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Abstract

Exponentially-fitted (EF) methods are special methods for ordinary differential equations that better catch periodic/oscillatory solutions. Such solutions often appear in Hamiltonian systems, and in view of this, symplectic or energy-preserving variants of EF methods have been intensively studied recently. In these studies, the symplectic variants have been further applied to Poisson systems, while such a challenge has not ever been done for the energy-preserving variants. In this paper, we propose an energy-preserving EF method for Poisson systems, with special emphasis on the second-/fourth-order schemes.

1 Introduction

In this paper[1], we consider numerical integration of Poisson systems of the form

$$\dot{y} = \Lambda(y)\nabla H(y), \quad y(t_0) = y_0,$$

where $y \in \mathbb{R}^n$, $\Lambda(y)$ is a skew-symmetric matrix and the dot on y stands for the differentiation with respect to time. It is well known that the energy (Hamiltonian) $H(y)$ is constant along the solution:

$$\dot{H}(y(t)) = \nabla H(y(t))^\top \dot{y}(t) = \nabla H(y(t))^\top \Lambda(y(t))\nabla H(y(t)) = 0,$$

and Poisson systems often have periodic or oscillatory solutions. With these considerations, the aim of this paper is to construct energy-preserving integrators specially tuned for periodic or oscillatory solutions.

There have been a lot of studies on energy-preserving methods for Hamiltonian systems in which the matrix Λ is independent of y. The average vector field (AVF) method [20]

$$y_1 = y_0 + h\Lambda \int_0^1 \nabla H(y_0 + \tau(y_1 - y_0))\,\mathrm{d}\tau$$

[1] This work is a full version of our recent report [16] with detailed discussions.

is a unified way of constructing energy-preserving integrators for Hamiltonian systems (see also [11] for the discrete gradient method). The AVF method is of order two. Hairer extended the method to higher-order [12] by introducing continuous stage Runge–Kutta (CSRK) methods. But Poisson systems require an additional technique, due to the dependence of the matrix Λ on $y(t)$. Cohen–Hairer [6] succeeded in constructing arbitrary high-order energy-preserving schemes for Poisson systems by introducing an idea of partitioned methods. The simplest second-order example reads

$$y_1 = y_0 + h\Lambda\left(\frac{y_0+y_1}{2}\right)\int_0^1 \nabla H(y_0+\tau(y_1-y_0))\,\mathrm{d}\tau. \tag{1}$$

Note that in (1) $\Lambda(y)$ and $\nabla H(y)$ are discretised in a different manner.

On the other hand, for ordinary differential equations with periodic or oscillatory solutions, there have been a lot of research activities. For example, trigonometric methods for second-order ODEs and exponentially-fitted methods for first-order ODEs have been studies in the last few decades. Among them, we focus on exponentially-fitted (EF) methods [7, 8, 9, 14, 17, 18, 19, 23, 24, 25, 27]. The idea of EF methods is to construct numerical one-step methods which exactly solve problems whose solution belongs to the linear space spanned by

$$\mathcal{F} = \{\exp(\lambda_1 t), \ldots, \exp(\lambda_r t)\}, \quad \lambda_i \in \mathbb{C}.$$

Recently, symplectic EF methods have been developed for Hamiltonian systems (see, e.g., [1, 2, 3, 4, 5, 10, 21, 22, 26]). Moreover, the methods can be (at least formally) applied to Poisson systems, and in some papers above, this has been done. For both systems, numerical experiments showed the effectiveness of the methods, though strictly speaking, for Poisson systems, there has been no rigorous, theoretical discussions as far as the author understands. From the perspective of geometric numerical integration, these works motivate us to consider energy-preserving EF counterparts. The present author proposed an energy-preserving EF method for Hamiltonian systems by introducing an algebraic condition of energy-preservation in terms of CSRK methods [15]. Although this method can be simply applied to Poisson systems like as in the symplectic versions, in this paper we do not take this approach since such a formal application should obviously destroy the energy-preservation of Poisson systems, and thus such an approach does not make good sense. Instead, in this paper, we aim at a stronger result: namely, we give a new, rigorous framework for constructing EF schemes which exactly inherit the energy-preservation property of Poisson systems. When the new method is applied to Hamiltonian systems, it reduces to the existing energy-preserving EF method.

The construction of the new method is based on the so called partitioned CSRK (PCSRK) methods. We show characterisations of energy-preservation and symmetry properties in terms of PCSRK methods, and based on them we give the new framework. There, the biggest difficulty is that a PCSRK method contains more parameters than a CSRK method, and all parameters have to be determined in terms of elementary functions. We illustrate how this can be done taking the derivation of second- and fourth-order schemes as our working examples.

This paper is organised as follows. In Section 2.1, we give energy-preservation and symmetry conditions for Poisson systems in terms of partitioned CSRK (PCSRK) methods. In Section 2.2, we introduce the concept of EF methods by illustrating the derivation of energy-preserving EF CSRK methods for Hamilton systems. In Section 3, we derive second- and forth-order energy-preserving EF schemes. Section 4 is devoted to numerical experiments, where we consider the Euler equations. In Section 5, we conclude this paper.

2 Preliminaries

2.1 Characterisations of energy-preserving methods

We first summarise energy-preservation and symmetry characterisations for Hamiltonian systems [15] in terms of CSRK methods. After that we give characterisations of PCSRK methods being energy-preserving and symmetric for Poisson systems.

We consider an s-degree CSRK method defined by

$$\begin{aligned} Y_\tau &= y_0 + h\Lambda \int_0^1 A_{\tau,\sigma} \nabla H(Y_\sigma)\, \mathrm{d}\sigma, \\ y_1 &= Y_1, \end{aligned}$$

where Y_τ ($\tau \in [0,1]$) is a polynomial of degree s with respect to τ satisfying $Y_0 = y_0$, and $A_{\tau,\sigma}$ is a polynomial with respect to the variables in the subscripts.

Theorem 1 ([15]). *A CSRK method solving Hamiltonian systems is energy-preserving if* $\frac{\partial}{\partial \tau} A_{\tau,\sigma}$ *is symmetric*[2].

In fact, coefficient polynomials $A_{\tau,\sigma}$ derived in [12] satisfy this condition. This theorem also indicates that $A_{\tau,\sigma}$ is polynomial of degree s and $s-1$ with respect to τ and σ, respectively. Symmetry condition is also written in terms of the coefficient polynomial.

Theorem 2 ([12]). *A CSRK method is symmetric if*

$$A_{1-\tau,1-\sigma} + A_{\tau,\sigma} = A_{1,\sigma}.$$

Next, let us consider Poisson systems (the following characterisation was already pointed out in our recent report [15], but we here add a proof). We consider an s-degree PCSRK method defined by

$$\begin{aligned} Y_\tau &= y_0 + h \sum_{j=1}^{s} \int_0^1 A_{i\tau,j\sigma} \Lambda(Z_j) \nabla H(Y_\sigma)\, \mathrm{d}\sigma, \\ Z_i &= z_0 + h \sum_{j=1}^{s} \int_0^1 \widehat{A}_{i\tau,j\sigma} \Lambda(Z_j) \nabla H(Y_\sigma)\, \mathrm{d}\sigma \quad (i = 1, \dots, s), \\ y_1 &= y_0 + h \sum_{i=1}^{s} \int_0^1 B_{i\tau} \Lambda(Z_i) \nabla H(Y_\tau)\, \mathrm{d}\tau, \\ z_1 &= z_0 + h \sum_{i=1}^{s} \int_0^1 \widehat{B}_{i\tau} \Lambda(Z_i) \nabla H(Y_\tau)\, \mathrm{d}\tau, \end{aligned} \tag{2}$$

with $y_0 = z_0$, where

- Y_τ is a polynomial of degree s with respect to τ satisfying $Y_0 = y_0$,
- $A_{i\tau,j\sigma}$ is a polynomial with respect to the variables τ and σ,
- $0 \le c_1 < \cdots < c_s \le 1$,
- $\widehat{A}_{i\tau,j\sigma} = A_{c_i,j\sigma}$,

[2] In this paper, a polynomial $f(\tau,\sigma)$ is said to be symmetric if $f(\tau,\sigma) = f(\sigma,\tau)$.

- $B_{j\sigma} = \widehat{B}_{j\sigma} = A_{1,j\sigma}$.

The notation $A_{i\tau,j\sigma}$, which was introduced in [6], depends on $\tau,\ \sigma \in [0,1]$, $j = 1,\dots,s$ and i. In reality, it does not depend on i, but we leave it as it is because it becomes useful when considering order conditions. In other places, we can simply understand this as $A_{i\tau,j\sigma} = A_{\tau,j\sigma}$.

It is clear that $y_1 = z_1$ and (2) is equivalent to $Z_i = Y_{c_i}$. For example, when $s = 1$, $A_{\tau,1\sigma} = \tau$ and $c_1 = 1/2$, the PCSRK method reduces to (1). As mentioned in [6], these methods are consistent with the partitioned system of differential equations

$$\begin{aligned} \dot{y} &= \Lambda(z)\nabla H(y), \quad y(t_0) = y_0, \\ \dot{z} &= \Lambda(z)\nabla H(y), \quad z(t_0) = z_0, \end{aligned}$$

whose solutions satisfy $y(t) = z(t)$ if $y_0 = z_0$.

Theorem 3. *A PCSRK method solving Poisson systems is energy-preserving if $\frac{\partial}{\partial\tau}A_{i\tau,j\sigma}$ is symmetric for all $j = 1,\dots,s$.*

Proof. We can express each $\frac{\partial}{\partial\tau}A_{i\tau,j\sigma}$ as

$$\frac{\partial}{\partial\tau}A_{i\tau,j\sigma} = \sum_{l=0}^{s-1} a_j(l,l)\tau^l\sigma^l + \sum_{m<n}\left(a_j(m,n)\tau^m\sigma^n + a_j(n,m)\tau^n\sigma^m\right).$$

Note that the symmetry of $\frac{\partial}{\partial\tau}A_{i\tau,j\sigma}$ is equivalent to $a_j(m,n) = a_j(n,m)$ for all j,m,n. Thus we have

$$\begin{aligned} H(y_1) - H(y_0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}H(Y_\tau)\,\mathrm{d}\tau = \int_0^1 \dot{Y}_\tau^\top \nabla H(Y_\tau)\,\mathrm{d}\tau \\ &= h\int_0^1 \left(\sum_{j=1}^{s}\int_0^1 \frac{\partial}{\partial\tau}A_{i\tau,j\sigma}\Lambda(Z_j)\nabla H(Y_\tau)\,\mathrm{d}\sigma\right)^\top \nabla H(Y_\tau)\,\mathrm{d}\tau \\ &= h\sum_{j=1}^{s}\sum_{l=0}^{s-1} a_j(l,l)\left(\int_0^1 \sigma^l\nabla H(Y_\sigma)\,\mathrm{d}\sigma\right)^\top \Lambda^\top(Z_j)\int_0^1 \tau^l\nabla H(Y_\tau)\,\mathrm{d}\tau \\ &\quad + h\sum_{j=1}^{s}\sum_{m<n}\left\{a_j(m,n)\left(\int_0^1 \sigma^n\nabla H(Y_\sigma)\,\mathrm{d}\sigma\right)^\top \Lambda^\top(Z_j)\int_0^1 \tau^m\nabla H(Y_\tau)\,\mathrm{d}\tau\right. \\ &\quad \left. + a_j(n,m)\left(\int_0^1 \sigma^m\nabla H(Y_\sigma)\,\mathrm{d}\sigma\right)^\top \Lambda^\top(Z_j)\int_0^1 \tau^n\nabla H(Y_\tau)\,\mathrm{d}\tau\right\} = 0. \end{aligned}$$

□

Symmetry condition is also given as follows.

Theorem 4. *A PCSRK method is symmetric if $A_{i(1-\tau),(s+1-j)(1-\sigma)} + A_{i\tau,j\sigma} = A_{1,j\sigma}$, and the nodes c_i are symmetric, i.e., $c_{s+1-i} = 1 - c_i$.*

In fact, the coefficient polynomials proposed in [6] satisfy the conditions of the above theorems.

2.2 Energy-preserving EF CSRK methods for Hamiltonian systems

This subsection is devoted to introduce the concept of EF methods by illustrating the derivation of a fourth-order energy-preserving EF CSRK integrator for Hamilton systems [15]. The discussion here will be used in the derivation of energy-preserving EF integrators for Poisson systems in the next section.

We start with a two-degree CSRK formulation:

$$A_{\tau,\sigma} = a_{11}\tau + a_{12}\tau\sigma + a_{21}\tau^2 + a_{22}\tau^2\sigma$$

with four parameters. It is easy to check that the energy-preservation condition in Theorem 1 is equivalent to

$$a_{12} = 2a_{21},$$

and the symmetry condition in Theorem 2 is equivalent to

$$a_{22} + 2a_{21} = 0, \quad a_{22} + a_{12} = 0.$$

Next, we consider the EF conditions. We write Y_τ as a linear combination of y_0, Y_c and y_1, i.e.,

$$Y_\tau = y_0\frac{(\tau - c)(\tau - 1)}{c} + Y_c\frac{\tau(\tau - 1)}{c(c - 1)} + y_1\frac{\tau(\tau - c)}{1 - c}.$$

The CSRK method can be rewritten as

$$\begin{aligned} Y_c &= y_0 + h\int_0^1 A_{c,\sigma} f(Y_\sigma)\,\mathrm{d}\sigma, \\ y_1 &= y_0 + h\int_0^1 B_\sigma f(Y_\sigma)\,\mathrm{d}\sigma, \end{aligned}$$

where $B_\sigma = A_{1,\sigma}$. The solution y_1 in the standard CSRK method is independent of the node c. But the solution of EF methods depends on c. Since the polynomial Y_τ cannot coincide with an exponential function $\exp(\lambda_i\tau h)$ in the EF methods, the node c, at which the numerical solution is fitted to exponential functions, plays an important role.

Next we determine $A_{\tau,\sigma}$ and B_σ so that the method is exact for a problem whose solution is a linear combination of $\mathcal{F} = \{u_1(t), \ldots, u_r(t)\}$. The conditions for $u_i(t)$ are

$$\begin{aligned} u_i(t_0 + ch) &= u_i(t_0) + h\int_0^1 A_{c,\sigma}\widetilde{u}_i'(t_0 + \sigma h)\,\mathrm{d}\sigma, \\ u_i(t_0 + h) &= u_i(t_0) + h\int_0^1 B_\sigma\widetilde{u}_i'(t_0 + \sigma h)\,\mathrm{d}\sigma, \end{aligned}$$

where $\widetilde{u}_i'(t_0 + \sigma h)$ denotes a polynomial of degree 2 (s, in general), which is a linear combination of $u_i'(t_0)$, $u_i'(t_0 + ch)$ and $u_i'(t_0 + h)$. We call such conditions for $i = 1, \ldots, r$ the EF conditions. When we consider the set $\mathcal{F}_1 = \{\exp(\lambda t), \exp(-\lambda t)\}$, the EF conditions are given as

$$\int_0^1 A_{c,\sigma}\widetilde{\cosh}(\sigma z)\,\mathrm{d}\sigma = \frac{\sinh(cz)}{z}, \quad \int_0^1 A_{c,\sigma}\widetilde{\sinh}(\sigma z)\,\mathrm{d}\sigma = \frac{\cosh(cz) - 1}{z}, \tag{3}$$

$$\int_0^1 B_\sigma\widetilde{\cosh}(\sigma z)\,\mathrm{d}\sigma = \frac{\sinh(z)}{z}, \quad \int_0^1 B_\sigma\widetilde{\sinh}(\sigma z)\,\mathrm{d}\sigma = \frac{\cosh(z) - 1}{z}, \tag{4}$$

where $z = \lambda h$, $\widetilde{\cosh}(\sigma z)$ denotes a polynomial of degree two which is a linear combination of $\cosh(0)$, $\cosh(cz)$ and $\cosh(z)$, and similar notation is used for sinh. Since B_σ is independent of σ because of $a_{22} + a_{12} = 0$, the conditions (4) are equivalent to

$$B_\sigma \frac{(3c^2 - 4c + 1) - \cosh(cz) + (3c^2 - 2c)\cosh(z)}{6c^2 - 6c} = \frac{\sinh(z)}{z},$$
$$B_\sigma \frac{-\sinh(cz) + (3c^2 - 2c)\sinh(z)}{6c^2 - 6c} = \frac{\cosh(z) - 1}{z}.$$

We obtain $c = 1/2$ so that the two conditions are compatible, and then have

$$B_\sigma = a_{11} + a_{21} = \frac{6(\cosh(z) - 1)}{z(4\sinh(\frac{z}{2}) + \sinh(z))}.$$

It follows from (3) that

$$a_{11} = \frac{6(-7 + 4\cosh(\frac{z}{2}) + 3\cosh(z))}{z(4\sinh(\frac{z}{2}) + \sinh(z))}, \quad a_{21} = \frac{12(3 - 2\cosh(\frac{z}{2}) - \cosh(z))}{z(4\sinh(\frac{z}{2}) + \sinh(z))}. \tag{5}$$

The resulting method is of order four, which can be verified by checking the conditions for order three. It is also worth mentioning that in the limit $z \to 0$ the standard fourth-order energy-preserving method $A_{\tau,\sigma} = \tau((4 - 3\tau) - 6(1 - \tau)\sigma)$ [12] is recovered.

3 Energy-preserving exponentially-fitted schemes for Poisson systems

We shall derive second- and fourth-order energy-preserving EF schemes for Poisson systems. While the derivation of second-order scheme is relatively easy, the derivation of fourth-order one requires a more careful treatment.

3.1 Second order scheme

Let $s = 1$. In this case, we have $c = 1/2$ from the symmetry condition. The PCSRK method reduces to

$$y_1 = y_0 + a_{11} h \Lambda \left(\frac{y_0 + y_1}{2} \right) \int_0^1 \nabla H(y_0 + \tau(y_1 - y_0))\, \mathrm{d}\tau$$

with a parameter a_{11}. This method is always energy-preserving independently of a_{11}. When one consider the set $\mathcal{F}_1 = \{\exp(\lambda t), \exp(-\lambda t)\}$, the EF condition is given as

$$e^z = 1 + a_{11} z \frac{1 + e^z}{2}, \quad e^{-z} = 1 - a_{11} z \frac{1 + e^{-z}}{2},$$

where $z = \lambda h$, from which we immediately obtain

$$a_{11} = \frac{2\sinh(\frac{z}{2})}{z\cosh(\frac{z}{2})}.$$

The resulting scheme reads

$$y_1 = y_0 + \frac{2\sinh(\frac{z}{2})}{z\cosh(\frac{z}{2})} h \Lambda \left(\frac{y_0 + y_1}{2} \right) \int_0^1 \nabla H(y_0 + \tau(y_1 - y_0))\, \mathrm{d}\tau.$$

As is the case with standard EF methods, when we implement the scheme, if $|z| \ll 1$, we recommend the following expansion

$$a_{11} = 1 - \frac{1}{12}z^2 + \frac{1}{120}z^4 - \frac{17}{20160}z^6 + \frac{31}{362880}z^8 - \frac{691}{79833600}z^{10} + \frac{5461}{6227020800}z^{12} - \cdots.$$

3.2 Fourth order scheme

A more interesting, nontrivial example is a derivation of fourth order schemes. Setting $s = 2$, we consider coefficient polynomials of the form $A_{i\tau,j\sigma} = a^j_{11}\tau + a^j_{12}\tau\sigma + a^j_{21}\tau^2 + a^j_{22}\tau^2\sigma$ ($j = 1, 2$). Our aim is to determine these eight parameters and two nodes c_1, c_2 (thus, there are 10 unknowns) so that they satisfy conditions of energy-preservation, symmetry, exponential-fitting and order. Note that considering symmetry conditions makes the derivation simple, that is, we do not have to care conditions for odd order. The procedure consists of the following four steps.

Step 1 (Energy-preservation condition) From Theorem 3, the energy-preservation condition is equivalent to $a^j_{12} = 2a^j_{21}$ ($j = 1, 2$).

Step 2 (Symmetry condition) From Theorem 4, the symmetry condition is equivalent to

$$a^1_{22} = a^2_{22}, \quad a^1_{21} + a^2_{21} = -a^1_{22}, \quad a^1_{11} - a^2_{11} = 4a^2_{21} + 2a^2_{22}, \quad c_1 + c_2 = 1.$$

Step 3 (EF condition) Next we consider EF conditions, which make the scheme exact for ODEs whose solution belongs to $\mathcal{F}_1 = \{\exp(\lambda t), \exp(-\lambda t)\}$. Note that there is a linear ODE whose solution belongs to this space. Thus, we can reduce the discussion of EF conditions into a framework of CSRK methods with $A_{\tau,\sigma} = \sum_{j=1}^{2} A_{i\tau,j\sigma}$. From the discussion in Section 2.2, the parameters of $A_{\tau,\sigma}$ are given as (5). Hence, for the coefficients of PCSRK methods, we have

$$a^1_{11} + a^2_{11} = \frac{6(-7 + 4\cosh(\frac{z}{2}) + 3\cosh(z))}{z(4\sinh(\frac{z}{2}) + \sinh(z))}, \quad a^1_{21} + a^2_{21} = \frac{12(3 - 2\cosh(\frac{z}{2}) - \cosh(z))}{z(4\sinh(\frac{z}{2}) + \sinh(z))}.$$

Step 4 (Order condition) Finally, we consider order conditions. Let

$$TP = \{\bullet, \circ, \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}, \begin{smallmatrix}\circ\\\bullet\end{smallmatrix}, \begin{smallmatrix}\bullet\\\circ\end{smallmatrix}, \begin{smallmatrix}\circ\\\circ\end{smallmatrix}, \begin{smallmatrix}\bullet\\\bullet\bullet\end{smallmatrix}, \begin{smallmatrix}\circ\\\bullet\bullet\end{smallmatrix}, \begin{smallmatrix}\bullet\\\circ\bullet\end{smallmatrix}, \begin{smallmatrix}\bullet\\\bullet\circ\end{smallmatrix}, \begin{smallmatrix}\circ\\\circ\bullet\end{smallmatrix}, \begin{smallmatrix}\bullet\\\circ\circ\end{smallmatrix}, \begin{smallmatrix}\circ\\\circ\circ\end{smallmatrix}, \dots\}$$

be a set of rooted bi-coloured trees (see [13, Chapter III.2] for a more precise definition of this set, and the theory of P-series). We denote the subset of TP whose roots are $\bullet$ by TP_y, and the remains by TP_z. We also denote the elementary weights by $\phi(\tau)$ ($\tau \in TP$), which are well-defined as with the standard partitioned RK methods. For example,

$$\phi(\begin{smallmatrix}\circ\\\bullet\\\bullet\end{smallmatrix}) = \int_0^1 \int_0^1 \int_0^1 \sum_{i,j,k} B_{i\tau} A_{i\tau,j\sigma} \widehat{A}_{j\sigma,k\upsilon} \, \mathrm{d}\tau \mathrm{d}\sigma \mathrm{d}\upsilon$$

for PCSRK methods, while

$$\phi(\begin{smallmatrix}\circ\\\bullet\\\bullet\end{smallmatrix}) = \sum_{i,j,l} b_i a_{ij} \hat{a}_{jk}$$

for partitioned RK methods. Since the elementary weights depend on h for EF methods, we have to consider order conditions taking this effect into consideration. The order conditions are summarised as follows. An EF PCSRK method is of order p if

$$\phi(\tau) = \frac{1}{\gamma(\tau)} + \mathcal{O}(h^{p-|\tau|+1}) \quad \text{for} \quad \tau \in TP_y, \quad |\tau| \le p,$$

where $|\tau|$ denotes the order of τ (i.e., the number of vertices). Note that we do not have to consider TP_z because $y_1 = z_1$. Since symmetric methods always have even order, it is sufficient to consider only first- and third-order conditions:

$$\phi(\bullet) = 1 + \mathcal{O}(h^4),$$
$$\phi(\text{⁘}) = \frac{1}{3} + \mathcal{O}(h^2), \quad \phi(\text{⁘}) = \frac{1}{3} + \mathcal{O}(h^2), \quad \phi(\text{⁘}) = \frac{1}{3} + \mathcal{O}(h^2),$$
$$\phi(\text{⁘}) = \frac{1}{6} + \mathcal{O}(h^2), \quad \phi(\text{⁘}) = \frac{1}{6} + \mathcal{O}(h^2), \quad \phi(\text{⁘}) = \frac{1}{6} + \mathcal{O}(h^2), \quad \phi(\text{⁘}) = \frac{1}{6} + \mathcal{O}(h^2).$$

These conditions are all satisfied if the perturbations of all parameters from those of the standard fourth-order energy-preserving scheme

$$A_{i\tau,1\sigma} = \frac{1}{2\sqrt{3}} \left((4\sqrt{3}+6)\tau - 6(1+\sqrt{3})\tau\sigma - 3(1+\sqrt{3})\tau^2 + 6\sqrt{3}\tau^2\sigma \right), \tag{6}$$
$$A_{i\tau,2\sigma} = \frac{1}{2\sqrt{3}} \left((4\sqrt{3}-6)\tau + 6(1-\sqrt{3})\tau\sigma + 3(1-\sqrt{3})\tau^2 + 6\sqrt{3}\tau^2\sigma \right), \tag{7}$$
$$c_{1,2} = \frac{1}{2} \mp \frac{\sqrt{3}}{6} \tag{8}$$

are less than $\mathcal{O}(h^2)$. Note that we have obtained eight independent conditions for ten parameters in Step 3, and thus two freedoms still remain at this stage. If one introduce two additional constraints arbitrarily so that they are consistent with the above order conditions, all parameters are uniquely determined.

In the next section, we simply consider the choices

$$c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad 2a_{21} + a_{22} = -\sqrt{3}$$

as additional constraints in Step 4. Then all parameters are uniquely determined to be

$$
\begin{aligned}
a^1_{11} &= \frac{3(-7+4\cosh(\frac{z}{2})+3\cosh(z))}{z(4\sinh(\frac{z}{2})+\sinh(z))}+\sqrt{3},\\
a^1_{12} &= \frac{12(3-2\cosh(\frac{z}{2})-\cosh(z))}{z(4\sinh(\frac{z}{2})+\sinh(z))}-\sqrt{3},\\
a^1_{21} &= \frac{6(3-2\cosh(\frac{z}{2})-\cosh(z))}{z(4\sinh(\frac{z}{2})+\sinh(z))}-\frac{\sqrt{3}}{2},\\
a^1_{22} &= -\frac{12(3-2\cosh(\frac{z}{2})-\cosh(z))}{z(4\sinh(\frac{z}{2})+\sinh(z))},\\
a^2_{11} &= \frac{3(-7+4\cosh(\frac{z}{2})+3\cosh(z))}{z(4\sinh(\frac{z}{2})+\sinh(z))}-\sqrt{3}\\
a^2_{12} &= \frac{12(3-2\cosh(\frac{z}{2})-\cosh(z))}{z(4\sinh(\frac{z}{2})+\sinh(z))}+\sqrt{3},\\
a^2_{21} &= \frac{6(3-2\cosh(\frac{z}{2})-\cosh(z))}{z(4\sinh(\frac{z}{2})+\sinh(z))}+\frac{\sqrt{3}}{2},\\
a^2_{22} &= -\frac{12(3-2\cosh(\frac{z}{2})-\cosh(z))}{z(4\sinh(\frac{z}{2})+\sinh(z))},\\
c_1 &= \frac{1}{2}-\frac{\sqrt{3}}{6},\\
c_2 &= \frac{1}{2}+\frac{\sqrt{3}}{6}.
\end{aligned}
$$

It is checked that the perturbations of these parameters from those in (6), (7) and (8) are less than $\mathcal{O}(h^2)$.

4 Numerical experiments

In this section, we test the derived schemes numerically. For a problem whose period is estimated to $T = 2\pi/\omega$, we consider the set $\mathcal{F}_1 = \{\exp(\lambda t), \exp(-\lambda t)\}$ with $\lambda = i\omega$, which is equivalent to $\{\sin(\omega t), \cos(\omega t)\}$. All computations were done in the computation environment: 2 GHz Inter Core i7, 8GB memory, OS X 10.9.3. We used Python 2.7.5 and its numpy and scipy packages.

We consider the Euler equations

$$\dot{q} = f(q) = ((\alpha-\beta)q_2q_3, (1-\alpha)q_3q_1, (\beta-1)q_1q_2)^\top,$$

which describe the motion of a rigid body under no forces. This system can be seen as a Poisson system

$$\dot{q} = \begin{pmatrix} 0 & \alpha q_3 & -\beta q_2 \\ -\alpha q_3 & 0 & q_1 \\ \beta q_2 & -q_1 & 0 \end{pmatrix} \nabla H(q), \quad H(q) = \frac{q_1^2+q_2^2+q_3^2}{2}.$$

We set the initial value to $q(0) = (0,1,1)^\top$, and the parameters $\alpha = 1+(1/\sqrt{1.51})$, $\beta = 1-(0.51/\sqrt{1.51})$, which are employed in [3]. The exact solution is given by

$$q(t) = (\sqrt{1.51}\,\mathrm{sn}(t,0.51), \mathrm{cn}(t,0.51), \mathrm{dn}(t,0.51))^\top,$$

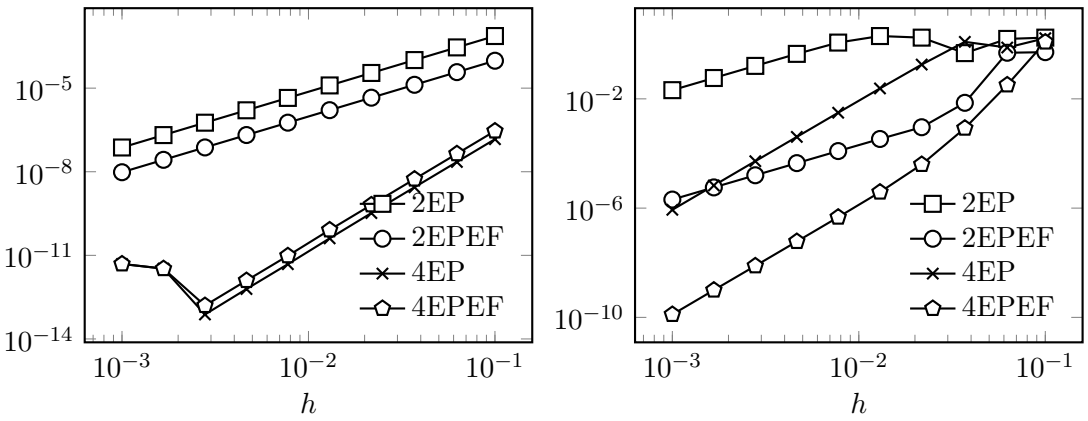

Figure 1: Global error of (left) the first and (right) second examples for Euler equations. 2EP: standard second-order energy-preserving scheme, 2EPEF: second-order energy-preserving EF scheme, 4EP: standard fourth-order energy-preserving scheme, 4EPEF: fourth-order energy-preserving EF scheme.

where $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ are the Jacobi elliptic functions. This solution is periodic with the period $T = 4K(0.51) = 7.450563209330954$, where $K(k)$ stands for the complete elliptic integral of the first kind defined by

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \, \mathrm{d}\theta = \int_0^1 \frac{1}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \, \mathrm{d}t.$$

The left figure in Fig. 1 plots the global error, from which one can see that the solution by the second-order energy-preserving EF scheme is better than that by the standard second-order energy-preserving scheme.

We also consider a more anomalous case. When $\beta \approx 1$, we expect, at least intuitively, $\dot{q}_3 \approx 0$ and thus $q_3(t) \approx 1$. Therefore, the variables q_1 and q_2 seem to behave like harmonic oscillator with period $T = 2\pi/(\alpha - 1)$. We set $\alpha = 51$ and $\beta = 1.01$. The global error is shown in the right figure of Fig. 1. One can see that EF schemes produce much better solutions than the same order, standard energy-preserving schemes.

5 Concluding remarks

In this paper, we derived energy-preserving exponentially-fitted integrators for Poisson systems. In the derivation, we used the energy-preservation and symmetry conditions in terms of partitioned continuous stage Runge–Kutta methods, EF conditions and order conditions. Through the numerical experiments for the Euler equations, we observed that the derived schemes gave better numerical solutions than the standard energy-preserving schemes, even if the period was roughly estimated.

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