

**MATHEMATICAL ENGINEERING
TECHNICAL REPORTS**

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with Applications to Statistics**

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METR 2014-13

May 2014

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WWW page: <http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html>

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Properties of powers of functions satisfying second-order linear differential equations with applications to statistics

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May, 2014

Abstract

We derive properties of powers of a function satisfying a second-order linear differential equation. In particular we prove that the n -th power of the function satisfies an $(n + 1)$ -th order differential equation and give a simple method for obtaining the differential equation. Also we determine the exponents of the differential equation and derive a bound for the degree of the polynomials, which are coefficients in the differential equation. The bound corresponds to the order of differential equation satisfied by the n -fold convolution of the Fourier transform of the function. These results are applied to some probability density functions used in statistics.

Keywords and phrases: characteristic function, exponents, holonomic function, indicial equation, skewness

1 Introduction

In statistics it is important to study the distribution of a sum (i.e. convolution) of n independent random variables. Usually the distribution is studied through the characteristic function, because the convolution of probability density functions corresponds to the product of characteristic functions. If the random variables are identically distributed, then we study the n -th power of a characteristic function. The central limit theorem is proved by analyzing the limiting behavior of the n -th power of a characteristic function as $n \rightarrow \infty$. Often the technique of asymptotic expansion is employed to improve the approximation for large n . However for finite n , the exact distribution of the sum of random variables is often difficult to treat. Hence it is important to develop methodology for studying properties of the n -th power of a function.

Recently techniques based on holonomic functions ([9], Chapter 6 of [6]) have been introduced to statistics and successfully applied to some difficult distributional problems (e.g.

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[12], [5]). In this paper we investigate the case that the function satisfies a second-order linear differential equation with rational function coefficients, which we call *holonomic differential equation*. In Section 2 we prove that the n -th power satisfies an $(n + 1)$ -th order differential equation and give a simple method for obtaining the differential equation. Also we determine the exponents of the differential equation and derive a bound for the degree of the polynomials which appear as coefficients of the differential equation.

As shown in Section 3, there are some important examples in statistics which falls into this case. We discuss sum of beta random variables and sum of cubes of standard normal random variables. The differential equations reveal many interesting properties of the characteristic function and the probability density function of the sum of random variables. These properties are hard to obtain by other methods. We end the paper with some discussions in Section 4.

2 Main results

In this section we present our main results in Theorems 2.4, 2.8 and 2.12. Theorem 2.4 gives the differential equation satisfied by the n -th power. Theorem 2.8 bounds the degree of coefficient polynomials. Theorem 2.12 derives exponents of the differential equation.

Let $C(x)$ denote the field of rational functions in x with complex coefficients and let

$$R = C(x)\langle\partial_x\rangle, \quad \partial_x = \frac{d}{dx},$$

denote the ring of differential operator with rational function coefficients. In R , the product of ∂_x and $a(x) \in C(x)$ is defined as $\partial_x a(x) = a(x)\partial_x + a'(x)$, where $a'(x)$ is the derivative of $a(x)$ with respect to x . In order to distinguish the product in R and the action of ∂_x to a function, we denote the latter by the symbol \bullet .

Example 2.1. If we write $\partial_x x$, both ∂_x and x are the elements of R . Hence $\partial_x x = x\partial_x + 1$. On the other hand, if we write $\partial_x \bullet x$, this x is a function. Hence $\partial_x \bullet x = 1$.

In this paper we study $f(x)$ which is a holonomic function satisfying a second-order differential equation:

$$\left[\partial_x^2 - a_1(x)\partial_x - a_0(x)\right] \bullet f(x) = 0, \quad a_0(x), a_1(x) \in C(x). \quad (1)$$

2.1 Order of the differential equation of the n -th power and its Fourier transform

Let $\mathbf{q}_0 = (1, 0, \dots, 0)^\top$ be an $(n + 1)$ dimensional column vector and let

$$A(x) = \begin{pmatrix} 0 & a_0(x) & & & & \\ n & a_1(x) & 2a_0(x) & & & \\ & n-1 & 2a_1(x) & \ddots & & \\ & & \ddots & \ddots & na_0(x) & \\ & & & & 1 & na_1(x) \end{pmatrix} \quad (2)$$

be an $(n + 1) \times (n + 1)$ tridiagonal matrix with entries from $\mathbb{C}(x)$. Furthermore define

$$\begin{aligned} \tilde{A}(x, \partial_x) &= A(x) + \partial_x I \tag{3} \\ &= \begin{pmatrix} \partial_x & a_0(x) & & & & & \\ n & \partial_x + a_1(x) & 2a_0(x) & & & & \\ & n-1 & \partial_x + 2a_1(x) & \ddots & & & \\ & & \ddots & \ddots & (n-1)a_0(x) & & \\ & & & & 2 & \partial_x + (n-1)a_1(x) & na_0(x) \\ & & & & & 1 & \partial_x + na_1(x) \end{pmatrix} \end{aligned}$$

with entries from R . Let

$$Q(x) = (q_{ij}(x))_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n+1}} = (\mathbf{q}_0, \tilde{A}(x, \partial_x) \bullet \mathbf{q}_0, \tilde{A}(x, \partial_x)^2 \bullet \mathbf{q}_0, \dots, \tilde{A}(x, \partial_x)^{n+1} \bullet \mathbf{q}_0)$$

be an $(n+1) \times (n+2)$ matrix with entries from $\mathbb{C}(x)$. If we write $\mathbf{q}_j = \tilde{A}(x, \partial_x)^j \bullet \mathbf{q}_0$, $j = 0, \dots, n+1$, then

$$\mathbf{q}_{j+1} = \tilde{A}(x, \partial_x) \bullet \mathbf{q}_j,$$

or writing down the elements we have

$$q_{i,j+1}(x) = (n+1-i)q_{i-1,j}(x) + (\partial_x + ia_1(x)) \bullet q_{i,j}(x) + (i+1)a_0(x)q_{i+1,j}(x), \tag{4}$$

where $q_{-1,j}(x) = q_{n+1,j}(x) = 0$. Hence it is easy to compute the elements of the columns of $Q(x)$ recursively, starting from the first column.

Define

$$[n]_i = \prod_{k=0}^{i-1} (n-k), \quad ([n]_0 = 1). \tag{5}$$

From (4) we can easily prove that $Q(x)$ is an upper-triangular matrix with non-zero diagonal elements, although $Q(x)$ is not a square matrix (cf. Example 2.3 below).

Lemma 2.2. $q_{ij}(x) = 0$ if $i > j$. $q_{ii}(x) = [n]_i \neq 0$ ($i = 0, 1, \dots, n$).

Proof. We use induction on j . The result is trivial for $j = 0$. Assume $q_{ij}(x) = 0$ ($i > j$) and $q_{j,j}(x) = [n]_j$. Then by (4) we have $q_{i,j+1}(x) = 0$ ($i > j+1$) and $q_{j+1,j+1}(x) = (n-j)[n]_j = [n]_{j+1}$. \square

This lemma implies $\text{rank } Q(x) = n+1$, or $\dim \text{Ker } Q(x) = 1$. Hence the element of $\text{Ker } Q(x)$ is unique up to the multiplication of a rational function. Here note that we are using the linear algebra over $\mathbb{C}(x)$.

Let

$$\mathbf{v}(x) = (v_i(x))_{0 \leq i \leq n+1} \in \text{Ker } Q(x), \quad \mathbf{v}(x) \neq \mathbf{0}, \tag{6}$$

where $v_i(x) \in \mathbb{C}(x)$, $i = 0, \dots, n+1$. Once we set $v_{n+1}(x) \neq 0$, then by the triangularity of $Q(x)$, $v_n(x), v_{n-1}(x), \dots, v_0(x)$ are successively determined. Moreover, if we set $v_{n+1}(x) = 0$, then we obtain $v_n(x) = \dots = v_0(x) = 0$. Hence $v_{n+1}(x) \neq 0$ for $\mathbf{v}(x) \neq \mathbf{0}$. Often we set $v_{n+1}(x) = 1$. For theoretical investigation it is convenient to clear the common denominators of $v_i(x)$'s and take $v_i(x)$'s as polynomials.

Example 2.3. Let $n = 3$ and let $a_0(x) = 1 + x^{-2}$, $a_1(x) = -x^{-1}$. Then

$$Q(x) = \begin{pmatrix} 1 & 0 & 3 + 3x^{-2} & -3x^{-1} - 9x^{-3} & 21 + 51x^{-2} + 54x^{-4} \\ 0 & 3 & -3x^{-1} & 21 + 27x^{-2} & -66x^{-1} - 144x^{-3} \\ 0 & 0 & 6 & -18x^{-1} & 60 + 126x^{-2} \\ 0 & 0 & 0 & 6 & -36x^{-1} \end{pmatrix}$$

If we set $v_4(x) = 1$, we successively obtain

$$v_3(x) = 6x^{-1}, \quad v_2(x) = -10 - 3x^{-2}, \quad v_1(x) = -30x^{-1} - 9x^{-3}, \quad v_0(x) = 9 + 6x^{-2} + 9x^{-4}.$$

Multiplying by x^4 we obtain \mathbf{v} with polynomial elements.

We now derive a holonomic differential equation satisfied by the n -th power of the holonomic function $f(x)$.

Theorem 2.4. *The n -th power of $f(x)$ satisfies the following $(n + 1)$ -th order holonomic differential equation:*

$$\left[v_{n+1}(x)\partial_x^{n+1} + v_n(x)\partial_x^n + \cdots + v_1(x)\partial_x + v_0(x) \right] \bullet f(x)^n = 0, \quad (7)$$

where $v_i(x)$'s are given in (6).

Proof. By induction we prove

$$\partial_x^k \bullet f^n = (f^n, f^{n-1}f', \dots, ff^{m-1}, f^m) \tilde{A}(x, \partial_x)^k \bullet \mathbf{q}_0 \quad (8)$$

for any $k \geq 0$. It is obvious for $k = 0$. By (1),

$$\begin{aligned} & \partial_x \bullet (f(x)^{n-j} f'(x)^j) \\ &= jf(x)^{n-j} f'(x)^{j-1} f''(x) + (n-j)f(x)^{n-j-1} f'(x)^{j+1} \\ &= ja_0(x)f(x)^{n-j+1} f'(x)^{j-1} + ja_1(x)f(x)^{n-j} f'(x)^j + (n-j)f(x)^{n-j-1} f'(x)^{j+1} \end{aligned}$$

holds for all $j = 0, 1, \dots, n$, and this leads to

$$\partial_x \bullet (f^n, f^{n-1}f', \dots, ff^{m-1}, f^m) = (f^n, f^{n-1}f', \dots, ff^{m-1}, f^m) A(x), \quad (9)$$

where $A(x)$ is given in (2).

Hence, assuming (8) for k , we obtain

$$\begin{aligned} \partial_x^{k+1} \bullet f^n &= \left(\partial_x \bullet (f^n, f^{n-1}f', \dots, ff^{m-1}, f^m) \right) \tilde{A}(x, \partial_x)^k \bullet \mathbf{q}_0 \\ &\quad + (f^n, f^{n-1}f', \dots, ff^{m-1}, f^m) \partial_x \bullet \left(\tilde{A}(x, \partial_x)^k \bullet \mathbf{q}_0 \right) \\ &= (f^n, f^{n-1}f', \dots, ff^{m-1}, f^m) A(x) \tilde{A}(x, \partial_x)^k \bullet \mathbf{q}_0 \quad (\text{by (9)}) \\ &\quad + (f^n, f^{n-1}f', \dots, ff^{m-1}, f^m) \partial_x \tilde{A}(x, \partial_x)^k \bullet \mathbf{q}_0 \\ &= (f^n, f^{n-1}f', \dots, ff^{m-1}, f^m) (A(x) + \partial_x I) \tilde{A}(x, \partial_x)^k \bullet \mathbf{q}_0 \\ &= (f^n, f^{n-1}f', \dots, ff^{m-1}, f^m) \tilde{A}(x, \partial_x)^{k+1} \bullet \mathbf{q}_0. \quad (\text{by (3)}) \end{aligned}$$

Thus (8) is proved. By arranging (8) for $k = 0, 1, \dots, n + 1$, we have

$$(f^n, \partial_x \bullet f^n, \dots, \partial_x^{n+1} \bullet f^n) = (f^n, f^{n-1} f', \dots, f f^{m-1}, f^m) Q(x).$$

By multiplying it by $\nu(x) \in \text{Ker } Q(x)$ from the right, we obtain (7). \square

Remark 2.5. If we just want to show the existence of a holonomic differential equation of order $n + 1$, then consider

$$M = \mathbf{C}(x)f^n + \mathbf{C}(x)f^{n-1}f' + \dots + \mathbf{C}(x)ff^{m-1} + \mathbf{C}(x)f^m.$$

Then M is a left R -module of dimension at most $n + 1$. Hence $n + 2$ elements, $f^n, \partial_x \bullet f^n, \dots, \partial_x^{n+1} \bullet f^n$, of M are linearly dependent over $\mathbf{C}(x)$. Similarly, we see that when $f(x)$ satisfies a holonomic differential equation of order $r (\geq 3)$, $f(x)^n$ satisfies a holonomic differential equation of order $\binom{n+r-1}{r-1}$.

There exists a function $f(x)$ satisfying a second-order holonomic differential equation, such that $f(x)^n$ does not satisfy any holonomic differential equation of order less than $n + 1$.

Example 2.6. Let $f(x) = \sin x$, with $f''(x) + f(x) = 0$. We prove by contradiction that $f^n, f^{n-1}f', \dots, f^m$, or $\sin^n x, \sin^{n-1}x \cos x, \dots, \cos^n x$ are linearly independent over $\mathbf{C}(x)$. It is obvious for $n = 0$. Let $m \geq 1$ be the smallest integer such that $\sin^m x, \dots, \cos^m x$ are linearly dependent. Then, there exist rational functions $q_0(x), \dots, q_m(x)$, not all zero, such that

$$q_0(x) \sin^m x + q_1(x) \sin^{m-1} x \cos x + \dots + q_{m-1}(x) \sin x \cos^{m-1} x + q_m(x) \cos^m x = 0. \quad (10)$$

By putting $x = k\pi$ ($k = 0, 1, \dots$), $q_m(x)$ has infinite number of zeros, and therefore $q_m(x)$ is identically zero. Divide the equation (10) by $\sin x$, and we obtain $q_0(x) \sin^{m-1} x + \dots + q_{m-1}(x) \cos^{m-1} x = 0$, which is a contradiction.

Since $f^n, f^{n-1}f', \dots, f^m$ are linearly independent and the matrix $\tilde{Q}(x) = (q_0, \tilde{A} \bullet q_0, \dots, \tilde{A}^n \bullet q_0)$ is non-singular by Lemma 2.2 over $\mathbf{C}(x)$, $(f^n, \partial_x \bullet f^n, \dots, \partial_x^n \bullet f^n) = (f^n, f^{n-1}f', \dots, f^m) \tilde{Q}(x)$ are linearly independent over $\mathbf{C}(x)$. Thus, there does not exist a holonomic differential equation of order less than $n + 1$ satisfied by $f(x)^n = \sin^n x$.

We have already remarked that we can take $\nu_i(x)$, $0 \leq i \leq n + 1$, as polynomials in (7). Also we can cancel common factors in them. Hence we can assume that they are coprime polynomials. We now investigate the highest degree of these polynomials, which is important when the differential equation is Fourier transformed, because it is equal to the order of the transformed equation.

For the rest of this subsection we assume that $a_0(x), a_1(x)$ are Laurent polynomials. Here, we define mindeg and maxdeg of a Laurent polynomial.

Definition 2.7. For a non-zero Laurent polynomial $f(x) = \sum_{k=m}^M c_k x^k$ ($m < M$, $c_m \neq 0$, $c_M \neq 0$), we define

$$\text{mindeg } f(x) = m, \quad \text{maxdeg } f(x) = M.$$

We define $\text{mindeg } 0 = \infty$, $\text{maxdeg } 0 = -\infty$.

Note that for a polynomial $f(x)$, $\max\deg f(x) = \deg f(x)$.

Now we state the following theorem on the largest degree of the polynomials.

Theorem 2.8. *Assume that $a_0(x), a_1(x)$ in (1) are Laurent polynomials and let $m_i = \min\deg a_i(x)$, $M_i = \max\deg a_i(x)$, $i = 0, 1$. Let $v_0(x), v_1(x), \dots, v_{n+1}(x)$ be coprime polynomials in (7). If $m_1 \leq -1$, $M_1 \geq -1$, $m_0 \geq 2m_1$, $M_0 \leq 2M_1$, then*

$$\max_{0 \leq k \leq n+1} \deg v_k(x) \leq \max\{M_0 + (n-1)M_1, nM_1, 0\} - \min\{m_0, m_1\} - (n-1)m_1. \quad (11)$$

Proof. Let m_{ij} denote $\min\deg q_{ij}(x)$. We prove

$$m_{0j} \geq m_0 + (j-2)m_1, \quad m_{ij} \geq (j-i)m_1, \quad (0 < i < j), \quad (12)$$

for $j = 2, 3, \dots, n+1$ by induction. It is easy to check them for $j = 2$. Assuming them up to j , by (4), we have

$$\begin{aligned} m_{0,j+1} &\geq \min\{m_{0j} + m_1, m_{1j} + m_0\} \\ &= m_0 + (j-1)m_1, \\ m_{i,j+1} &\geq \min\{m_{i-1,j}, m_{ij} + m_1, m_{i+1,j} + m_0\} \\ &\geq \min\{m_{i-1,j}, m_{ij} + m_1, m_{i+1,j} + 2m_1\} \\ &= (j-i+1)m_1, \quad (0 < i < j+1). \end{aligned}$$

Thus, the results are shown by induction.

Hence choosing an element $\tilde{v}(x) = (\tilde{v}_0(x), \dots, \tilde{v}_n(x), 1)^\top \in \text{Ker } Q(x)$, we successively obtain

$$\begin{aligned} \min\deg \tilde{v}_n(x) &\geq m_1, \quad \min\deg \tilde{v}_{n-1}(x) \geq 2m_1, \\ \dots, \quad \min\deg \tilde{v}_1(x) &\geq nm_1, \quad \min\deg \tilde{v}_0(x) \geq m_0 + (n-1)m_1. \end{aligned}$$

This implies that $\min_k \min\deg \tilde{v}_k(x) \geq \min\{m_0 + (n-1)m_1, nm_1, 0\} = \min\{m_0, m_1\} + (n-1)m_1$. By regarding as Laurent polynomials of x as those of x^{-1} , we also have $\max_k \max\deg \tilde{v}_k(x) \leq \max\{M_0 + (n-1)M_1, nM_1, 0\}$. Therefore, clearing the denominators of \tilde{v} of (7), we obtain (11) for the polynomials $v_i(x)$ of \mathcal{V} . \square

Let $D = C\langle x, \partial_x \rangle$ denote the polynomial ring in x and ∂_x with complex coefficients. The Fourier transform \mathcal{F} , which is a ring isomorphism of D , is defined by (Section 6.10 of [6])

$$\mathcal{F} : x \mapsto i\partial_x, \quad \mathcal{F} : \partial_x \mapsto ix, \quad (i = \sqrt{-1}). \quad (13)$$

Hence the Fourier transform $\hat{L}(x, \partial_x)$ of $L(x, \partial_x) \in D$ is given by $L(i\partial_x, ix)$.

This definition is based on the fact that if a function $f(x)$ satisfies the differential equation $L(x, \partial_x) \bullet f(x) = 0$, then the Fourier transform $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x) dx$ satisfies the differential equation $\hat{L}(x, \partial_x) \bullet \hat{f}(x) = 0$ under some regularity conditions. If f is a rapidly decreasing holonomic function, then the correspondence (13) is immediate (Section 5.1.4 of [16]). The

correspondence can be justified in the class of slowly increasing functions. See Chapter 5 of [4].

We take $v_i(x)$'s as coprime polynomials in (7) and then take the Fourier transform. By the correspondence (13), the highest degree of the coefficient polynomials of L equals the order of \hat{L} . Hence we have the following corollary.

Corollary 2.9. *Under the condition of Theorem 2.8, there exists a holonomic differential equation satisfied by the n -th convolution of $\mathcal{F}[f(x)]$ whose order is less than or equal to the right-hand side of (11):*

$$\max\{M_0 + (n - 1)M_1, nM_1, 0\} - \min\{m_0, m_1\} - (n - 1)m_1.$$

2.2 Exponents for the differential equation of the n -th power and the Fourier transformed equation

Consider an r -th order differential equation

$$\left[(x - a)^r \partial_x^r + (x - a)^{r-1} b_{r-1}(x) \partial_x^{r-1} + \cdots + (x - a) b_1(x) \partial_x + b_0(x) \right] \bullet f(x) = 0. \quad (14)$$

If $b_0(x), \dots, b_{r-1}(x)$ are all analytic at $x = a$, then a is said to be a regular singular point for the equation. If $a = 0$ and $b_0(1/x), \dots, b_{r-1}(1/x)$ are all analytic at $x = 0$, then ∞ is said to be a regular singular point for the equation.

When the equation (14) is holonomic, a is a regular singular point if the denominators of $b_0(x), \dots, b_{r-1}(x)$ do not have a factor $(x - a)$, and ∞ is a regular singular point if $a = 0$ and $b_0(x), \dots, b_{r-1}(x)$ are all proper.

When $x_0 \in \mathbb{C} \cup \{\infty\}$ is a regular singular point for the equation, the r -th degree equation

$$b(\lambda) = [\lambda]_r + b_{r-1}(x_0)[\lambda]_{r-1} + \cdots + b_1(x_0)[\lambda]_1 + b_0(x_0) = 0,$$

where $[\lambda]_i = \lambda(\lambda - 1) \dots (\lambda - i + 1)$ (cf. (5)), is called the *indicial equation* (Section 9.5 of [7], Chapter 15 of [8]) for (14) relative to the regular singular point x_0 . The roots of the indicial equation are called the *exponents*.

The case $x_0 \neq \infty$ can be reduced to the case $x_0 = 0$ by the transform $x - x_0 \mapsto x$ and the case $x_0 = \infty$ can be reduced to $x_0 = 0$ by $x \mapsto 1/x$. Hence in the following we put $x_0 = 0$.

The equation (14) is equal to

$$\left[[\theta_x]_r + b_{r-1}(x)[\theta_x]_{r-1} + \cdots + b_1(x)[\theta_x]_1 + b_0(x) \right] \bullet f(x) = 0,$$

where $\theta_x = x \partial_x$ is the Euler operator, since $x^k \partial_x^k = [\theta_x]_k$. This shows that $b(\lambda)$ is obtained by expressing the differential equation in terms of x and θ_x , and substituting $x = 0$ and $\theta_x = \lambda$ formally.

In this subsection we assume that $x_0 \in \mathbb{C} \cup \{\infty\}$ is a regular singular point for the equation (1) for $f(x)$. Let λ_1, λ_2 be the exponents for (1) relative to the regular singular point x_0 .

We show the following lemma on the eigenvalues of a matrix before the proof of Theorem 2.12 on the exponents for (7) relative to x_0 .

Lemma 2.10. *The eigenvalues of an $(n + 1) \times (n + 1)$ tridiagonal matrix*

$$M = \begin{pmatrix} 0 & -\lambda_1\lambda_2 & & & & & \\ n & \lambda_1 + \lambda_2 & -2\lambda_1\lambda_2 & & & & \\ & n-1 & 2(\lambda_1 + \lambda_2) & \ddots & & & \\ & & \ddots & \ddots & & & \\ & & & 2 & -(n-1)\lambda_1\lambda_2 & & \\ & & & & (n-1)(\lambda_1 + \lambda_2) & -n\lambda_1\lambda_2 & \\ & & & & 1 & n(\lambda_1 + \lambda_2) & \end{pmatrix}$$

are

$$(n - k)\lambda_1 + k\lambda_2, \quad (k = 0, 1, \dots, n).$$

Proof. The eigenvalues of M are equal to those of the matrix

$$M' = \begin{pmatrix} 0 & \lambda_1 & & & & & \\ -n\lambda_2 & \lambda_1 + \lambda_2 & 2\lambda_1 & & & & \\ & -(n-1)\lambda_2 & 2(\lambda_1 + \lambda_2) & \ddots & & & \\ & & \ddots & \ddots & (n-1)\lambda_1 & & \\ & & & -2\lambda_2 & (n-1)(\lambda_1 + \lambda_2) & n\lambda_1 & \\ & & & & -\lambda_2 & n(\lambda_1 + \lambda_2) & \end{pmatrix}.$$

because the determinant of a tridiagonal matrix $T = (t_{ij})$ depends only on the diagonal elements t_{ii} and the products of off-diagonal elements $t_{i,i+1}t_{i+1,i}$.

If $\lambda_1 = 0$, it is obvious that the eigenvalues are $0, \lambda_2, \dots, n\lambda_2$. Otherwise, putting $z = \lambda_2/\lambda_1$, we prove that the eigenvalues of the matrix M'/λ_1 are $\mu_k = kz + (n - k)$ ($k = 0, 1, \dots, n$).

For $z \neq 1$, all of μ_k 's are different. We show that the eigenvector corresponding to μ_k is $\mathbf{v}^k = (v_l^k)_{0 \leq l \leq n}$ where

$$v_l^k = \sum_j \binom{n-k}{l-j} \binom{k}{j} z^j. \quad (15)$$

Here, the summation for j is over the finite interval $\max\{0, k + l - n\} \leq j \leq \min\{k, l\}$.

The l -th entry ($0 \leq l \leq n$) of $(\mu_k I - M'/\lambda_1)\mathbf{v}^k$ equals

$$\begin{aligned} & \sum_j \left[(n-l+1) \binom{n-k}{l-j} \binom{k}{j-1} + (n-k-l) \binom{n-k}{l-j} \binom{k}{j} \right. \\ & \quad \left. + (k-l) \binom{n-k}{l-j+1} \binom{k}{j-1} - (l+1) \binom{n-k}{l-j+1} \binom{k}{j} \right] z^j. \end{aligned}$$

The first two terms equal

$$\begin{aligned} & \left[(k+1) \binom{k}{j-1} + (n-k-l) \binom{k+1}{j} \right] \binom{n-k}{l-j} \\ & = (n-k-l+j) \binom{n-k}{l-j} \binom{k+1}{j} = (n-k) \binom{n-k}{l-j+1} \binom{k+1}{j}, \end{aligned}$$

by the relations $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$, $k\binom{n}{k} = n\binom{n-1}{k-1}$ and $\binom{n}{k} = \binom{n}{n-k}$. Similarly, the last two terms equal $-(n-k)\binom{n-k}{l-j+1}\binom{k+1}{j}$. Those show that $(\mu_k I - M'/\lambda_1)\mathbf{v}^k = \mathbf{0}$.

For $z = 1$, all of μ_k 's are identical. Let $\mathbf{v}^k = (v_l^k)_{0 \leq l \leq n}$ ($k = 0, 1, \dots, n$) be $(n+1)$ dimensional vectors where

$$v_l^k = \binom{n-k}{l}, \quad (0 \leq l \leq n-k), \quad v_l^k = 0, \quad (n-k < l \leq n).$$

Then, we can show $(nI - M'/\lambda_1)\mathbf{v}^0 = \mathbf{0}$ and $(nI - M'/\lambda_1)\mathbf{v}^k = k\mathbf{v}^{k-1}$ ($k = 1, \dots, n$) as above. Hence \mathbf{v}^k 's, which are linearly independent, are the generalized eigenvectors of the matrix. \square

Remark 2.11. (15) can be formally written as follows:

$$v_l^k = \binom{n-k}{l} {}_2F_1(-k, -l, n-k-l+1; z),$$

where

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (a)_n = \prod_{k=0}^{n-1} (a+k).$$

Then, the l -th entry ($0 \leq l \leq n$) of $(\mu_k I - M'/\lambda_1)\mathbf{v}^k = \mathbf{0}$ is equivalent to

$$\begin{aligned} c(c-1) {}_2F_1(-k, -l-1, c-1; z) - c(c-1+(k-l)z) {}_2F_1(-k, -l, c; z) \\ - l(c+k)z {}_2F_1(-k, -l+1, c+1; z) = 0, \end{aligned} \quad (16)$$

where $c = n - k - l + 1$. The recursion (16) can be confirmed by `HolonomicFunction` ([10]), a package of `Mathematica`.

We now show the following theorem on the exponents for (7).

Theorem 2.12. *If x_0 is a regular singular point for (1), then x_0 is a regular singular point for (7). Moreover, its exponents for (7) are*

$$(n-k)\lambda_1 + k\lambda_2, \quad (k = 0, 1, \dots, n).$$

Proof. We put $x_0 = 0$ without loss of generality by translation. Then, the equation (1) can be rearranged to

$$\left[\theta_x^2 - b_1(x)\theta_x - b_0(x) \right] \bullet f(x) = 0,$$

where $b_0(x) = x^2 a_0(x)$ and $b_1(x) = x a_1(x) + 1$ are analytic at $x = 0$.

Let

$$B(x) = \begin{pmatrix} 0 & b_0(x) & & & \\ n & b_1(x) & 2b_0(x) & & \\ & n-1 & 2b_1(x) & \ddots & \\ & & \ddots & \ddots & nb_0(x) \\ & & & 1 & nb_1(x) \end{pmatrix}$$

be an $(n + 1) \times (n + 1)$ tridiagonal matrix and let $\tilde{B}(x, \theta_x) = B(x) + \theta_x I$, similarly to (2) and (3). Then, as in (8) we have

$$\theta_x^k \bullet f^n = (f^n, f^{n-1}(\theta_x \bullet f), \dots, f(\theta_x \bullet f)^{n-1}, (\theta_x \bullet f)^n) \tilde{B}(x, \theta_x)^k \bullet \mathbf{q}_0.$$

Let

$$P(x) = (p_{ij}(x))_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n+1}} = (\mathbf{q}_0, \tilde{B}(x, \theta_x) \bullet \mathbf{q}_0, \dots, \tilde{B}(x, \theta_x)^{n+1} \bullet \mathbf{q}_0)$$

be an $(n + 1) \times (n + 2)$ matrix and let $\mathbf{w}(x) = (w_0(x), \dots, w_n(x), 1) \in \text{Ker } P(x)$. Then, the differential equation

$$\left[\theta_x^{n+1} + w_n(x) \theta_x^n + \dots + w_1(x) \theta_x + w_0(x) \right] \bullet f(x)^n = 0$$

is equal to (7).

Every entry of $P(x)$ is analytic at $x = 0$ since $b_0(x)$ and $b_1(x)$ are analytic. Moreover $w_0(x), \dots, w_n(x)$ are all analytic because each of $p_{ii}(x)$ is a constant and $w_{n+1}(x) = 1$. Thus, $x_0 = 0$ is a regular singular point for (7) since θ_x^k is a linear combination of $1, x\partial_x, \dots, x^k \partial_x^k$. Furthermore, the indicial equation for (7) is

$$\lambda^{n+1} + w_n(0)\lambda^n + \dots + w_1(0)\lambda + w_0(0) = 0.$$

On the other hand, since

$$\tilde{B}(x, \theta_x)^k \bullet \mathbf{q}_0 \Big|_{x=0} = \tilde{B}(x, \theta_x)^{k-1} \bullet B(0) \mathbf{q}_0 \Big|_{x=0} = \dots = \tilde{B}(x, \theta_x) \bullet B(0)^{k-1} \mathbf{q}_0 \Big|_{x=0} = B(0)^k \mathbf{q}_0,$$

we have

$$P(0) = (\mathbf{q}_0, B(0)\mathbf{q}_0, \dots, B(0)^{n+1} \mathbf{q}_0).$$

Hence, by $P(0)\mathbf{w}(0) = \mathbf{0}$, we obtain

$$\left(B(0)^{n+1} + w_n(0)B(0)^n + \dots + w_1(0)B(0) + w_0(0)I \right) \mathbf{q}_0 = \mathbf{0}.$$

Then by the Cayley–Hamilton theorem and by the uniqueness of $\mathbf{w}(0)$, the characteristic equation of the matrix $B(0)$ is equal to the indicial equation for (7).

On the other hand, by Vieta's formula, we have $b_0(0) = -\lambda_1 \lambda_2$ and $b_1(0) = \lambda_1 + \lambda_2$. Hence the matrix $B(0)$ is equal to the matrix M in Lemma 2.10. Thus the exponents for (7) are proved to be $(n - k)\lambda_1 + k\lambda_2$ ($k = 0, 1, \dots, n$). \square

We have described the exponents for the differential equation satisfied by $f(x)^n$. From now on, we investigate the exponents for the Fourier transformed equation.

Consider a differential equation

$$L \bullet f(x) = 0, \quad L = p_r(x) \partial_x^r + p_{r-1}(x) \partial_x^{r-1} + \dots + p_1(x) \partial_x + p_0(x),$$

where $p_0(x), \dots, p_r(x)$ are coprime polynomials. Let d be the degree of $p_r(x)$. We assume that $\deg p_k(x) \leq d$ ($k = 0, 1, \dots, r-1$).

By the definition of a regular singular point, if $x = 0$ is a regular singular point, then $d \leq r$. Similarly, if $x = \infty$ is a regular singular point, then $d \geq r$.

If $x = 0$ is a regular singular point, the main terms of the differential equation in the neighborhood of $x = 0$ are $x^d \partial_x^r, x^{d-1} \partial_x^{r-1}, \dots, \partial_x^{r-d}$. Because of the relation $x^a \partial_x^b = x^{a-b} \theta_x (\theta_x - 1) \cdots (\theta_x - b + 1)$, the indicial equation has $0, 1, \dots, r-d-1$ as its roots.

The regular singular point and its exponents are transformed by the Fourier transform as follows.

Proposition 2.13. *Suppose that $p_r(x) = x^d$, $\deg p_k(x) \leq d$ ($k = 0, 1, \dots, r-1$), $x = 0$ is a regular singular point for the equation $L \bullet f(x) = 0$ and its exponents are μ_1, \dots, μ_d and $0, 1, \dots, r-d-1$. Then $x = \infty$ is a regular singular point for the Fourier transformed equation $\hat{L} \bullet \hat{f}(x) = 0$ and its exponents are $-\mu_1 - 1, -\mu_2 - 1, \dots, -\mu_d - 1$.*

Suppose that $\deg p_k(x) \leq \deg p_r(x)$ ($k = 0, 1, \dots, r-1$), $x = \infty$ is a regular singular point for the equation $L \bullet f(x) = 0$ and its exponents are μ_1, \dots, μ_r . Then $x = 0$ is a regular singular point for the transformed equation and its exponents are $-\mu_1 - 1, -\mu_2 - 1, \dots, -\mu_r - 1$ and $0, 1, \dots, d-r-1$.

Proof. By the assumption there exists a term with the highest degree both in x and ∂_x . By the Fourier transform $x \mapsto i\partial_x, \partial_x \mapsto ix$, the highest degree is not changed and their weights are reversed. Hence if $x = 0$ (∞) is a regular singular point for $L \bullet f = 0$, then $x = \infty$ (0) is a regular singular point for $\hat{L} \bullet \hat{f} = 0$, and the main terms in the neighborhood of the singular point are not changed by the Fourier transform.

If $x = 0$ is a regular singular point for $L \bullet f(x) = 0$, its main term equals $x^{-(r-d)} \theta_x (\theta_x - 1) \cdots (\theta_x - (r-d) + 1) (\theta_x - \mu_1) \cdots (\theta_x - \mu_d)$. The main term is Fourier transformed to

$$\begin{aligned} & \partial_x^{-(r-d)} (\theta_x + 1) (\theta_x + 2) \cdots (\theta_x + (r-d)) (\theta_x + \mu_1 + 1) \cdots (\theta_x + \mu_d + 1) \\ &= x^{r-d} (\theta_x + \mu_1 + 1) \cdots (\theta_x + \mu_d + 1), \end{aligned}$$

by the formula $\partial_x^k x^k = (\theta_x + 1) (\theta_x + 2) \cdots (\theta_x + k)$. This gives the exponents for the Fourier transformed equation at $x = \infty$.

If $x = \infty$ is a regular singular point for $L \bullet f(x) = 0$, its main term equals $x^{d-r} (\theta_x - \mu_1) \cdots (\theta_x - \mu_r)$. The main term is Fourier transformed to

$$\begin{aligned} & \partial_x^{d-r} (\theta_x + \mu_1 + 1) \cdots (\theta_x + \mu_r + 1) \\ &= x^{-(d-r)} \theta_x (\theta_x - 1) \cdots (\theta_x - (d-r) + 1) (\theta_x + \mu_1 + 1) \cdots (\theta_x + \mu_r + 1), \end{aligned}$$

by the formula $x^k \partial_x^k = \theta_x (\theta_x - 1) \cdots (\theta_x - k + 1)$. This gives the exponents for the Fourier transformed equation at $x = 0$. \square

3 Applications to statistics

3.1 Sum of beta random variables

Let $f_n(a, b; x)$ be the probability density function of sum of n beta random variables $\text{Beta}(a, b)$. The moment generating function of the beta random variable is

$$M(a, b; t) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 e^{tu} u^{a-1} (1-u)^{b-1} du.$$

Since $M(a, b; t)$ equals the confluent hypergeometric function ${}_1F_1(a, a+b; t)$ (c.f. [15]), the characteristic function $\phi(a, b; t) = M(a, b; it)$ satisfies the following second order differential equation:

$$\left[\partial_t^2 - \left(i - \frac{a+b}{t} \right) \partial_t - \frac{ia}{t} \right] \bullet \phi(a, b; t) = 0, \quad (i = \sqrt{-1}). \quad (17)$$

An $(n+1)$ -th order differential equation satisfied by $M(a, b; t)^n$ is derived by Theorem 2.4, and by the Fourier transform, we obtain a holonomic differential equation satisfied by $f_n(a, b; x)$. By Corollary 2.9, putting $m_0 = M_0 = -1$, $m_1 = -1$, $M_1 = 0$, the equation for $f_n(x)$ is at most of the n -th order. In fact, the equation derived by the procedure of Section 2 is exactly of the n -th order.

We define the initial term of a formal power series, before the proposition on the order.

Definition 3.1. For a formal power series $f(x) = \sum_{k=0}^{\infty} c_n x^{\lambda+k}$ ($c_0 \neq 0$), we define

$$\text{in } f(x) = c_0 x^{\lambda}.$$

We denote the matrix (vector) whose (i, j) entry is $\text{in } f_{ij}(x)$ by $\text{in } F(x)$, where $F(x) = (f_{ij}(x))$.

Proposition 3.2. *The differential equation for $f_n(a, b; x)$ derived by the procedure of Section 2 is of the n -th order.*

Proof. We prove

$$\text{in } q_{0,j}(t) = i(-1)^j n \frac{a}{b} (a+b)_{j-1} t^{-(j-1)}, \quad \text{in } q_{1,j}(t) = (-1)^{j-1} n (a+b)_{j-1} t^{-(j-1)},$$

for $j = 2, 3, \dots, n+1$ by induction based on (4). It is easy to check them for $j = 2$. Assuming them up to j , we have

$$\begin{aligned} \text{in } q_{0,j+1}(t) &= \partial_t q_{0,j}(t) + a_0(t) \text{in } q_{1,j}(t) = i(-1)^{j+1} n \frac{a}{b} (a+b)_j t^{-j}, \\ \text{in } q_{1,j+1}(t) &= (\partial_t - bt^{-1}) \text{in } q_{1,j}(t) = (-1)^j n (a+b)_j t^{-j}, \end{aligned}$$

since $\text{mindeg } q_{2,j} \geq -(j-2)$ by (12). Thus, the results are shown by induction.

Hence, we obtain $\text{mindeg } q_{0,n+1}(t) = -n$. As in the proof of Theorem 2.8, by clearing the denominators, we see that the highest degree of t of the equation (7) for $\phi(a, b; t)^n$ is n . \square

Example 3.3. $f_3(a, b; x)$ satisfies the differential equation

$$\begin{aligned} & \left[x(x-1)(x-2)(x-3)\partial_x^3 + (-6(a+b-2)x^3 + 2(16a+11b-27)x^2 - 6(8a+3b-11)x \right. \\ & + 18(a-1))\partial_x^2 + ((a+b-2)(11(a+b)-18)x^2 - (48a^2 + 66ab + 18b^2 - 145a - 95b \\ & + 108)x + 3(a-1)(15a+12b-22))\partial_x - (a+b-2)(2(a+b)-3)(3(a+b)-4)x \\ & \left. + 3(a-1)(2(a+b)-3)(3(a+b)-4) \right] \bullet f_3(a, b; x) = 0. \end{aligned}$$

Note that $x = 0$ is a regular singular point for (17), and its exponents are $0, 1 - (a + b)$ since

$$\partial_t^2 - \left(i - \frac{a+b}{t} \right) \partial_t - \frac{ia}{t} = t^{-2} \theta_t (\theta_t - 1 + a + b) - it^{-1} (\theta_t + a).$$

Hence we can obtain the exponents for the equation satisfied by $f_n(a, b; x)$ relative to regular singular point ∞ . However, it is not informative since $f_n(a, b; x)$ has a compact support.

On the other hand, the equation in Example 3.3 has regular singular points at $x = 0, 1, 2, 3$. In general, the degree of the coefficient polynomial of the highest order term ∂_x^n is less than or equal to $n + 1$. On the other hand, a differential equation satisfied by f_n , or the n -th convolution of f_1 , has to have singular points at $x = 0, 1, \dots, n$, because f_1 has singular points at $x = 0, 1$. Therefore, the highest order term of the differential equation derived as mentioned above is $x(x-1)\cdots(x-n)\partial_x^n$, and this implies that $x = 0, 1, \dots, n$ are all regular singular points.

Especially, in the case of $a = b = 1$ (then, the beta distribution becomes the uniform distribution), the differential equation is simply $x(x-1)\cdots(x-n)\partial_x^n \bullet f_n(1, 1; x) = 0$. This is because $\phi(1, 1; t) = (e^{it} - 1)/(it)$ and thus

$$\begin{aligned} & 2\pi\mathcal{F}^{-1}[x(x-1)\cdots(x-n)\partial_x^n \bullet f_n(1, 1; x)] \\ & = -i\partial_t(-i\partial_t - 1)\cdots(-i\partial_t - n) \bullet (-1)^n (e^{it} - 1)^n \\ & = -i\partial_t(-i\partial_t - 1)\cdots(-i\partial_t - (n-1)) \bullet (-1)^n n (e^{it} - 1)^{n-1} \\ & = \cdots = -i\partial_t \bullet (-1)^n n! = 0. \end{aligned}$$

This shows that $f_n(1, 1; x)$ has to be a piece-wise $(n-1)$ -th degree polynomial.

The exact form of $f_n(1, 1; x)$ is given in Section 1.9 of [2]. For $n = 2$, $f_2(1, 1; x)$ is a continuous piece-wise linear function and $f_2 \in C^0$. By induction it follows that $f_n(1, 1; x) = \int_0^1 f_{n-1}(1, 1; x-y)dy$ belongs to C^{n-2} . Hence we can put $f_n(1, 1; x) = c_0 x^{n-1}$ ($0 \leq x \leq 1$) by the smoothness at $x = 0$. We can also put $f_n(1, 1; x) = c_0 x^{n-1} + c_1 (x-1)^{n-1}$ ($1 \leq x \leq 2$) by the smoothness at $x = 1$. In the same way, we can put $f_n(1, 1; x) = \sum_{j=0}^k c_j (x-j)^{n-1}$ ($k \leq x \leq k+1$). By the smoothness at $x = n$, the k -th ($k = 0, 1, \dots, n-2$) derivative of $g_n(x) = \sum_{j=0}^{n-1} c_j (x-j)^{n-1}$ at $x = n$ is zero. Moreover, we have

$$\sum_{j=0}^{n-1} \int_j^n c_j (x-j)^{n-1} dx = \sum_{j=0}^{n-1} c_j \frac{(n-j)^n}{n} = 1,$$

because of $\int_0^n f_n(1, 1; x)dx = 1$. Hence, c_0, \dots, c_{n-1} satisfy the equation

$$\begin{pmatrix} n^n & (n-1)^n & \cdots & 1^n \\ n^{n-1} & (n-1)^{n-1} & \cdots & 1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n^1 & (n-1)^1 & \cdots & 1^1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} n \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix on the left is invertible by the Vandermonde determinant. Therefore, we can determine the probability density function $f_n(1, 1; x)$.

Remark 3.4. The k -th moment of the beta random variable is $(\alpha)_k/(\alpha + \beta)_k$. The moments of sum of n beta random variables are the coefficients of

$$\left(\frac{(\alpha)_0}{(\alpha + \beta)_0} + \frac{(\alpha)_1}{(\alpha + \beta)_1}t^1 + \frac{(\alpha)_2}{(\alpha + \beta)_2}t^2 + \cdots \right)^n.$$

The probability density function can be approximated in terms of orthogonal polynomials by fitting the moments. However the information provided by the differential equation can not be easily derived from the moments.

3.2 Sum of cubes of standard normal random variables

In this section we study characteristic functions and probability density functions of sum of cubes of standard normal variables. Concerning the probability distribution of sample skewness from normal population, Geary ([3]) and Mulholland ([11]) give very detailed results. However the distribution of the sum of cubes of standard normal variables, which is a more basic quantity than the sample skewness, has not been studied in detail.

Let $f_n(x)$ denote the probability density function of sum of cubes of n standard normal variables. The characteristic function of the cube of a standard normal variable is

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{itx^3} dx, \quad (i = \sqrt{-1}).$$

Let

$$I_j(t) = \frac{i^j}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^j e^{-x^2/2} e^{itx^3} dx,$$

for $j = 0, 1, 2, \dots$. $I_j(t)$ satisfies the recursion

$$\begin{aligned} 3tI_j(t) &= \frac{i^{j-1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{j-2} e^{-x^2/2} (e^{itx^3})' dx = -\frac{i^{j-1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx^3} (x^{j-2} e^{-x^2/2})' dx \\ &= I_{j-1}(t) + (j-2)I_{j-3}(t). \end{aligned}$$

Since $\phi(t) = I_0(t)$, $\partial_t \phi(t) = -I_3(t)$, $\partial_t^2 \phi(t) = I_6(t)$, we can express $\partial_t \phi(t), \partial_t^2 \phi(t)$ in terms of $I_0(t), I_1(t)$. By eliminating $I_0(t), I_1(t)$, we obtain the second-order differential equation for $\phi(t)$

$$\left[27t^3 \partial_t^2 + (81t^2 + 1) \partial_t + 15t \right] \bullet \phi(t) = 0. \quad (18)$$

This can be also derived by integration algorithm ([13]).

A differential equation satisfied by $\phi(t)^n$ is derived by Theorem 2.4, and by Fourier transform, we obtain a differential equation satisfied by $f_n(x)$. In Corollary 2.9, putting $m_0 = M_0 = -2$, $m_1 = -3$, $M_1 = -1$, the equation for $f_n(x)$ is at most of the $3n$ -th order. In fact, it is exactly of the $3n$ -th order.

Proposition 3.5. *The differential equation for $f_n(x)$ derived by the procedure of Section 2 is of the $3n$ -th order.*

Proof. As in the proof of Proposition 3.2, we can show

$$\text{in } q_{0,j}(t) = (-1)^{j-1} 5n(3t)^{-3(j-1)+1}, \quad \text{in } q_{1,j}(t) = (-1)^{j-1} n(3t)^{-3(j-1)},$$

for $j = 2, 3, \dots, n+1$. Hence, we obtain $\text{mindeg } q_{0,n+1}(t) = -3n$, and thus the highest degree of t of the equation (7) is $3n$. \square

Example 3.6. $f_4(x)$ satisfies the differential equation

$$\begin{aligned} & \left[177147x^5 \partial_x^{12} + 5314410x^4 \partial_x^{11} + 52455195x^3 \partial_x^{10} + (65610x^4 + 202242825x^2) \partial_x^9 \right. \\ & + (1180980x^3 + 278372295x) \partial_x^8 + (6145470x^2 + 89579520) \partial_x^7 + (8505x^3 + 9950850x) \partial_x^6 \\ & \left. + (76545x^2 + 3408480) \partial_x^5 + 155655x \partial_x^4 + (450x^2 + 56160) \partial_x^3 + 1350x \partial_x^2 + 480 \partial_x + 8x \right] \\ & \bullet f_4(x) = 0. \end{aligned}$$

Note that $x = \infty$ is a regular singular point of the differential equation (18), and its exponents are $-1/3, -5/3$ since

$$27x^3 \partial_x^2 + (81x^2 + 1) \partial_x + 15x = 3x(3\theta_x + 1)(3\theta_x + 5) + x^{-1} \theta_x.$$

This implies that there exists a differential equation satisfied by $\phi(t)^n$ which is regular at $x = \infty$ and its exponents are $-n/3, -(n+4)/3, \dots, -(5n-4)/3, -5n/3$. Moreover, there exists a differential equation satisfied by $f_n(x)$ which is regular at $x = 0$ and its exponents are $n/3 - 1, (n+4)/3 - 1, \dots, (5n-4)/3 - 1, 5n/3 - 1$ and $0, 1, \dots, 2n - 2$.

We now briefly discuss issues in numerical evaluation of f_n based on our differential equation and computation of initial values. For numerically solving the differential equation satisfied by f_n , an initial value of $(f_n, \partial_x \bullet f_n, \dots, \partial_x^{3n-1} \bullet f_n)$ at $x = x_0 \neq 0$ is needed. Note that we can not use $x = 0$ as the initial point, because it is the singular point of the differential equation.

By dividing the interval of integration of the inversion formula for the characteristic function and integrating by parts repeatedly, for any integer $m (\geq 0)$, we obtain

$$f_n(x) = \frac{1}{\pi} \text{Re} \left[\int_0^T \phi_n(t) e^{itx} dt + e^{iT x} \sum_{j=1}^m \phi_n^{(j-1)}(T) \left(\frac{i}{x} \right)^j + \left(\frac{i}{x} \right)^m \int_T^\infty \phi_n^{(m)}(t) e^{itx} dt \right],$$

where $\phi_n(t) = \phi(t)^n$. Integration by parts is needed for numerical evaluation of derivatives of f_n .

The formal k -th derivative of $f_n(x)$ at $x = x_0$ is

$$f_n^{(k)}(x_0) = \frac{1}{\pi} \operatorname{Re} \left[\int_0^T (it)^k \phi_n(t) e^{ix_0 t} dt + e^{iT x_0} \sum_{j=1}^m \phi_n^{(j-1)}(T) \sum_{l=0}^k \binom{k}{l} \frac{(j+l-1)!}{(j-1)!} \frac{i^{k+j+l}}{x_0^{j+l}} T^{k-l} \right. \\ \left. + \sum_{l=0}^k \binom{k}{l} \frac{(m+l-1)!}{(m-1)!} \frac{i^{m+k+l}}{x_0^{m+l}} \int_T^\infty t^{k-l} \phi_n^{(m)}(t) e^{ix_0 t} dt \right]. \quad (19)$$

Define

$$n!m = \prod_{0 \leq k < n/m} (n - km), \quad (n > 0), \quad n!m = 1, \quad (n \leq 0).$$

Lemma 3.7. *The following expansion of $\phi(t)$ holds at any $t (\neq 0)$:*

$$\phi(t) = \frac{\sqrt{2\pi}}{3\Gamma(2/3)} |t|^{-1/3} \sum_{k=0}^{\infty} \frac{(6k-5)!6}{(6k)!6(6k-4)!6} (3t)^{-2k} - \frac{\sqrt{2\pi}}{9\Gamma(1/3)} |t|^{-5/3} \sum_{k=0}^{\infty} \frac{(6k-1)!6}{(6k)!6(6k+4)!6} (3t)^{-2k}.$$

Proof. Airy function $\operatorname{Ai}(x)$ can be written as follows ([1], [17], Chapter 9 of [15]):

$$\operatorname{Ai}(x) = \frac{1}{\pi} e^{-\frac{2}{3}x^{3/2}} \int_0^\infty e^{-\sqrt{x}u^2} \cos(u^3/3) du \quad (x > 0).$$

By the transform $u = 2^{-1/2}x^{-1/4}s$, we obtain

$$\operatorname{Ai}(x) = \frac{1}{\sqrt{2\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} \int_0^\infty e^{-s^2/2} \cos\left(\frac{x^{-3/4}}{2^{3/2}} \frac{s^3}{3}\right) ds.$$

From this and

$$\phi(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2} \cos(tx^3) dx,$$

we obtain the following relation between $\phi(t)$ and $\operatorname{Ai}(t)$:

$$\phi(t) = \frac{\sqrt{2\pi}}{3^{1/3}} t^{-1/3} e^{t^{-2}/108} \operatorname{Ai}\left(\frac{t^{-4/3}}{4 \cdot 3^{4/3}}\right) \quad (t > 0). \quad (20)$$

Since the Maclaurin expansion of $\operatorname{Ai}(t)$ is known, the expansion of $\phi(t)$ in $1/t$ can be derived. \square

By (20) we see that $\phi(t) = O(t^{-1/3})$ ($t \rightarrow \infty$). Hence, (19) is justified for $k < m + n/3$ since the integral on $[T, \infty)$ converges for $k < m + n/3$.

In (19) the integral on $[0, T]$ can be numerically computed by using the relation (20).

The integral on $[T, \infty)$ can be computed without numerical integral. The values of $J_l = \int_T^\infty e^{it}/t^{l/3} dt$ ($l = 1, 2, \dots$), where $T' = T x_0$, is needed for computation of the integral on $[T, \infty)$. J_l satisfies the following recurrence relation:

$$J_{l+3} = \frac{3}{l} \left(\frac{e^{iT'}}{T^{l/3}} + iJ_l \right),$$

and thus it is sufficient to compute J_1 , J_2 and J_3 . J_1 and J_2 can be reduced to the integral on $[0, T]$ from the formulae (cf. Section 5.9 of [15])

$$\int_0^{\infty} \frac{\sin x}{x^p} dx = \frac{\pi}{2\Gamma(p) \sin(p\pi/2)}, \quad \int_0^{\infty} \frac{\cos x}{x^p} dx = \frac{\pi}{2\Gamma(p) \cos(p\pi/2)} \quad (0 < p < 1).$$

J_3 can be computed by Maclaurin expansion of trigonometric integrals.

The second term of the right side of (19) can be computed by Lemma 3.7.

From the above, the value of $f_n^{(k)}(x_0)$ ($n = 0, 1, \dots, 3n - 1$) can be computed and the differential equation satisfied by $f_n(x)$ can be numerically solved.

4 Some discussions

In this paper we investigated properties of powers of functions satisfying a second-order holonomic differential equation. Our motivating example was the distribution of convolutions of cubes of standard normal random variables presented in Section 3.2, which was in turn motivated by the algorithm given in [14]. In the course of our study of distribution of cubes of standard normal random variables, we noticed some remarkable properties satisfied by the characteristic function of the cube of a standard normal random variable. Based on this example, we developed more general theory presented in Section 2, which may be relevant to problems in other areas of applied mathematics.

From a mathematical viewpoint, it is of interest to generalize the results of Section 2 to the case of powers of a general holonomic function. From a statistical viewpoint, it is of interest to investigate the distribution of the sum of the r -th power ($r \geq 4$) of standard normal random variables.

Acknowledgment

We are grateful to C. Koutschan for computation of (16).

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