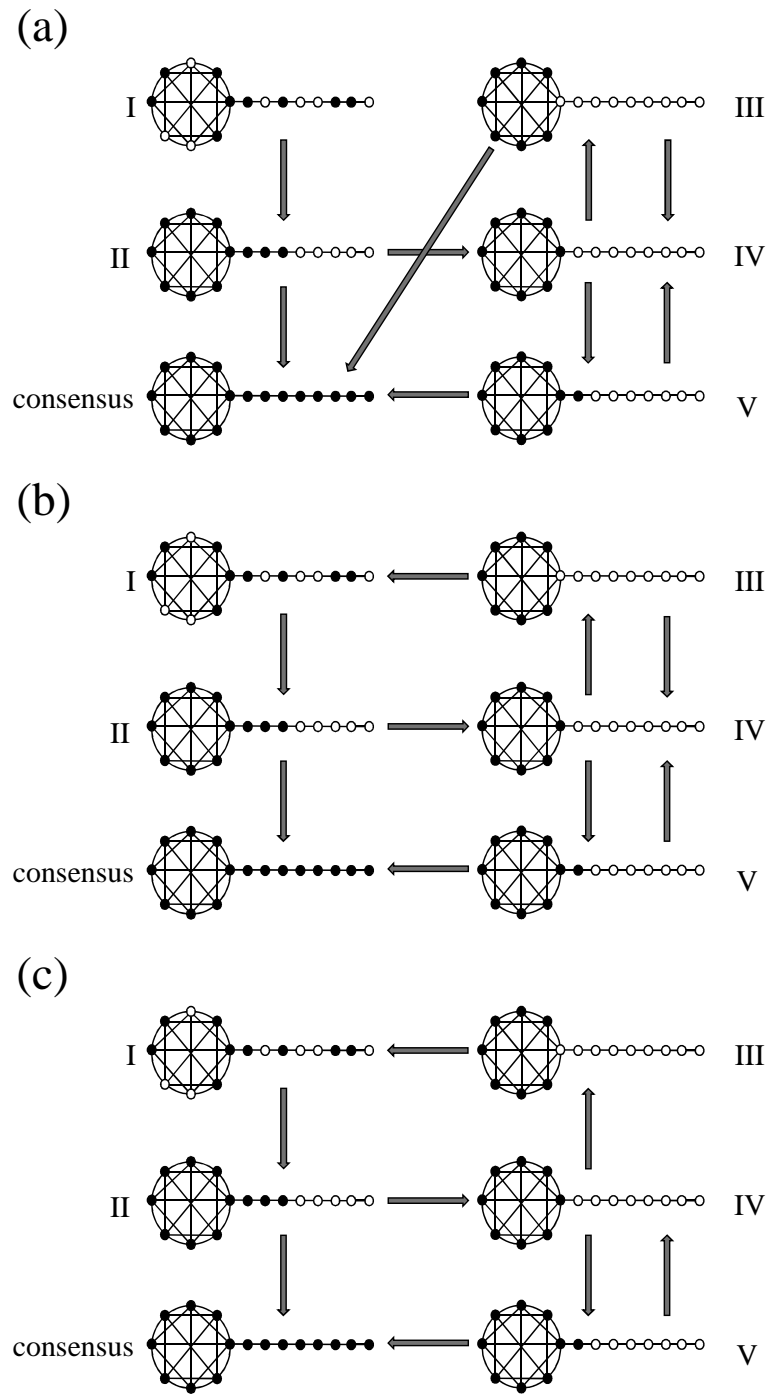
Figure 6: Transitions among typical configurations for evaluating approximate lower and upper bounds of $\langle T \rangle$ for the lollipop graph. (a) Schematic for evaluating a lower bound. (b) Schematic for evaluating an upper bound under LD and IP. (c) Schematic for evaluating an upper bound under VM.
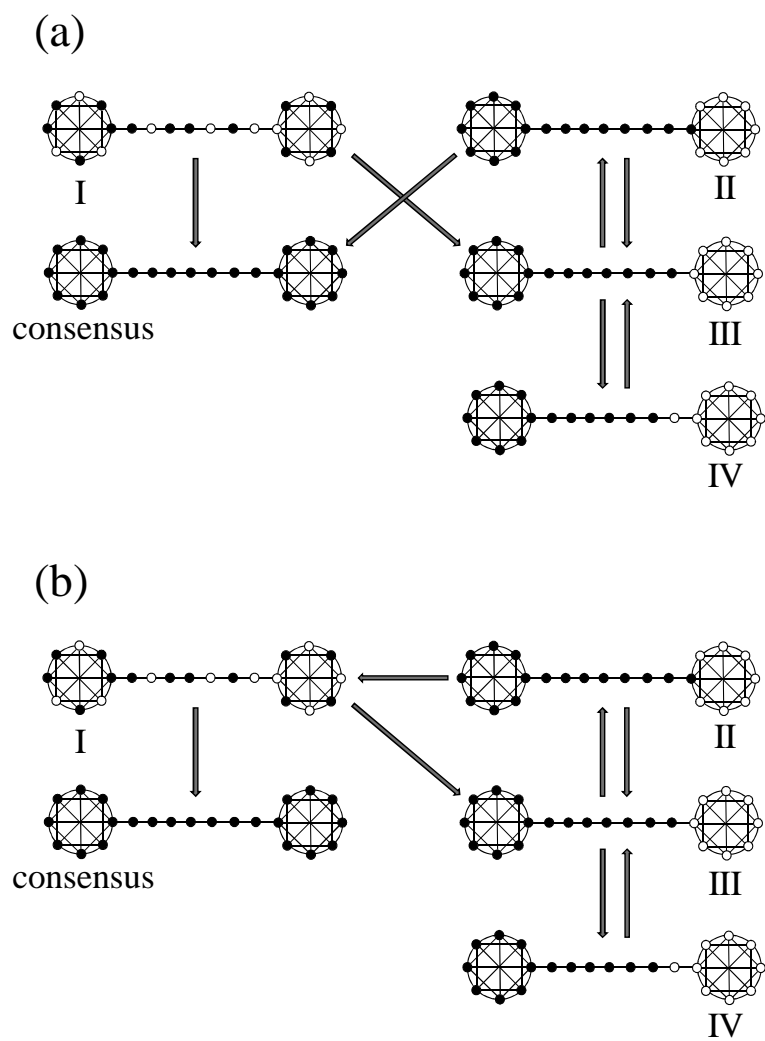
Figure 7: Transitions among typical configurations for evaluating approximate lower and upper bounds of $\langle T \rangle$ for the barbell graph. (a) Schematic for evaluating a lower bound. (b) Schematic for evaluating an upper bound under LD and IP. (c) Schematic for evaluating an upper bound under VM.
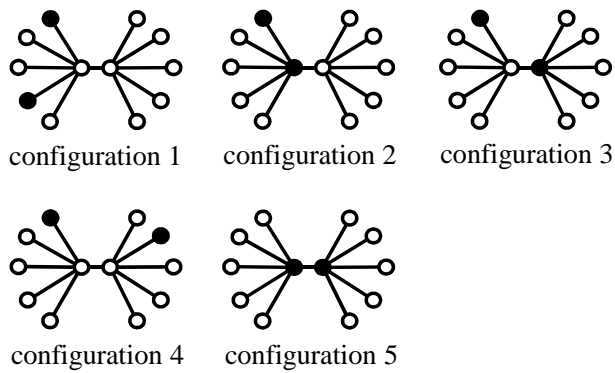
(c)

I

consensus

V

II

III

IV

configuration 1   configuration 2   configuration 3

configuration 4   configuration 5
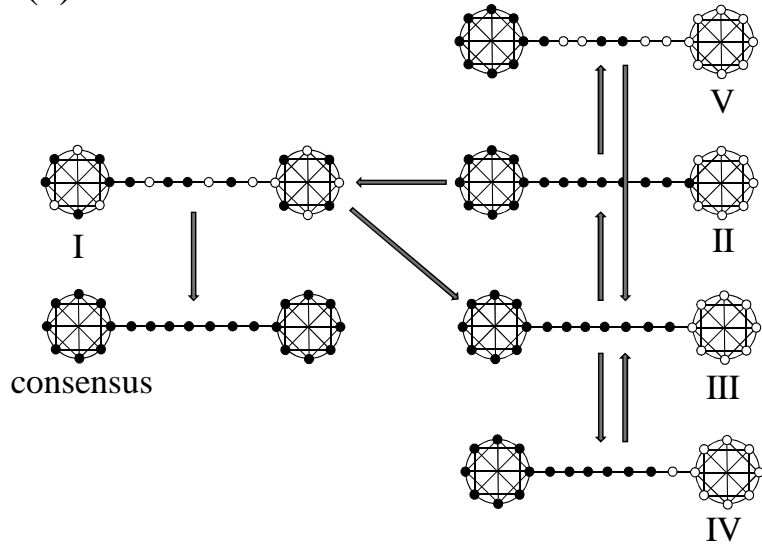
Figure 8: Five configurations in the double star graph that define $p_1(t)$, ..., $p_5(t)$.
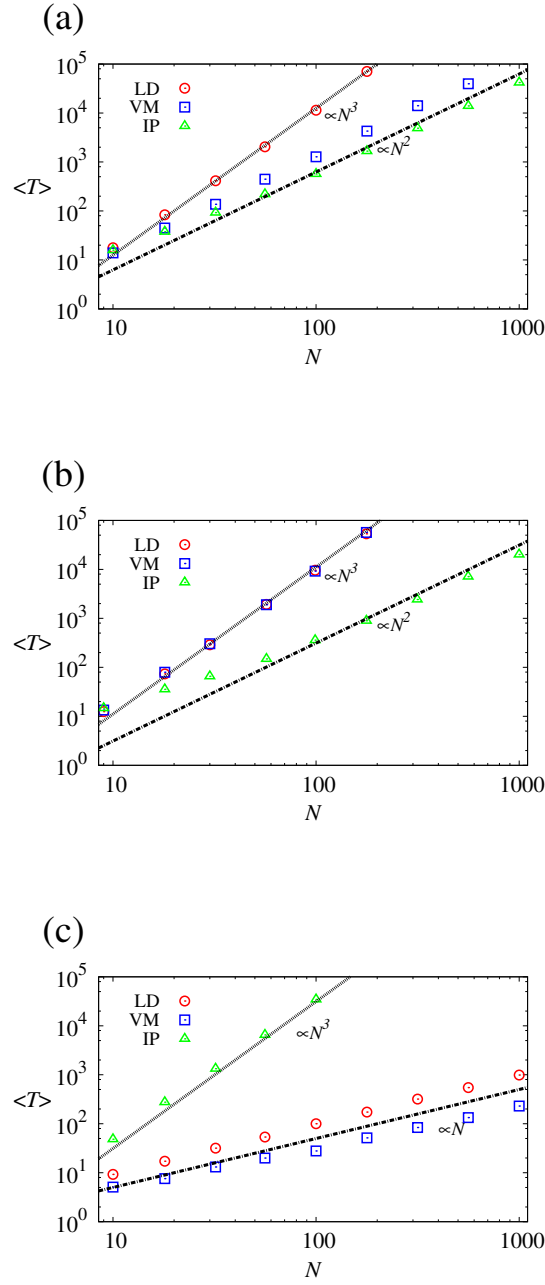
Figure 9: Relationship between the mean consensus time, $\langle T \rangle$, and the number of nodes, N, under different networks and update rules. (a) Lollipop graph. (b) Barbell graph. (c) Double star graph.

Table 1: Summary of the results. The dependence of the mean consensus time on the number of nodes is shown for each combination of the network and update rule.

	LD	VM	IP
Lollipop	$O(N^3)$	$O(N^2)$	$O(N^2)$
Barbell	$O(N^3)$	$O(N^3)$	$O(N^2)$
Double star	$O(N)$	$O(N)$	$O(N^3)$