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A game-theoretic proof of Erdős-Feller-Kolmogorov-Petrowsky law of the iterated logarithm for fair-coin tossing

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Abstract

We give a game-theoretic proof of the celebrated Erdős-Feller-Kolmogorov-Petrowsky law of the iterated logarithm for fair coin tossing. Our proof, based on Bayesian strategy, is explicit as many other game-theoretic proofs of the laws in probability theory.

Keywords and phrases: Bayesian strategy, constant-proportion betting strategy, lower class, upper class.

1 Introduction

Let $x_n = \pm 1,\ n = 1, 2, \dots,$ be independent symmetric Bernoulli random variables with $P(x_n = -1) = P(x_n = 1) = 1/2$. Let $S_n = x_1 + x_2 + \cdots + x_n$. Concerning the behavior of S_n, the celebrated Erdős-Feller-Kolmogorov-Petrowsky law of the iterated logarithm (EFKP-LIL [17, Chapter 5.2]) states the following:

$$P(S_n \ge \sqrt{n}\psi(n)\ \ i.o.) = 0 \text{ or } 1 \quad \text{according as} \quad \int_1^\infty \psi(\lambda) e^{-\psi(\lambda)^2/2} \frac{d\lambda}{\lambda} < \infty \text{ or } = \infty, \qquad (1)$$

where ψ is a positive non-decreasing continuous function defined on $[1, \infty)$. The set of functions ψ such that $P(S_n \ge \sqrt{n}\psi(n)\ \ i.o.) = 0$ is called the *upper class* and the set of functions ψ such that $P(S_n \ge \sqrt{n}\psi(n)\ \ i.o.) = 1$ is called the *lower class* [17, pp.33-34].

As the name indicates, this is an extension of the LIL. The first one who showed this result seems Kolmogorov, which has been stated in Lévy's book [13] without proof. Erdős [4] has given a complete proof, which has been generalized by Feller [5, 6] (see also Bai [1]). Petrowsky [16] has proved the statement for Brownian motion (see also Itô and McKean [8, Section 1.8 and 4.12] and Knight [10, Section 5.4]). Further developments can be seen in the literature such

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as similar statements for self-normalized sums [7, 3], for weighted sums [2] and for Brownian motion [9].

In order to state a game-theoretic version of EFKP-LIL, consider the following fair-coin game.

Fair-Coin Game
Players: Skeptic, Reality
Protocol:
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots$:
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in \{-1, 1\}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.
Collateral Duties: Skeptic must keep $\mathcal{K}_n$ non-negative. Reality must keep $\mathcal{K}_n$ from tending to infinity.

Let

$$I(\psi) = \int_1^\infty \psi(\lambda) e^{-\psi(\lambda)^2/2} \frac{d\lambda}{\lambda}. \tag{2}$$

The goal of this paper is to prove the game-theoretic statement of EFKP-LIL in the following form.

Theorem 1.1. *Let ψ be a positive non-decreasing continuous function defined on $[1, \infty)$. In the fair-coin game,*

$$I(\psi) < \infty \Rightarrow \textit{Skeptic can force } S_n < \sqrt{n}\psi(n) \ \textit{a.a.} \tag{3}$$

$$I(\psi) = \infty \Rightarrow \textit{Skeptic can force } S_n \geq \sqrt{n}\psi(n) \ \textit{i.o.} \tag{4}$$

The first statement is the *validity* and the second statement is the *sharpness* of EFKP-LIL. For terminology and notions of game-theoretic probability see [18]. As shown in Chapter 8 of [18], game-theoretic statement of EFKP-LIL in (3) and (4) implies the measure-theoretic statement in (1). Furthermore our proof gives a clear relation between the almost-sure events and the integrability.

In Section 2 we give a proof of the validity and in Section 3 we give a proof of the sharpness. We discuss some topics for further research in Section 4.

2 Validity

As is often seen in the upper-lower class theory (see Feller [6, Lemma 1]), we can restrict our attention to ψ such that

$$\psi^L(n) \leq \psi(n) \leq \psi^U(n) \text{ for all sufficiently large } n, \tag{5}$$

where

$$\psi^L(n) = \sqrt{2 \ln\ln n + 3 \ln\ln\ln n}, \quad \psi^U(n) = \sqrt{2 \ln\ln n + 4 \ln\ln\ln n}.$$

Here L means the lower class and U means the upper class. It can be verified that $I(\psi^U) < \infty$ and $I(\psi^L) = \infty$. Note that if a function $\psi(n)$ belongs to the upper class, then any function larger than $\psi(n)$ belongs to the upper class, and a similar statement holds for the lower class.

We discretize the integral in (2) as

$$\sum_{k=1}^{\infty} \frac{\psi(k)}{k} e^{-\psi(k)^2/2} < \infty. \tag{6}$$

Since $xe^{-x^2/2}$ is decreasing for $x \geq 1$, the function $\lambda \mapsto \frac{\psi(\lambda)}{\lambda} e^{-\psi(\lambda)^2/2}$ is decreasing for λ such that $\psi(\lambda) \geq 1$ and convergences of the integral in (2) and the sum in (6) are equivalent.

2.1 Constant-proportion betting strategy

Our proof highly depends on constant-proportion betting strategy (and its mixture). Here we give basic properties.

We fix a small positive real δ for the rest of this paper. For instance, $\delta < 0.01$ is good enough.

A constant-proportion betting strategy S^γ with the parameter γ sets

$$M_n = \gamma \mathcal{K}_{n-1}$$

for a constant $\gamma \in (-1, 1)$. For the rest of this paper we assume $0 \leq \gamma \leq \delta$. The capital process with this strategy is denoted by $\mathcal{K}_n^\gamma$. Note that $\mathcal{K}_n^\gamma$ is always positive. With the condition of $\mathcal{K}_0^\gamma = 1$, the value $\mathcal{K}_n^\gamma$ can be evaluated as

$$\mathcal{K}_n^\gamma = \prod_{i=1}^{n} (1 + \gamma x_i) = \left(\frac{1+\gamma}{1-\gamma} \right)^{S_n/2} (1 - \gamma^2)^{n/2}. \tag{7}$$

Note that $\mathcal{K}_n^\gamma$ is determined (except n and γ) by S_n and is monotone increasing in S_n. In particular, by (7), we have

$$S_n \leq 0 \;\Rightarrow\; \mathcal{K}_n^\gamma \leq 1. \tag{8}$$

By the fact that

$$t - \frac{1}{2}t^2 - |t|^3 \leq \ln(1+t) \leq t - \frac{1}{2}t^2 + |t|^3$$

for $|t| \leq \delta$, taking the logarithm of $\prod_{i=1}^n (1 + \gamma x_i)$ we have

$$\gamma S_n - \frac{1}{2}\gamma^2 n - \gamma^3 n \leq \ln \mathcal{K}_n^\gamma \leq \gamma S_n - \frac{1}{2}\gamma^2 n + \gamma^3 n$$

and

$$e^{-\gamma^3 n} e^{\gamma S_n - \gamma^2 n/2} \leq \mathcal{K}_n^\gamma \leq e^{\gamma^3 n} e^{\gamma S_n - \gamma^2 n/2}. \tag{9}$$

For the proof of validity, we only use the lower bound in (9).

2.2 Proof of validity

From now on for notational simplicity we write $\psi_k = \psi(k)$. The convergence of the infinite series in (6) implies the existence of a non-decreasing sequence of positive reals a_k diverging to infinity ($a_k \uparrow \infty$), such that the series multiplied term by term by a_k is still convergent:

$$Z := \sum_{k=1}^{\infty} a_k \frac{\psi_k}{k} e^{-\psi_k^2/2} < \infty. \tag{10}$$

This is easily seen by dividing the infinite series into blocks of sums less than or equal to $1/2^k$ and multiplying the k-th block by k (see also [14, Lemma 4.15]).

For $k \geq 1$ let

$$p_k = \frac{1}{Z} a_k \frac{\psi_k}{k} e^{-\psi_k^2/2}$$

and consider the capital process of a countable mixture of constant-proportion strategies

$$\mathcal{K}_n = \sum_{k=1}^{\infty} p_k \mathcal{K}_n^{\gamma_k}, \qquad \text{where} \qquad \gamma_k = \frac{\psi_k}{\sqrt{k}}. \tag{11}$$

Obviously $\mathcal{K}_n$ is never negative. By the upper bound in (5), as $k \to \infty$ we have

$$\gamma_k \leq \sqrt{\frac{2 \ln\ln k + 4 \ln\ln\ln k}{k}} \to 0.$$

Hence $\gamma_k < \delta$ for sufficiently large k.

We now confirm that $\limsup_n \mathcal{K}_n = \infty$ if $S_n \geq \sqrt{n}\psi_n$ infinitely often. By (9) and (10), we have

$$\begin{aligned} Z\mathcal{K}_n &\geq Z \sum_{k:\gamma_k<\delta} p_k \exp(\gamma_k S_n - \frac{\gamma_k^2 n}{2} - \gamma_k^3 n) \\ &= \sum_{k:\gamma_k<\delta} a_k \frac{\psi_k}{k} \exp(-\frac{\psi_k^2}{2} + \gamma_k S_n - \frac{\gamma_k^2 n}{2} - \gamma_k^3 n). \end{aligned}$$

We consider n and k such that $S_n \geq \sqrt{n}\psi_n$, $\gamma_k < \delta$, $\lfloor n - n/\psi_n \rfloor \leq k \leq n$ and $\psi_n/(\psi_n - 1) \leq 1 + \delta/2$. By (11), we have

$$\begin{aligned} -\frac{\psi_k^2}{2} + \gamma_k S_n - \frac{\gamma_k^2 n}{2} &\geq -\frac{\psi_k^2}{2} + \sqrt{n}\psi_n \frac{\psi_k}{\sqrt{k}} - \frac{\psi_k^2}{k}\frac{n}{2} \\ &= \psi_k \left(-\frac{1}{2}\left(1 + \frac{n}{k}\right)\psi_k + \sqrt{\frac{n}{k}}\psi_n \right) \\ &\geq -\frac{\psi_n^2}{2}\left(\sqrt{\frac{n}{k}} - 1\right)^2 \geq -\frac{\psi_n^2}{2}\left(\frac{n}{k} - 1\right)^2 \\ &\geq -\frac{1}{2}\left(\frac{\psi_n}{\psi_n - 1}\right)^2 \geq -\frac{1}{2} - \delta. \end{aligned}$$

For sufficiently large n, we have

$$\psi_n \le \psi^U(n) < \psi^U(2k) = \sqrt{2\ln\ln 2k + 4\ln\ln\ln 2k} < 2\sqrt{2\ln\ln k + 3\ln\ln k} = 2\psi^L(k) \le 2\psi_k.$$

Thus,

$$\begin{aligned}
Z\mathcal{K}_n &\ge \sum_{k=\lfloor n-n/\psi_n\rfloor}^{n} a_k \frac{\psi_k}{k}\exp(-\frac{1}{2}-\delta-\gamma_k^3 n) \\
&\ge a_{\lfloor n-n/\psi_n\rfloor}\frac{\psi_n}{2n}\sum_{k=\lfloor n-n/\psi_n\rfloor}^{n}\exp(-\frac{1}{2}-\delta-\gamma_n^3 n) \\
&\ge a_{\lfloor n-n/\psi_n\rfloor}\frac{\psi_n}{2n}\left(\frac{n}{\psi_n}-1\right)\exp(-\frac{1}{2}-\delta-\gamma_n^3 n) \\
&= a_{\lfloor n-n/\psi_n\rfloor}\left(\frac{1}{2}-\frac{\psi_n}{2n}\right)\exp(-\frac{1}{2}-\delta-\gamma_n^3 n).
\end{aligned}$$

Since $a_{\lfloor n-n/\psi_n\rfloor} \to \infty$, $\psi_n/n \to 0$ and $\gamma_n^3 n \to 0$, we have shown

$$S_n \ge \sqrt{n}\psi_n \ i.o. \ \Rightarrow \ \limsup_{n\to\infty} \mathcal{K}_n = \infty.$$

3 Sharpness

In this section we prove the sharpness (4) of EFKP-LIL in game-theoretic probability. We divide our proof into several subsections.

3.1 Change of time scale

The first key of our proof is a change of time scale from λ to k:

$$\lambda = C^{4k\ln k}.$$

Remark 3.1. Here C is sufficiently large such that (24) below is satisfied. For instance $C = 15$ is good enough. In our proof, given ψ, $C = 15$ and δ, which we have already fixed, we take k to be sufficiently large. Also "4" in $4k\ln k$ is not essential, because we can replace C by C^4.

By taking the derivative of $\ln\lambda = 4k(\ln k)(\ln C)$, we have

$$\frac{d\lambda}{\lambda} = 4(\ln k + 1)dk(\ln C).$$

Hence the integrability condition is written as

$$\int_1^\infty \psi(\lambda)e^{-\psi(\lambda)^2/2}\frac{d\lambda}{\lambda} = \infty \ \Leftrightarrow \ \int_1^\infty (\ln k)\psi(C^{4k\ln k})e^{-\psi(C^{4k\ln k})^2/2}dk = \infty.$$

Let $f(x) = \psi(C^{4x\ln x})e^{-\psi(C^{4x\ln x})^2/2}$. Since $xe^{-x^2/2}$ is decreasing for $x \geq 1$, the function $f(x)$ is decreasing for x such that $\psi(C^{4x\ln x}) \geq 1$. Thus, for sufficiently large k and x such that $k \leq x \leq k+1$, we have

$$\ln x f(x) \geq \ln k f(x+1) \geq \frac{1}{2}\ln(k+1)f(k+1),$$
$$\ln x f(x) \leq \ln(k+1)f(x) \leq 2\ln k f(k).$$

Hence, we have

$$\int_1^\infty (\ln k)\psi(C^{4k\ln k})e^{-\psi(C^{4k\ln k})^2/2}dk = \infty \;\Leftrightarrow\; \sum_{k=1}^\infty (\ln k)\psi(C^{4k\ln k})e^{-\psi(C^{4k\ln k})^2/2} = \infty.$$

Then, it suffices to show that

$$\sum_{k=1}^\infty (\ln k)\psi(C^{4k\ln k})e^{-\psi(C^{4k\ln k})^2/2} = \infty \;\Rightarrow\; \text{Skeptic can force } S_n > \sqrt{n}\psi(n) \; i.o.$$

Recall that we can assume (5) here again.

3.2 Bounding relevant capital processes

In this section we introduce mixtures of constant-proportion betting strategies and bound their capital processes. We discuss relevant capital processes in further subsections.

3.2.1 Uniform mixture of constant-proportion betting strategies

We consider a continuous uniform mixture of constant-proportion strategies with the betting proportion $u\gamma$, $0 \leq u \leq 1$. This is a Bayesian strategy, a similar one to which has been considered in [11].

Define

$$\mathcal{L}_n^\gamma = \int_0^1 \mathcal{K}_n^{u\gamma} du = \int_0^1 \prod_{i=1}^n (1 + u\gamma x_i) du.$$

At round n this strategy bets $M_n = \int_0^1 u\gamma \prod_{i=1}^{n-1}(1 + u\gamma x_i)du$. Then by induction on n, the capital process is indeed written as

$$\mathcal{L}_n^\gamma = \mathcal{L}_{n-1}^\gamma + M_n x_n = \int_0^1 \prod_{i=1}^{n-1}(1 + u\gamma x_i)du + x_n \int_0^1 u\gamma \prod_{i=1}^{n-1}(1 + u\gamma x_i)du$$
$$= \int_0^1 \prod_{i=1}^n (1 + u\gamma x_i)du.$$

Applying (9) and noting $u^3\gamma^3 \leq \gamma^3$, we have

$$e^{-\gamma^3 n}\int_0^1 e^{u\gamma S_n - u^2\gamma^2 n/2}du \leq \mathcal{L}_n^\gamma \leq e^{\gamma^3 n}\int_0^1 e^{u\gamma S_n - u^2\gamma^2 n/2}du.$$

We further bound the integral in the following lemma.

Lemma 3.2. *We have*

$$S_n > 0 \;\Rightarrow\; \mathcal{L}_n^\gamma \le e^{\gamma^3 n} \min\left\{ e^{S_n^2/(2n)} \frac{\sqrt{2\pi}}{\gamma\sqrt{n}}, e^{\gamma S_n} \right\}, \tag{12}$$

$$S_n \le 0 \;\Rightarrow\; \mathcal{L}_n^\gamma \le \min\left\{ 1, e^{\gamma^3 n} \frac{\sqrt{2\pi}}{\gamma\sqrt{n}} \right\}. \tag{13}$$

Proof. Completing the square we have

$$-\frac{1}{2}u^2\gamma^2 n + u\gamma S_n = -\frac{\gamma^2 n}{2}\left(u - \frac{S_n}{\gamma n}\right)^2 + \frac{S_n^2}{2n}.$$

Hence by the change of variables

$$v = \gamma\sqrt{n}\left(u - \frac{S_n}{\gamma n}\right), \qquad du = \frac{dv}{\gamma\sqrt{n}},$$

we obtain

$$\begin{aligned}\int_0^1 e^{u\gamma S_n - u^2\gamma^2 n/2} du &= e^{S_n^2/(2n)} \int_0^1 \exp\left(-\frac{\gamma^2 n}{2}\left(u - \frac{S_n}{\gamma n}\right)^2\right) du \\ &= e^{S_n^2/(2n)} \frac{1}{\gamma\sqrt{n}} \int_{-S_n/\sqrt{n}}^{\gamma\sqrt{n} - S_n/\sqrt{n}} e^{-v^2/2} dv.\end{aligned}$$

Then we can bound $\mathcal{L}_n^\gamma$ from above as

$$\mathcal{L}_n^\gamma \le e^{\gamma^3 n + S_n^2/(2n)} \frac{\sqrt{2\pi}}{\gamma\sqrt{n}}. \tag{14}$$

If $S_n \le 0$, then

$$\int_0^1 e^{u\gamma S_n - u^2\gamma^2 n/2} du \le \int_0^1 e^{-u^2\gamma^2 n/2} du \le \frac{\sqrt{2\pi}}{\gamma\sqrt{n}}.$$

Hence we have

$$S_n \le 0 \;\Rightarrow\; \mathcal{L}_n^\gamma \le e^{\gamma^3 n} \frac{\sqrt{2\pi}}{\gamma\sqrt{n}}.$$

Also by (8), $\mathcal{L}_n^\gamma \le 1$ if $S_n \le 0$. Hence we have (13).

From now we consider the case $S_n > 0$. Without change of variables, we can also bound the integral $\int_0^1 g(u)du$, $g(u) = e^{u\gamma S_n - u^2\gamma^2 n/2}$, directly as

$$\int_0^1 g(u)du \le \max_{0\le u\le 1} g(u).$$

Note that $g(0) = 1, g(1) = e^{\gamma S_n - \gamma^2 n/2}$. By the unimodality of $g(u)$, depending on the sign of $\gamma n - S_n$, we have the following two cases.

Case 1 $S_n \geq \gamma n$. Then $\gamma n - S_n \leq 0$ and $\max_{0\leq u\leq 1} g(u) = g(1)$. Hence

$$\mathcal{L}_n^\gamma \leq e^{\gamma^3 n} e^{\gamma S_n - \gamma^2 n/2}. \tag{15}$$

Case 2 $0 < S_n < \gamma n$. Then $-S_n < 0$, $\gamma n - S_n > 0$, and $\max_{0\leq u\leq 1} g(u) = g(S_n/(\gamma n)) = e^{S_n^2/(2n)}$. Hence

$$\mathcal{L}_n^\gamma \leq e^{\gamma^3 n} e^{S_n^2/(2n)}. \tag{16}$$

Furthermore in this case $S_n^2 < \gamma n S_n$ and $S_n^2/(2n) < \gamma S_n/2$. Therefore we also have

$$\mathcal{L}_n^\gamma \leq e^{\gamma^3 n} e^{\gamma S_n/2}. \tag{17}$$

By (15) and (17) we have

$$\mathcal{L}^{\gamma_n} \leq e^{\gamma^3 n} e^{\gamma S_n}.$$

Combining this with (14) we obtain (12). This proves the lemma. □

We show three more inequalities. For the case $0 < S_n \leq \min\{\sqrt{n}, \gamma n\}$, by the right-hand side of (16), we have

$$0 < S_n \leq \min\{\sqrt{n}, \gamma n\} \ \Rightarrow\ \mathcal{L}_n^\gamma \leq e^{\gamma^3 n} e^{1/2}. \tag{18}$$

Concerning lower bounds for $\mathcal{L}_n^\gamma$, when $\sqrt{n} \leq S_n \leq \gamma n$, we have

$$\sqrt{n} \leq S_n \leq n\gamma \ \Rightarrow\ \mathcal{L}_n^\gamma \geq e^{-\gamma^3 n + S_n^2/(2n)} \frac{1}{\gamma\sqrt{n}} \int_{-1}^{0} e^{-v^2/2} dv \geq e^{-\gamma^3 n + S_n^2/(2n)} \frac{1}{\gamma\sqrt{n}} e^{-1/2}. \tag{19}$$

and when $\gamma\sqrt{n} - S_n/\sqrt{n} > 1$ and $S_n \geq 0$, we have

$$\begin{aligned} &\gamma\sqrt{n} - S_n/\sqrt{n} > 1 \text{ and } S_n \geq 0 \\ &\quad\Rightarrow\ \mathcal{L}_n^\gamma \geq e^{-\gamma^3 n + S_n^2/(2n)} \frac{1}{\gamma\sqrt{n}} \int_0^1 e^{-v^2/2} dv \geq e^{-\gamma^3 n + S_n^2/(2n)} \frac{1}{\gamma\sqrt{n}} e^{-1/2}. \end{aligned} \tag{20}$$

3.2.2 Buying a process and selling two processes

Next we consider the following capital process.

$$\mathcal{Q}_n^\gamma = 3\mathcal{L}_n^\gamma - \mathcal{L}_n^{\gamma/C} - \mathcal{K}_n^{\gamma C}$$

This capital process consists of buying three units of $\mathcal{L}_n^\gamma$ and selling one unit each of $\mathcal{L}_n^{\gamma/C}$ and $\mathcal{K}_n^{\gamma C}$. This combination of selling and buying is essential in the game-theoretic proof of the law of the iterated logarithm in Chapter 5 of [18] and [15].

We want to bound $\mathcal{Q}_n^\gamma$ from above. For easy cases we can just use $\mathcal{Q}_n^\gamma \leq 3\mathcal{L}_n^\gamma$. First by (13) we have

$$S_n \leq 0 \quad\Rightarrow\quad \mathcal{Q}_n^\gamma \leq 3\min\left\{1, e^{\gamma^3 n}\frac{\sqrt{2\pi}}{\gamma\sqrt{n}}\right\}. \tag{21}$$

Also by (18)

$$0 < S_n \leq \min\{\sqrt{n}, \gamma n\} \ \Rightarrow\ \mathcal{Q}_n^\gamma \leq 3e^{\gamma^3 n} e^{1/2}. \tag{22}$$

In other cases we have the following bound from above.

Lemma 3.3. *Suppose that $S_n \geq \min\{\sqrt{n}, \gamma n\}$. There exist C and C_1 such that*

$$Q_n^\gamma \leq \begin{cases} 0, & \text{if} \quad S_n \leq \gamma n/C, \\ 3e^{\gamma^3 n} \min\left\{e^{S_n^2/(2n)} \frac{\sqrt{2\pi}}{\gamma\sqrt{n}}, e^{\gamma S_n}\right\}, & \text{if} \quad \gamma n/C < S_n < C\gamma n, \\ C_1, & \text{if} \quad S_n \geq C\gamma n. \end{cases} \tag{23}$$

Remark 3.4. In this lemma, C and C_1 depend on γ and n through $\gamma^3 n$. However from Section 3.3 on, we take $\gamma^3 n$ to be sufficiently small. Hence C and C_1 can be taken to be constants not depending on γ and n. In particular we can take $C = 15$, as discussed in Remark 3.1. See the discussion at the beginning of Section 3.3.

Proof. We distinguish three cases

$$\text{(i) } S_n \leq \gamma n/C, \quad \text{(ii) } \gamma n/C < S_n < C\gamma n, \quad \text{(iii) } S_n \geq C\gamma n,$$

and bound either $3\mathcal{L}_n^\gamma - \mathcal{L}_n^{\gamma/C}$ or $3\mathcal{L}_n^\gamma - \mathcal{K}_n^{\gamma C}$ from above.

Case (i) For this case $\min\{\sqrt{n}, \gamma n\} \leq S_n \leq \gamma n/C$. Since $C > 1$ and $\gamma n/C < \gamma n$, this case occurs only if $\sqrt{n} \leq \gamma n$. Hence $\sqrt{n} \leq S_n \leq \gamma n/C$. We now consider $3\mathcal{L}_n^\gamma - \mathcal{L}_n^{\gamma/C}$. Then by Lemma 3.2 and (19)

$$\begin{aligned} Q_n^\gamma &\leq 3\mathcal{L}_n^\gamma - \mathcal{L}_n^{\gamma/C} \\ &\leq 3e^{\gamma^3 n + S_n^2/(2n)} \frac{1}{\gamma\sqrt{n}} \sqrt{2\pi} - e^{-\gamma^3 n/C^3 + S_n^2/(2n)} \frac{C}{\gamma\sqrt{n}} e^{-1/2} \\ &= \frac{e^{S_n^2/(2n)}}{\gamma\sqrt{n}} \left(3e^{\gamma^3 n}\sqrt{2\pi} - e^{-\gamma^3 n/C^3} C e^{-1/2}\right). \end{aligned}$$

We check when this is negative. If

$$C \geq 3e^{2\gamma^3 n}\sqrt{2\pi e}, \tag{24}$$

then

$$\begin{aligned} 3e^{\gamma^3 n}\sqrt{2\pi} - e^{-\gamma^3 n/C^3} C e^{-1/2} &\leq 3e^{\gamma^3 n}\sqrt{2\pi} - 3e^{2\gamma^3 n - \gamma^3 n/C^3}\sqrt{2\pi} \\ &\leq 3e^{\gamma^3 n}\sqrt{2\pi} - 3e^{\gamma^3 n}\sqrt{2\pi} = 0. \end{aligned}$$

Hence if $C \geq 3e^{2\gamma^3 n}\sqrt{2\pi e}$ then $Q_n^\gamma \leq 0$ for Case (i).

Case (ii) Again use $Q_n^\gamma \leq 3\mathcal{L}_n^\gamma$ and Lemma 3.2.

Case (iii) We consider $\mathcal{L}_n^\gamma - \mathcal{K}_n^{\gamma C}$. Since $S_n \geq Cn\gamma > n\gamma$, by (15) we have $\mathcal{L}_n^\gamma \leq e^{\gamma^3 n} e^{\gamma S_n - \gamma^2 n/2}$ and

$$Q_n^\gamma \leq 3\mathcal{L}_n^\gamma - \mathcal{K}_n^{\gamma C} \leq 3e^{\gamma^3 n} e^{\gamma S_n - \gamma^2 n/2} - e^{-\gamma^3 C^3 n} e^{\gamma C S_n - \gamma^2 C^2 n/2}$$

$$= 3e^{\gamma^3 n} e^{\gamma S_n - \gamma^2 n/2} \left(1 - \frac{1}{3} e^{-\gamma^3(1+C^3)n} e^{\gamma(C-1)S_n - (C^2-1)\gamma^2 n/2}\right).$$

Hence if the right-hand side is non-positive we have $Q_n^\gamma \leq 0$:

$$S_n \geq Cn\gamma \text{ and } -\gamma^3(1+C^3)n - \ln 3 + \gamma(C-1)S_n - \frac{1}{2}(C^2-1)\gamma^2 n \geq 0$$
$$\Rightarrow \quad Q_n^\gamma \leq 0. \tag{25}$$

Otherwise, write $A = \gamma^3(1+C^3)n + \ln 3$ and consider the case

$$\gamma(C-1)S_n - \frac{1}{2}(C^2-1)\gamma^2 n \leq A.$$

Dividing this by $C-1$ and also considering $S_n \geq Cn\gamma$, we have

$$\gamma S_n - \frac{1}{2}(C+1)\gamma^2 n \leq \frac{A}{C-1}, \tag{26}$$
$$-S_n + Cn\gamma \leq 0. \tag{27}$$

$\gamma \times (27) + (26)$ gives

$$\frac{1}{2}(C-1)\gamma^2 n \leq \frac{A}{C-1} \quad \text{or} \quad \frac{1}{2}\gamma^2 n \leq \frac{A}{(C-1)^2}.$$

Then by (26)

$$\gamma S_n - \frac{1}{2}\gamma^2 n \leq \frac{A}{C-1} + \frac{C}{2}\gamma^2 n \leq \frac{A}{C-1} + \frac{CA}{(C-1)^2} = \frac{(2C-1)A}{(C-1)^2}.$$

Hence just using $Q_n^\gamma \leq 3\mathcal{L}_n^\gamma$ and (15) in this case, we obtain

$$Q_n^\gamma \leq 3e^{\gamma^3 n} \exp\left(\frac{(2C-1)(\gamma^3(1+C^3)n + \ln 3)}{(C-1)^2}\right). \tag{28}$$

Since the right-hand side is positive, it also covers (25). Hence we have (28) for the whole case (iii).

Finally, take C to satisfy (24). Then by (22) and (28), let

$$C_1 = 3e^{\gamma^3 n} \max\left\{e^{1/2}, \exp\left(\frac{(2C-1)(\gamma^3(1+C^3)n + \ln 3)}{(C-1)^2}\right)\right\}. \tag{29}$$

□

3.2.3 Further continuous mixture of processes

We finally introduce another continuous mixture of capital processes. Define a capital process

$$\mathcal{M}_n^{\gamma,k} = \frac{1}{\ln k}\int_0^{\ln k} Q_n^{\gamma C^{-w}} dw = \frac{1}{\ln k}\int_0^{\ln k} (3\mathcal{L}_n^{\gamma C^{-w}} - \mathcal{L}_n^{\gamma C^{-w-1}} - \mathcal{K}_n^{\gamma C^{-w+1}})dw. \tag{30}$$

For example $(1/\ln k)\int_0^{\ln k} \mathcal{L}_n^{\gamma C^{-w}} dw$ is the capital process for the strategy betting

$$M_n = \frac{1}{\ln k}\int_0^{\ln k}\int_0^1 uC^{-w}\gamma \prod_{i=1}^{n-1}(1 + uC^{-w}\gamma x_i)\, du\, dw$$

at round n. Under the same notation as in Lemma 3.3, for the case $S_n > 0$, we can bound $\mathcal{M}_n^{\gamma,k}$ from above as follows:

$$\mathcal{M}_n^{\gamma,k} \le C_1 + \frac{2}{\ln k}\max_{\gamma'\in[\gamma C^{-\ln k},\gamma]}\left(3e^{\gamma'^3 n}\min\{e^{S_n^2/(2n)}\frac{\sqrt{2\pi}}{\gamma'\sqrt{n}}, e^{\gamma' S_n}\}\right), \tag{31}$$

because the length of the interval

$$\left\{w \mid \frac{S_n}{nC} < \gamma C^{-w} < \frac{S_n C}{n}\right\}$$

is equal to 2.

If $S_n \le 0$, integrating (21), we have

$$\begin{aligned}
\mathcal{M}_n^{\gamma,k} &\le \frac{1}{\ln k}\int_0^{\ln k} 3\min\left\{1, e^{\gamma^3 C^{-3w} n}\frac{\sqrt{2\pi}}{\gamma C^{-w}\sqrt{n}}\right\} dw \\
&\le \max_{w\in[0,\ln k]}\min\left\{3, e^{\gamma^3 C^{-3w} n}\frac{3\sqrt{2\pi}}{\gamma C^{-w}\sqrt{n}}\right\} \\
&\le \min\left\{3, \frac{3\sqrt{2\pi}}{\gamma\sqrt{n}}e^{\gamma^3 n}C^{\ln k}\right\}.
\end{aligned}$$

Hence

$$S_n \le 0 \;\Rightarrow\; \mathcal{M}_n^{\gamma,k} \le \min\left\{3, \frac{3\sqrt{2\pi}}{\gamma\sqrt{n}}e^{\gamma^3 n}C^{\ln k}\right\}. \tag{32}$$

3.3 Dynamic strategy forcing the sharpness

Write

$$n_k = \left\lceil C^{4k\ln k}\right\rceil,\ \psi_k = \psi(n_k).$$

Note that ψ_k here is different from ψ_k in Section 2. As in Chapter 5 of [18] and [15], we divide the time axis into "cycles" $[n_k, n_{k+1}]$, $k \ge 1$. Betting strategy for the k-th cycle is based on the following betting proportion:

$$\gamma_k = \frac{\psi_{k+1}}{\sqrt{C^{\ln k} n_k}}.$$

As a preliminary consideration we check the relation between γ_k and C and the growth of ψ_k and γ_k.

First we confirm that we can take $C = 15$ to satisfy (24). Consider $\gamma_k^3(1+C^3)n_{k+1}$. By $\psi(n) \le \psi^U(n)$ and for sufficiently large k such that $\ln k/\ln(k+1) \ge 1-\delta$

$$\begin{aligned}\gamma_k^3(1+C^3)n_{k+1} &\le \frac{(2\ln\ln C^{4(k+1)\ln(k+1)} + 4\ln\ln\ln C^{4(k+1)\ln(k+1)})^{3/2}}{(C^{\ln k}C^{4k\ln k})^{3/2}}(1+C^3)C^{4(k+1)\ln(k+1)} \\ &= \frac{(2\ln\ln C^{4(k+1)\ln(k+1)} + 4\ln\ln\ln C^{4(k+1)\ln(k+1)})^{3/2}}{C^{\left\{(2k+\frac{1}{2})(3\frac{\ln k}{\ln(k+1)}-2)-3\right\}\ln(k+1)}}(1+C^3) \\ &\le \frac{(2\ln\ln C^{4(k+1)\ln(k+1)} + 4\ln\ln\ln C^{4(k+1)\ln(k+1)})^{3/2}}{C^{\left\{(2k+\frac{1}{2})(1-3\delta)-3\right\}\ln(k+1)}}(1+C^3).\end{aligned}$$

With $C = 15$, for sufficiently large k we have

$$\gamma_k^3(1+C^3)n_{k+1} \le (1+15^3)\frac{(2\ln\ln 15^{4(k+1)\ln(k+1)} + 4\ln\ln\ln 15^{4(k+1)\ln(k+1)})^{3/2}}{15^{\left\{(2k+\frac{1}{2})(1-3\delta)-3\right\}\ln(k+1)}} \le \delta.$$

Hence we choose k_0 such that $\gamma_k^3(1+C^3)n_{k+1} \le \delta$ for $k \ge k_0$. Then in our formulas in the previous section, in the k-th cycle, we have

$$e^{\gamma_k^3 n} \le e^{\gamma_k^3 n_{k+1}} \le e^{\delta}.$$

Also with $\delta < 0.01$ and $C = 15$, (24) is satisfied. Furthermore C_1 in (29) is bounded as

$$C_1 \le 3e^{\delta}\max\left\{e^{1/2}, \exp\left(\frac{29(\delta+\ln 3)}{14^2}\right)\right\} = 3e^{\delta+1/2}.$$

Now we check the growth of ψ_k and γ_k. Note that all of $\psi^U(C^{4k\ln k})$, $\psi^U(C^{4(k+1)\ln(k+1)})$, $\psi^L(C^{4k\ln k})$, $\psi^L(C^{4(k+1)\ln(k+1)})$ are of the order

$$\sqrt{2\ln\ln C^{4k\ln k}}(1+o(1)) = \sqrt{2\ln k}(1+o(1))$$

as $k\to\infty$. In particular by (5),

$$\lim_{k\to\infty}\frac{\psi^U(n_k)}{\psi_{k+1}} = 1. \tag{33}$$

Hence given $C = 15$, we choose k sufficiently large such that $\psi^U(n_k)/\psi_{k+1} \le 1+\delta$. We increase k_0 if needed. Furthermore, as $k\to\infty$

$$\gamma_k\sqrt{n_k}\psi_{k+1} = \frac{\psi_{k+1}^2}{\sqrt{C^{\ln k}}} \to 0. \tag{34}$$

For each cycle $[n_k, n_{k+1}]$, $k \ge k_0$, we apply the following capital process to x_n's in the cycle.

$$\mathcal{N}_n^{\gamma_k} = 1 + \frac{1}{D}(\ln k)\psi_{k+1}e^{-\psi_{k+1}^2/2}(1-\mathcal{M}_{n-n_k}^{\gamma_k,k}). \tag{35}$$

Since the strategy for $\mathcal{M}^{\gamma_k,k}_{n-n_k}$ is applied only to x_n's in the cycle, $1 = \mathcal{N}^{\gamma_k}_{n_k} = \mathcal{M}^{\gamma_k}_0$. Concerning $\mathcal{N}^{\gamma_k}_n$ we prove the following Proposition 3.5.

Recall that we have a fixed $\delta < 0.01$, $C = 15$, and chose k_0 such that $\gamma_k^3(1 + C^3)n_{k+1} \le \delta$ and $\psi^U(n_k)/\psi_{k+1} \le 1 + \delta$ for $k \ge k_0$. Now, we set $D = 12\sqrt{2\pi}/(1-\delta)$ in (35) and increase k_0 to satisfy all of (39)–(45), if needed.

Proposition 3.5. *Suppose that*

$$-\sqrt{n}\psi^U(n) \le S_n \le \sqrt{n}\psi(n), \qquad \forall n \in [n_k, n_{k+1}]. \tag{36}$$

Then there exists k_0 such that for all $k \ge k_0$

$$\mathcal{N}^{\gamma_k}_n > \delta^2, \ \forall n \in [n_k, n_{k+1}], \tag{37}$$

and

$$\mathcal{N}^{\gamma_k}_{n_{k+1}} \ge 1 + \delta(\ln k)\psi_{k+1}e^{-\psi^2_{k+1}/2}. \tag{38}$$

Proof. In our proof, for notational simplicity we write n instead of $n - n_k$.

For proving (37), we use (31) for $S_n - S_{n_k}$. We want to bound $\mathcal{M}^{\gamma_k,k}_n$ from above. By the term $\dfrac{2}{\ln k}$ on the right-hand side of (31), it suffices to show the following:

For sufficiently large k such that

$$\frac{C_1}{D}(\ln k)\psi_{k+1}e^{-\psi^2_{k+1}/2} \le \frac{1-\delta^2}{2}, \tag{39}$$

$$\begin{aligned} &S_n \le \sqrt{n_k}\psi^U(n_k) + \sqrt{n_k + n}\psi(n_k + n) \\ &\quad \Rightarrow \psi_{k+1}e^{-\psi^2_{k+1}/2} \max_{\gamma \in [\gamma_k C^{-\ln k}, \gamma_k]} \left(3e^{\gamma^3 n} \min\{e^{S_n^2/(2n)}\frac{\sqrt{2\pi}}{\gamma\sqrt{n}}, e^{\gamma S_n}\}\right) < \frac{D}{4}(1-\delta^2), \forall n \in [0, n_{k+1} - n_k]. \end{aligned}$$

Note that $(\ln k)\psi_{k+1}e^{-\psi^2_{k+1}/2} \to 0$ as $k \to \infty$. Hence the case $S_n \le 0$ is trivial because $\mathcal{M}^{\gamma_k,k}_n$ is bounded from above by (32). Otherwise, we prove

$$\begin{aligned} 0 < S_n &\le \sqrt{n_k}\psi^U(n_k) + \sqrt{n}\psi(n_k + n) \\ &\Rightarrow \psi_{k+1}e^{-\psi^2_{k+1}/2} \times 3e^{\gamma^3 n}\min\{e^{S_n^2/(2n)}\frac{\sqrt{2\pi}}{\gamma\sqrt{n}}, e^{\gamma S_n}\} \le \frac{D}{4}(1-\delta^2), \\ &\qquad \forall n \in [0, n_{k+1} - n_k], \forall \gamma \in [\gamma_k C^{-\ln k}, \gamma_k]. \end{aligned}$$

Let $c_1 > 4/(1 + 2\delta)^2$, so that

$$\frac{1}{2} - \frac{1}{\sqrt{c_1}} - \delta > 0.$$

We distinguish two cases:

$$\text{(a) } n \le \frac{\psi^2_{k+1}}{c_1\gamma^2}, \quad \text{(b) } \frac{\psi^2_{k+1}}{c_1\gamma^2} < n \le n_{k+1} - n_k.$$

For case (a), $\sqrt{n_k}\psi^U(n_k) \le (1+\delta)\sqrt{n_k}\psi_{k+1}$ by (33) for sufficiently large k. Also $\psi(n_k+n) \le \psi(n_{k+1})$. Hence in this case

$$S_n \le \left((1+\delta)\sqrt{n_k} + \sqrt{n_k + \psi_{k+1}^2/(c_1\gamma^2)}\right)\psi_{k+1}$$

and

$$\gamma S_n \le \left((1+\delta)\gamma\sqrt{n_k} + \sqrt{\gamma^2 + \psi_{k+1}^2/c_1}\right)\psi_{k+1}.$$

Then for $\gamma \le \gamma_k$, by (34)

$$\gamma S_n \le \left((1+\delta)\gamma_k\sqrt{n_k} + \sqrt{\gamma_k^2 + \psi_{k+1}^2/c_1}\right)\psi_{k+1} = \frac{\psi_{k+1}^2}{\sqrt{c_1}}(1+\delta) \tag{40}$$

for sufficiently large k. Noting that for sufficiently large k

$$\psi_{k+1}e^{-\psi_{k+1}^2/2}3e^{\gamma^3 n}e^{\gamma S_n} \le \psi_{k+1}\exp\left(-\psi_{k+1}^2(\frac{1}{2} - \frac{1}{\sqrt{c_1}} - \delta)\right)3e^{\gamma^3 n} < \frac{D}{4}(1-\delta^2) \tag{41}$$

we have $\mathcal{N}_n^{\gamma_k} > (1-\delta)^2/2$ uniformly in $\gamma \in [\gamma_k C^{-\ln k}, \gamma_k]$ and in $n \le \psi_{k+1}^2/c_1\gamma^2$.

For case (b), $\psi_{k+1}/\sqrt{c_1} \le \gamma\sqrt{n}$ and $S_n \le ((1+\delta)\sqrt{n_k} + \sqrt{n_k+n})\psi_{k+1}$. Hence

$$\begin{aligned}
\psi_{k+1}e^{-\psi_{k+1}^2/2} \times 3e^{\gamma^3 n}e^{S_n^2/(2n)}\frac{\sqrt{2\pi}}{\gamma\sqrt{n}} &\le \psi_{k+1}e^{-\psi_{k+1}^2/2} \times \frac{3e^\delta\sqrt{2\pi}\sqrt{c_1}}{\psi_{k+1}}\exp\left(\frac{((1+\delta)\sqrt{n_k} + \sqrt{n_k+n})^2}{2n}\psi_{k+1}^2\right) \\
&= 3e^\delta\sqrt{2\pi}\sqrt{c_1}\exp\left(\frac{(1+(1+\delta)^2)n_k + 2(1+\delta)\sqrt{n_k}\sqrt{n_k+n}}{2n}\psi_{k+1}^2\right) \\
&= 3e^\delta\sqrt{2\pi}\exp\left((\frac{(1+(1+\delta)^2)n_k}{2n} + (1+\delta)\sqrt{\frac{n_k}{n}}\sqrt{1+\frac{n_k}{n}})\psi_{k+1}^2\right).
\end{aligned}$$

For $\gamma \le \gamma_k$

$$\frac{\psi_{k+1}^2}{c_1\gamma^2} < n \Rightarrow \frac{n_k}{n} < \frac{c_1\gamma^2 n_k}{\psi_{k+1}^2} \le \frac{c_1\gamma_k^2 n_k}{\psi_{k+1}^2}.$$

Hence $n_k/n \to 0$ as $k \to \infty$. Also by (34), as $k \to \infty$

$$\sqrt{\frac{n_k}{n}}\psi_{k+1}^2 \le \sqrt{c_1}\gamma_k\sqrt{n_k}\psi_{k+1} \le \frac{3\sqrt{c_1}\ln\ln C^{4(k+1)\ln(k+1)}}{\sqrt{C^{\ln k}}} \to 0.$$

Therefore, for sufficiently large k,

$$\begin{aligned}
\psi_{k+1}e^{-\psi_{k+1}^2/2} \times 3e^{\gamma^3 n}e^{S_n^2/(2n)}\frac{\sqrt{2\pi}}{\gamma\sqrt{n}} &\le 3e^\delta\sqrt{2\pi}\exp\left(\frac{1+(1+\delta)^2}{2}(\frac{n_k}{n} + \sqrt{\frac{n_k}{n}}\sqrt{1+\frac{n_k}{n}})\psi_{k+1}^2\right) \\
&< \frac{D}{4}(1-\delta^2).
\end{aligned} \tag{42}$$

Hence the left-hand side is bounded uniformly in $\gamma \le \gamma_k$ and $n \in [n_k/\gamma^2, n_{k+1} - n_k]$. This proves (37).

Now we prove (38). Write $n_k^* = n_{k+1} - n_k$. We bound $\mathcal{M}_{n_k^*}^{\gamma_k,k}$ from above.

We first consider the case $S_{n_k^*} \leq 0$. Note that

$$n_k^* = \left\lceil C^{4(k+1)\ln(k+1)} \right\rceil - \left\lceil C^{4k\ln k} \right\rceil \geq C^{4k\ln k}(C^{4\ln(k+1)} - 1) - 2 \geq n_k(C^{4\ln(k+1)} - 1) - 2.$$

Then the product $\gamma_k C^{-\ln k}\sqrt{n_k^*}$ is bounded from below as

$$\begin{aligned}\gamma_k C^{-\ln k}\sqrt{n_k^*} &\geq \frac{\psi_{k+1}}{\sqrt{C^{\ln k} n_k}} C^{-\ln k}\sqrt{n_k(C^{4\ln(k+1)} - 1) - 2} \\ &\geq \psi_{k+1}\sqrt{C^{\ln(k+1)} - C^{-3\ln(k+1)}\left(1 + \frac{2}{n_k}\right)}.\end{aligned}$$

The right-hand side diverges to $+\infty$ as $k \to \infty$. Then by (32), for sufficiently large k we have

$$S_{n_k^*} \leq 0 \quad \Rightarrow \quad \mathcal{M}_{n_k^*}^{\gamma_k,k} \leq e^{\gamma_k^3 n_k^*} \frac{3\sqrt{2\pi}}{\gamma_k C^{-\ln k}\sqrt{n_k^*}} \leq 1 - D\delta. \tag{43}$$

Now consider the case $S_{n_k^*} > 0$. In this case we show that $\mathcal{M}_{n_k^*}^{\gamma_k,k} \leq 0$ for sufficiently large k. Consider the integrand $Q_{n_k^*}^{\gamma_k C^{-w}}$ in the (30). For $\gamma \in [\gamma_k C^{-\ln k}, \gamma_k]$, by

$$\begin{aligned}\frac{n_{k+1}}{n_k} &= \frac{C^{(k+1)\ln(k+1)}}{C^{k\ln k}} \\ &\geq C^{4\ln(k+1)} \geq C^{4\ln k}\end{aligned}$$

we have

$$\begin{aligned}\gamma\sqrt{n_k^*}/C - S_{n_k^*}/\sqrt{n_k^*} &\geq \gamma_k C^{-\ln k}\sqrt{n_k^*}/C - S_{n_k^*}/\sqrt{n_k^*} \\ &\geq \psi_{k+1}\frac{1}{C}\sqrt{\frac{n_{k+1} - n_k}{C^{3\ln k} n_k}} - \psi_{k+1}\sqrt{\frac{n_{k+1}}{n_{k+1} - n_k}} \\ &\geq \psi_{k+1}\left(\frac{1}{C}\sqrt{\frac{n_{k+1}}{C^{3\ln k} n_k} - C^{-3\ln k}} - \sqrt{\frac{1}{1 - \frac{n_k}{n_{k+1}}}}\right) \\ &\geq \psi_{k+1}\left(\frac{1}{C}\sqrt{C^{\ln k} - C^{-3\ln k}} - \sqrt{\frac{1}{1 - C^{-4\ln k}}}\right).\end{aligned}$$

The right-hand side diverges to $+\infty$ as $k \to \infty$. Hence for sufficiently large k

$$\gamma\sqrt{n_k^*}/C - S_{n_k^*}/\sqrt{n_k^*} \geq 1 \tag{44}$$

for every $\gamma \in [\gamma_k C^{-\ln k}, \gamma_k]$. Now by (20), for $\gamma \in [\gamma_k C^{-\ln k}, \gamma_k]$, we have

$$Q_{n_k^*}^{\gamma} \leq 3\mathcal{L}_{n_k^*}^{\gamma} - \mathcal{L}_{n_k^*}^{\gamma/C}$$

$$\leq 3e^{\gamma^3 n_k^* + S_{n_k^*}^2/(2n_k^*)} \frac{\sqrt{2\pi}}{\gamma_k \sqrt{n_k^*}} - e^{-\gamma^3 n_k^*/C^3 + S_{n_k^*}^2/(2n_k^*)} \frac{C}{\gamma_k \sqrt{n_k^*}} e^{-1/2}$$

$$\leq \frac{e^{S_{n_k^*}^2/(2n_k^*)}}{\gamma_k \sqrt{n_k^*}} \left(3\sqrt{2\pi} e^{\gamma^3 n_k^*} - C e^{-\gamma^3 n_k^*/C^3} e^{-1/2}\right).$$

For sufficiently large k this is negative because, with $C = 15$,

$$3\sqrt{2\pi} e^{\gamma^3 n_k^*} - C e^{-\gamma^3 n_k^*/C^3} e^{-1/2} < 0 \ \Leftrightarrow\ 3\sqrt{2\pi} e^{1/2} e^{\gamma^3 n_k^* + \gamma^3 n_k^*/C^3} < C. \tag{45}$$

Hence $\mathcal{M}_{n_k^*}^{\gamma_k, k} \leq 0$. This proves (38). □

We assume that by the validity result, Skeptic already employs a strategy forcing $S_n \geq -\sqrt{n}\psi^U(n)$ *a.a.* In addition to this strategy, based on Proposition 3.5, consider the following strategy.

Start with initial capital $\mathcal{K}_0 = 1$.
Set $k = k_0$.
Do the followings repeatedly:
1. Apply the strategy in Proposition 3.5 for $n \in [n_k, n_{k+1}]$.
   If (36) holds, then go to 2. Otherwise go to 3.
2. Let $k = k + 1$. Go to 1.
3. Wait until $\exists k'$ such that $-\sqrt{n_{k'}}\psi^U(n_{k'}) \leq S_{n_{k'}} \leq \sqrt{n_{k'}}\psi(n_{k'})$.
   Set $k = k'$ and go to 1.

Since Skeptic already employs a strategy forcing $S_n \geq -\sqrt{n}\psi^U(n)$ *a.a.*, the lower bound in (36) violated only finite number of times. Hence if $S_n \leq \sqrt{n}\psi(n)$ *a.a.*, then Step 3 is performed only finite number of times. Also when Step 3 is performed, the overshoot of $|x_n| = 1$ does not make Skeptic bankrupt by (37). Now for each iteration of Step 2, Skeptic multiplies his capital at least by

$$1 + \delta(\ln k)\psi_{k+1} e^{-\psi_{k+1}^2/2}.$$

Then

$$\delta \sum_{k=k_0}^{\infty} (\ln k)\psi_{k+1} e^{-\psi_{k+1}^2/2} \leq \prod_{k=k_0}^{\infty} \left(1 + \delta(\ln k)\psi_{k+1} e^{-\psi_{k+1}^2/2}\right).$$

Since the left-hand side diverges to infinity, the above strategy forces the sharpness.

4 Discussion

In this paper we gave a game-theoretic proof of EFKP-LIL. Our validity proof is very short. Our sharpness proof is elementary, but it is still long. A simpler sharpness proof is desired.

In our sharpness proof we used the change of time scale $\lambda = C^{4k \ln k}$ and formed the cycles $[n_k, n_{k+1}]$ based on this time scale. There is a question whether this scale is the best. Actually we can prove the sharpness based on the change of time scale $\lambda = C^{4k \ln\ln k}$. Any sparser cycles than $n_k = C^{4k \ln\ln k}$ can be used for proving the sharpness.

It is interesting to consider a generalization of EFKP-LIL to games other than the fair-coin game. In particular the case of self-normalized sums discussed in [7, 3] is also important from game-theoretic viewpoint. Self-normalized sums in game-theoretic probability have been studied in [12].

Another possible extension is that $\psi(n)$ is sequentially given by a third player Forecaster at the beginning of each round of the game. From the game-theoretic viewpoint it is of interest to ask whether Skeptic can force

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n} e^{-\psi(n)^2/2} = \infty \quad \Leftrightarrow \quad S_n \geq \sqrt{n}\psi(n) \ \ i.o.$$

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