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Total dual integrality of the linear complementarity problem

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Abstract

In this paper, we introduce total dual integrality of the linear complementarity problem by analogy with the linear programming problem. The main idea of defining the notion is to propose the LCP with orientation, a variant of the LCP whose feasible complementary cones are specified by an additional input vector. This allows us to define naturally its dual problem and the total dual integrality of the LCP. We show that if the LCP is totally dual integral, then all basic solutions are integral. If the input matrix is sufficient or rank-symmetric, then this implies that the LCP always has an integral solution whenever it has a solution.

1 Introduction

Given a matrix $M \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, the *linear complementarity problem* (LCP) is to find a vector $z \in \mathbb{R}^n$ such that

$$Mz + q \geq 0, \quad z \geq 0, \quad z^\top(Mz + q) = 0. \tag{1}$$

We denote a problem instance of the LCP with M and q by LCP (M, q). We say that n is the *order* of LCP (M, q). The LCP, introduced by Cottle [5], Cottle and Dantzig [6], and Lemke [16], is one of the most widely studied mathematical programming problems, which, for example, contains linear and convex quadratic programming problems. The decision version of the LCP (i.e., deciding whether (1) has a solution $z \in \mathbb{R}^n$) is NP-complete [4]. For details of the LCP and related topics, see the books of Cottle, Pang, and Stone [7] and Murty [17].

In this paper, we focus on integral solutions to the LCP. Integral solutions to the LCP were first considered by Chandrasekaran [2] in the context of the least element theory. A class of the LCP having integral solutions was considered earlier by Pardalos and Nagurney [18], with some applications which need integral solutions. For example, the problem of finding a market equilibrium can

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be represented as the LCP [20], and its integral solution corresponds to an integral equilibrium. Another example is the polymatrix game in game theory. Computing a Nash equilibrium in the polymatrix game can be reduced to the LCP [14], and, if it has an integral solution, then the game has a pure-strategy Nash equilibrium.

We say that a square integral matrix M is *principally unimodular* if all principal submatrices have determinants 0 or ± 1. Principal unimodularity, introduced by Bouchet [1], arises as a generalization of total unimodularity. As total unimodularity characterizes integrality in the linear programming (see e.g., [19]), principal unimodularity is related to integrality in the LCP. Chandrasekaran, Kabadi and Sridhar [3], and Cunningham and Geelen [9] independently showed that a matrix M is principally unimodular if and only if all *basic* solutions in LCP (M, q) are integral for every integral vector q. Note that a basic solution does not always exist even if a solution exists. However, when a matrix M belongs to some special classes such as *column sufficient* matrices or *rank-symmetric* matrices, principal unimodularity of M guarantees that LCP (M, q) has an integral solution [3, 9].

In this paper, we introduce the notion of *total dual integrality* of the LCP in order to discuss a wider class of the LCP having an integral solution. Recall that total dual integrality of linear systems, introduced by Edmonds and Giles [11], gives a unified framework for linear programming problems having an integral optimal solution arising in combinatorial optimization. It is known that an integral matrix A is totally unimodular if and only if the linear system $Ax \leq b,\ x \geq 0$ is totally dual integral for every vector b. As this notion is defined using the LP dual problem, toward defining the total dual integrality of the LCP, we need the LCP dual problem. We remark that a dual problem of the LCP was introduced by Fukuda and Terlaky [12]. They provided a theorem of the alternative: exactly one of primal and dual LCP problems has a solution. However, their duality of the LCP is not suitable to define total dual integrality. In this paper, we introduce the *LCP with orientation*: the problem of finding a solution to the LCP whose feasible complementary cones are specified by an additional input vector. Then we can naturally define its dual problem and total dual integrality of the LCP by analogy with that of linear systems.

Our main result is to show that total dual integrality of the LCP implies that any basic solution is integral. When a matrix M is sufficient or rank-symmetric, we also show that total dual integrality certifies integrality of the LCP in the sense that, for any solution z, there exists an integral solution with basis identical to z. In that case, our results imply that M is principally unimodular if and only if LCP (M, q) is totally dual integral for every integral vector q. This gives an alternative proof of previous works [3, 9] as mentioned above. In addition, we reveal the computational complexity of testing the total dual integrality of a given LCP instance. We show that it is coNP-hard to decide if a given LCP instance is totally dual integral.

The rest of the paper is organized as follows. Section 2 defines the linear programming and complementarity problems, and fix notation needed in the subsequent sections. Section 3 proposes the LCP with orientation and total dual integrality of the LCP. Section 4 discusses integrality of the LCP by using the total dual integrality. In Section 5, we characterize integrality of the LCP in terms of principal unimodularity and total dual integrality for sufficient and rank-symmetric matrices. Section 6 shows that it is intractable to recognize total dual integrality of the LCP. Finally, Section 7 clarifies the relationship of matrix classes appearing in related works and this paper.

2 Preliminaries

In this section, we define the linear programming and complementarity problems, and review existing results on them.

For a positive integer n, let $[n] = \{1, \dots, n\}$. Let A be an $m \times n$ real matrix, where A has a row index set $[m]$ and a column index set $[n]$. For $S \subseteq [m]$ and $T \subseteq [n]$, we denote by A_{ST} the submatrix of A such that S and T are row and column index sets, respectively. We also define $A_{\cdot T}$ and $A_{S\cdot}$ by $A_{\cdot T} = A_{[m]T}$ and $A_{S\cdot} = A_{S[n]}$, respectively. If $T = \{j\}$, we simply write A_{Sj} and $A_{\cdot j}$ instead of $A_{S\{j\}}$ and $A_{\cdot\{j\}}$, respectively. We similarly define A_{iT} and $A_{i\cdot}$. Let z denote a vector in $\mathbb{R}^n$ with index set $[n]$. In this paper, we assume that all vectors are column. For index set $B \subseteq [n]$, let z_B denote the subvector of z with elements corresponding to B, i.e., z_B in $\mathbb{R}^B$. For i in $[n]$, we also denote by z_i the ith element of z.

2.1 Linear systems and linear programming problems

Let $Ax \le b$ be a system of linear inequalities, and P be a polyhedron defined by $Ax \le b$, i.e., $P = \{x \mid Ax \le b\}$. The affine hyperplane $\{x \mid c^\top x = \delta\}$ is called a *supporting hyperplane* of P if c is nonzero and $\delta = \max\{c^\top x \mid x \in P\}$. A subset F of P is called a *face* of P if $F = P$ or F is the intersection of P with some supporting hyperplane of P. Alternatively, F is a face of P if and only if F is nonempty and $F = P \cap \{x \mid A'x = b'\}$ for some subsystem $A'x \le b'$ of $Ax \le b$. In particular, a face is *minimal* if it contains no other face. It is known that a minimal face of P can be represented as $\{x \mid A'x = b'\}$ for some subsystem $A'x \le b'$ of $Ax \le b$. A zero-dimensional face is called a *vertex*.

A polyhedron P is called *integral* if P is the convex hull of the integral vectors in P. A polyhedron P is integral if and only if any face of P contains an integral vector, or equivalently, $\max\{c^\top x \mid x \in P\}$ has an integral optimal solution for each vector c such that the maximum exists.

Recall that a matrix is *totally unimodular* if all square submatrices have determinants 0 or ± 1. Totally unimodular matrices are well studied, since they characterize integral polyhedra.

Theorem 2.1 ([13])**.** *Let A be an integral matrix. Then A is totally unimodular if and only if the polyhedron $\{x \mid Ax \le b,\ x \ge 0\}$ is integral for each integral vector b.*

Total dual integrality is a weaker concept than total unimodularity for integral polyhedra.

Definition 2.2. *Let A be a rational matrix, and let b be a rational vector. A system $Ax \le b$ is* totally dual integral *if the dual problem of* $\max\{c^\top x \mid Ax \le b\}$*, that is,*

$$\min\{b^\top y \mid A^\top y = c,\ y \ge 0\} \tag{2}$$

has an integral optimal solution for each integral vector c such that (2) *is finite.*

Theorem 2.3 ([11])**.** *Let A be a rational matrix, and let b be an integral vector. If $Ax \le b$ is a totally dual integral system, then the polyhedron $\{x \mid Ax \le b\}$ is integral.*

It is known that if a matrix A is totally unimodular, then a linear system $Ax \le b$ is totally dual integral for every vector b. Hence, we can restate Theorem 2.1 as follows.

Corollary 2.4 ([13])**.** *Let A be an integral matrix. Then A is totally unimodular if and only if the system $Ax \leq b,\ x \geq 0$ is totally dual integral for each vector b.*

2.2 Linear complementarity problems

For a subset $B \subseteq [n]$, let $\overline{B}$ denote the complement of B, i.e., $\overline{B} = [n] \setminus B$. For an $n \times n$ real matrix M (with row and column index sets $[n]$) and index set $B \subseteq [n]$, define an $n \times n$ matrix $C_M(B)$ by

$$C_M(B)_{\cdot i} = \begin{cases} -M_{\cdot i} & \text{if } \ i \in B, \\ I_{\cdot i} & \text{if } \ i \notin B, \end{cases}$$

where I is the identity matrix. For a matrix A, let $\operatorname{pos} A$ denote a positive cone spanned by column vectors of A, i.e., $\operatorname{pos} A = \{Ax \mid x \geq 0\}$. $\operatorname{pos} C_M(B)$ is called the *complementary cone* of B relative to M. We say that a solution z to LCP (M, q) has a *basis* $B \subseteq [n]$ if it holds that $(Mz + q)_B = 0$ and $z_{\overline{B}} = 0$. Note that a vector q in $\mathbb{R}^n$ is contained in $\operatorname{pos} C_M(B)$ for some $B \subseteq [n]$ if and only if LCP (M, q) has a solution z with basis B. To see this, suppose that q is in $\operatorname{pos} C_M(B)$, i.e., there exists a vector x in $\mathbb{R}^n$ with $C_M(B)x = q$ and $x \geq 0$. Then a vector z defined by $z_B = x_B$ and $z_{\overline{B}} = 0$ is a solution to LCP (M, q), since it holds that $z \geq 0, z_{\overline{B}} = 0, (Mz+q)_B = M_{BB}x_B + q_B = 0$, and $(Mz + q)_{\overline{B}} = M_{\overline{B}B}x_B + q_{\overline{B}} \geq 0$. On the other hand, if LCP (M, q) has a solution z with basis B, then a vector x defined by $x_B = z_B$ and $x_{\overline{B}} = M_{\overline{B}B}z_B + q_{\overline{B}}$ is contained in $\operatorname{pos} C_M(B)$. Let $K(M) = \bigcup_{B \subseteq [n]} \operatorname{pos} C_M(B)$. We define two polyhedra associated with LCP (M, q) by

$$\begin{aligned} P(M, q) &= \{z \mid Mz + q \geq 0,\ z \geq 0\}, \text{ and} \\ P_B(M, q) &= \{z \in P(M, q) \mid (Mz + q)_B = 0,\ z_{\overline{B}} = 0\} \text{ for } B \subseteq [n]. \end{aligned}$$

Note that $P_B(M, q)$ is a face of $P(M, q)$. $\bigcup_{B \subseteq [n]} P_B(M, q)$ represents the set of solutions to LCP (M, q). Thus LCP (M, q) is equivalent to finding a nonempty set $P_B(M, q)$ for some B.

A solution z to LCP (M, q) is called a *basic solution* (*with respect to* B) if z is of the form

$$z_B = -M_{BB}^{-1} q_B, \quad z_{\overline{B}} = 0. \tag{3}$$

Let us notice that LCP (M, q) has a basic solution with respect to B if and only if $q \in \operatorname{pos} C_M(B)$ and M_{BB} is nonsingular. Moreover, this is equivalent to the condition that $P_B(M, q)$ is a vertex of $P(M, q)$.

Principally unimodular matrices play an important role in integrality of the LCP, which is analogous to totally unimodular matrices for integrality of the linear programming problem.

Theorem 2.5 ([3, 9])**.** *A square integral matrix M is principally unimodular if and only if any basic solution to* LCP (M, q) *is integral for any integral vector q.*

3 Linear complementarity problems with orientation

In this section, we introduce the LCP with orientation. We define the problem as follows: given a square matrix $M \in \mathbb{R}^{n \times n}$ and two vectors $q, r \in \mathbb{R}^n$, the LCP with orientation is the problem of finding a solution $z \in \mathbb{R}^n$ to LCP (M, q) with basis identical to the one of some solution to LCP $\left(-M^\top, r\right)$, i.e., a vector $z \in \mathbb{R}^n$ satisfying for some vector $y \in \mathbb{R}^n$ and $B \subseteq [n]$,

$$Mz + q \geq 0,\ z \geq 0, \tag{4}$$

$$-M^\top y + r \geq 0,\ y \geq 0, \tag{5}$$

$$(Mz + q)_B = z_{\overline{B}} = 0,\ (-M^\top y + r)_B = y_{\overline{B}} = 0. \tag{6}$$

We denote by LCP(M, q, r) a problem instance of the LCP with orientation, and (z, y) is called a *solution pair* if z and y satisfy (4), (5), and (6). Similarly to the LCP, we say that a solution z to LCP(M, q, r) is a *basic* solution with respect to B if z is of the form (3).

Recall that LCP (M, q) is to find an index set $B \subseteq [n]$ such that $q \in \operatorname{pos} C_M(B)$. LCP$(M, q, r)$ is equivalent to finding an index set B such that $q \in \operatorname{pos} C_M(B)$ and $r \in \operatorname{pos} C_{-M^\top}(B)$. Thus, in the LCP with orientation, we have an additional constraint that the vector r defines complementary cones which we can use.

You might see that the LCP with orientation is more difficult than the LCP, since it holds that LCP$(M, q, 0) =$ LCP (M, q) for any matrix M and vector q. However, they are polynomially equivalent, since LCP(M, q, r) can be reduced to the LCP (M', q') for M' and q' given by

$$M' = \begin{matrix} z & y & u & v \\ \end{matrix} \atop \begin{pmatrix} M & -M^\top & O & O \\ M & -M^\top & O & O \\ M & O & O & O \\ O & -M^\top & O & O \end{pmatrix} \in \mathbb{R}^{4n \times 4n},\ q' = \begin{pmatrix} q + r \\ q + r \\ q \\ r \end{pmatrix} \in \mathbb{R}^{4n}.$$

Indeed, LCP (M', q') is equivalent to finding vectors z and $y \in \mathbb{R}^n$ that satisfy

$$Mz - M^\top y + q + r \geq 0, \tag{7}$$

$$Mz + q \geq 0,\ -M^\top y + r \geq 0,\ z \geq 0,\ y \geq 0, \tag{8}$$

$$z^\top(Mz - M^\top y + q + r) = y^\top(Mz - M^\top y + q + r) = 0. \tag{9}$$

Note that (7) is implied by (8), and (8) and (9) imply

$$z^\top(Mz + q) = z^\top(-M^\top y + r) = 0, \tag{10}$$

$$y^\top(Mz + q) = y^\top(-M^\top y + r) = 0. \tag{11}$$

Therefore, we have (8), (10) and (11), which prove that LCP (M', q') is equivalent to LCP(M, q, r).

We here claim that the LCP with orientation characterizes integrality of LCP (M, q) in the following sense. LCP (M, q) is said to be *integral* if for any $B \subseteq [n]$ such that LCP (M, q) has a solution with basis B, LCP (M, q) has an integral solution with basis B. In other words, LCP (M, q) is integral if and only if for any $B \subseteq [n]$ with $P_B(M, q) \neq \emptyset$, $P_B(M, q)$ contains an integral vector.

Proposition 3.1. *Let M be an integral matrix, and let q be an integral vector.* LCP (M, q) *is integral if and only if* LCP(M, q, r) *has an integral solution for each integral vector r such that* LCP(M, q, r) *has a solution.*

To prove the proposition, we first observe that a solution to LCP(M, q, r) is represented as an optimal solution to a linear programming problem over $P(M, q)$.

Lemma 3.2. *A pair* (z, y) *is a solution pair of* LCP(M, q, r) *if and only if it satisfies the following three conditions:*

(a) z *is optimal to*

$$\max\{-r^\top z \mid Mz + q \geq 0,\ z \geq 0\}, \tag{12}$$

(b) y *is optimal to the dual problem of* (12), *i.e.,*

$$\min\{q^\top y \mid -M^\top y + r \geq 0,\ y \geq 0\}, \tag{13}$$

(c)

$$z^\top(Mz + q) = 0 \text{ and } y^\top(-M^\top y + r) = 0. \tag{14}$$

Proof. For the only-if part, let (z, y) be a solution pair to LCP(M, q, r), i.e., (4), (5), and (6) are satisfied for some $B \subseteq [n]$. Then (6) immediately implies (14), and by (4) and (5), z and y are feasible to (12) and (13), respectively. Moreover, (6) implies the complementarity conditions $(Mz+q)^\top y = 0$ and $z^\top(-M^\top y+r) = 0$ for (12) and (13). Therefore, z and y are optimal solutions to (12) and (13), respectively.

For the if part, let (z, y) satisfy (a), (b), and (c). Then it implies (4) and (5). Moreover, since z and y are optimal, it holds that $z^\top(-M^\top y + r) = 0$ and $y^\top(Mz + q) = 0$. In order to show that z and y satisfy (6) for some B, let $B = \{i \in [n] \mid (Mz + q)_i = 0 \text{ and } (-M^\top y + r)_i = 0\}$. For each index $i \notin B$, at least one of $(Mz + q)_i > 0$ and $(-M^\top y + r)_i > 0$ is satisfied. By $z^\top(-M^\top y + r) = y^\top(Mz + q) = 0$ and (14), we have $z_i = y_i = 0$. Therefore, it holds that $(Mz + q)_B = 0$, $(-M^\top y + r)_B = 0$, $z_{\overline{B}} = 0$ and $y_{\overline{B}} = 0$, and thus B satisfies (6). □

Proof of Proposition 3.1. First we assume that LCP (M, q) is integral. Let r be an integral vector such that LCP(M, q, r) has a solution. Then there exist a set $B \subseteq [n]$ and a solution pair (z, y) to LCP(M, q, r) satisfying (4), (5), and (6). We can see that $P_B(M, q)$ is nonempty since z is in $P_B(M, q)$. By the assumption, $P_B(M, q)$ has an integral vector. Since any vector in $P_B(M, q)$ can be a solution to LCP(M, q, r), this implies that LCP(M, q, r) has an integral solution.

Conversely, assume that LCP(M, q, r) has an integral solution for each integral vector r such that LCP(M, q, r) has a solution. Let B be an index set such that $P_B(M, q)$ is nonempty. Define $r = \sum_{i \in B}(M_{i\cdot})^\top + \sum_{i \notin B} e^{(i)}$, where $e^{(i)}$ is the ith unit vector, i.e., $e^{(i)}_i = 1$ and $e^{(i)}_j = 0$ for $j \neq i$. Then, any vector in $P_B(M, q)$ is a solution to LCP(M, q, r), and hence it has an integral solution z^* by the assumption.

Let Q be the set of optimal solutions to $\max\{-r^\top z \mid z \in P(M, q)\}$. By Lemma 3.2, z^* is contained in Q, and also $P_B(M, q) \subseteq Q$ holds. We claim that $Q = P_B(M, q)$.

Suppose that we have a vector x in $Q \setminus P_B(M, q)$. Then there exists an index j such that either ($j \in B$ and $(Mx + q)_j > 0$), or ($j \notin B$ and $x_j > 0$). Hence, since $Mx + q \geq 0$ and $x \geq 0$, we have

$$-r^\top x = -\sum_{i \in B}(M_{i\cdot})x - \sum_{i \notin B} e_i^\top x < \sum_{i \in B} q_i,$$

where the last strict inequality follows from the existence of j. On the other hand, for any $z \in P_B(M, q)$, it holds that

$$-r^\top z = -\sum_{i \in B} (M_{i\cdot})z - \sum_{i \notin B} e_i^\top z = \sum_{i \in B} q_i > -r^\top x,$$

which contradicts that x is contained in Q. Thus $Q = P_B(M, q)$.

From our claim, the vector z^* is contained in $P_B(M, q)$, which means that $P_B(M, q)$ has an integral vector. Therefore, LCP (M, q) is integral. □

4 Integrality of the linear complementarity problem

Given LCP(M, q, r), we define the *dual LCP with orientation* to be LCP$(-M^\top, r, q)$. In other words, the dual problem is the problem of finding a vector y satisfying (4), (5), and (6) for some z and B. We now introduce the total dual integrality of the LCP as follows.

Definition 4.1. *Let M be a rational matrix, and let q be a rational vector.* LCP (M, q) *is* totally dual integral *if* LCP$(-M^\top, r, q)$ *has an integral solution for each integral vector r such that* LCP$(-M^\top, r, q)$ *has a solution.*

As we mentioned in the introduction, Fukuda and Terlaky [12] proposed another definition of duality for the LCP with sufficient matrices. They showed a theorem of the alternative: exactly one of primal and dual LCP problems has a solution. We note that our definition of the duality is quite different from theirs.

The following theorem is the main result of this section.

Theorem 4.2. *Let M be an integral matrix, and let q be an integral vector. If* LCP (M, q) *is totally dual integral, then any basic solution to* LCP(M, q, r) *is integral for any integral vector r.*

When setting r to be zero, we have the following corollary for LCP (M, q).

Corollary 4.3. *Let M be an integral matrix, and let q be an integral vector. If* LCP (M, q) *is totally dual integral, then all basic solutions to* LCP (M, q) *are integral.*

By definition, if $Mz + q \geq 0,\ z \geq 0$ is totally dual integral *in terms of linear systems*, then the linear programming problem (13) of (b) in Lemma 3.2 has an integral optimal solution for each integral vector r such that (13) is finite, and hence LCP (M, q) is totally dual integral. However, total dual integrality of LCP (M, q) does not necessarily imply that of $Mz + q \geq 0,\ z \geq 0$. Indeed by Lemma 3.2, if LCP (M, q) is totally dual integral, then (13) has an integral optimal solution for each integral vector r such that LCP$(-M^\top, r, q)$ has a solution. It does not imply total dual integrality of $Mz+q \geq 0,\ z \geq 0$, since all vectors such that (13) has optimal solutions are not taken as r. This motivates us to weaken the total dual integrality of linear systems to prove Theorem 4.2.

In Section 4.1, we introduce a weaker variant of the total dual integrality of linear systems, and then provide the proof of Theorem 4.2 in Section 4.2.

4.1 S-dual integrality of linear systems

In this subsection, we define S-dual integrality of linear systems for a given cone S.

Definition 4.4. *Let A be a rational matrix, and let b be a rational vector. For a cone S, a system of linear inequalities $Ax \leq b$ is called* S-dual integral *if the dual problem of* $\max\{c^\top x \mid Ax \leq b\}$, *that is,*

$$\min\{b^\top y \mid A^\top y = c,\ y \geq 0\}$$

has an integral optimal solution for each integral vector c in S such that the minimum exists.

We then show the following proposition, which is useful in the proof of Theorem 4.2. Recall that $\mathrm{pos}\, A = \{A\alpha \mid \alpha \geq 0\}$ for a matrix A.

Proposition 4.5. *Let A be a rational matrix, and let b be an integral vector. Let $P = \{x \mid Ax \leq b\}$ and $F = \{x \mid A'x = b'\}$ be a minimal face of P, where $A'x \leq b'$ is a subsystem of $Ax \leq b$. If $Ax \leq b$ is $\mathrm{pos}\,(A')^\top$-dual integral, then the face F contains an integral vector.*

Remark 4.6. *$\mathbb{R}^n$-dual integrality coincides with the total dual integrality. Hence, Proposition 4.5 leads to Theorem 2.3.*

To prove Proposition 4.5, we make use of the following lemma.

Lemma 4.7 ([15]). *Let A be a rational matrix, and let b be a rational vector. A linear equation $Ax = b$ has an integral solution if and only if $y^\top b$ is an integer for each rational vector y such that $y^\top A$ is integral.*

We show the following two lemmas, where the first one is a well-known fact on optimal solutions to $\max\{c^\top x \mid x \in P\}$.

Lemma 4.8. *Let A be a real matrix, and let b be a real vector. Let $P = \{x \mid Ax \leq b\}$, and let $F = P \cap \{x \mid A'x = b'\}$ be a nonempty face of P, where $A'x \leq b'$ is a subsystem of $Ax \leq b$. For any vector c in $\mathrm{pos}\,(A')^\top$, any vector in F is an optimal solution to* $\max\{c^\top x \mid Ax \leq b\}$.

Proof. Choose c in $\mathrm{pos}\,(A')^\top$ arbitrarily. Since $c = (A')^\top \alpha$ for some $\alpha \geq 0$, for any vectors x in P, we have

$$c^\top x = \alpha^\top A'x \leq \alpha^\top b'.$$

Note that the last inequality above becomes equal only when x is contained in F, which completes the proof. □

Lemma 4.9. *Let A be a rational matrix, and let b be an integral vector. Let $P = \{x \mid Ax \leq b\}$, and let $F = P \cap \{x \mid A'x = b'\}$ be a face of P, where $A'x \leq b'$ is a subsystem of $Ax \leq b$. If $Ax \leq b$ is $\mathrm{pos}\,(A')^\top$-dual integral, then the supporting hyperplane $H = \{x \mid c^\top x = \delta\}$ of P contains an integral vector for each integral vector c in $\mathrm{pos}\,(A')^\top$.*

Proof. Suppose to the contrary that supporting hyperplane $H = \{x \mid c^\top x = \delta\}$ of P contains no integral vector for some integral vector c in $\mathrm{pos}\,(A')^\top$. From $\mathrm{pos}\,(A')^\top$-dual integrality, $\min\{b^\top y \mid A^\top y = c,\ y \geq 0\}$ has an integral optimal solution. Since δ is the optimal value of $\max\{c^\top x \mid$

$Ax \leq b\} = \min\{b^\top y \mid A^\top y = c,\ y \geq 0\}$ by Lemma 4.8, δ is integral. Since $c^\top x = \delta$ has no integral solution, there exists a rational number α such that αc is integral and $\alpha\delta$ is not an integer by Lemma 4.7. We may assume that $\alpha > 0$ by adding a positive integer to α. Then we have $\alpha c \in \mathrm{pos}\,(A')^\top$. We can see that

$$\max\{\alpha c^\top x \mid Ax \leq b\} = \alpha \max\{c^\top x \mid Ax \leq b\} = \alpha\delta$$

is not an integer. This contradicts $\mathrm{pos}\,(A')^\top$-dual integrality. □

We are now ready to prove Proposition 4.5.

Proof of Proposition 4.5. Suppose that F contains no integral vector to derive a contradiction. By Lemma 4.7, there exists a vector y such that $(A')^\top y$ is an integral vector and $y^\top b'$ is not an integer. We may assume that $y > 0$ by adding a large positive integer γ to each element of y for which $(A')^\top(\gamma 1)$ is integral. Let $c = (A')^\top y$ and $\delta = y^\top b'$. Then c is contained in $\mathrm{pos}\,(A')^\top$. Hyperplane $H = \{x \mid c^\top x = \delta\}$ contains no integral vector, since c is an integer vector whereas δ is not an integer. Since $H = \{x \mid c^\top x = \delta\}$ is a supporting hyperplane of P with c in $\mathrm{pos}\,(A')^\top$, this contradicts Lemma 4.9. Thus F contains an integral vector. □

We note that Proposition 4.5 requires the minimality of F. Suppose that $F = P \cap \{x \mid A'x = b'\}$ is a non-minimal face of P. Then, even if $Ax \leq b$ is $\mathrm{pos}\,(A')^\top$-dual integral, and c is an integral vector in $\mathrm{pos}\,(A')^\top$, $\max\{c^\top x \mid Ax \leq b\}$ does not always have an integral optimal solution. For example, let

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & -6 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and let A' and b' be the first row of A and b, respectively, i.e., $A' = \begin{bmatrix} 2 & -1 \end{bmatrix}$ and $b' = 0$. Then we have $\max\{c^\top x \mid Ax \leq b\} = 0$ for any integral vector c in $\mathrm{pos}\,(A')^\top = \left\{ \begin{bmatrix} 2 & -1 \end{bmatrix}^\top \alpha \mid \alpha \geq 0 \right\}$, since, as seen in Figure 1, the optimal solution set is represented as the convex combination of $\begin{bmatrix} 1/3 & 2/3 \end{bmatrix}^\top$ and $\begin{bmatrix} 1/6 & 1/3 \end{bmatrix}^\top$, which contains no integral vector. On the other hand, for any integral vector $c = \begin{bmatrix} 2 & -1 \end{bmatrix}^\top \alpha$ with $\alpha \geq 0$, the dual problem $\min\{b^\top y \mid A^\top y = c,\ y \geq 0\}$ has an optimal solution $y = \begin{bmatrix} \alpha & 0 & 0 \end{bmatrix}^\top$. Thus $Ax \leq b$ is $\mathrm{pos}\,(A')^\top$-dual integral, while any integral c in $\mathrm{pos}\,(A')^\top$ provides linear programming problem $\max\{c^\top x \mid Ax \leq b\}$ with no integral optimal solution.

4.2 Proof of Theorem 4.2

Lemma 4.10. *Let M be an integral matrix in $\mathbb{Z}^{n \times n}$, and let q be an integral vector in $\mathbb{Z}^n$. Let B be a subset of $[n]$ such that $\mathrm{pos}\, C_M(B)$ contains q. Let $S = \mathrm{pos}\,(-C_{-M^\top}(B))$. If $\mathrm{LCP}\,(M, q)$ is totally dual integral, then the linear system $Mz + q \geq 0,\ z \geq 0$ is S-dual integral.*

Proof. Take an integral vector r in S arbitrarily. Then $\mathrm{LCP}(-M^\top, -r, q)$ has a solution with respect to B. Since $\mathrm{LCP}\,(M, q)$ is totally dual integral, $\mathrm{LCP}(-M^\top, -r, q)$ has an integral solution y^*. By Lemma 3.2, y^* is an optimal solution to the linear programming problem

$$\min\{q^\top y \mid -M^\top y - r \geq 0,\ y \geq 0\}.$$

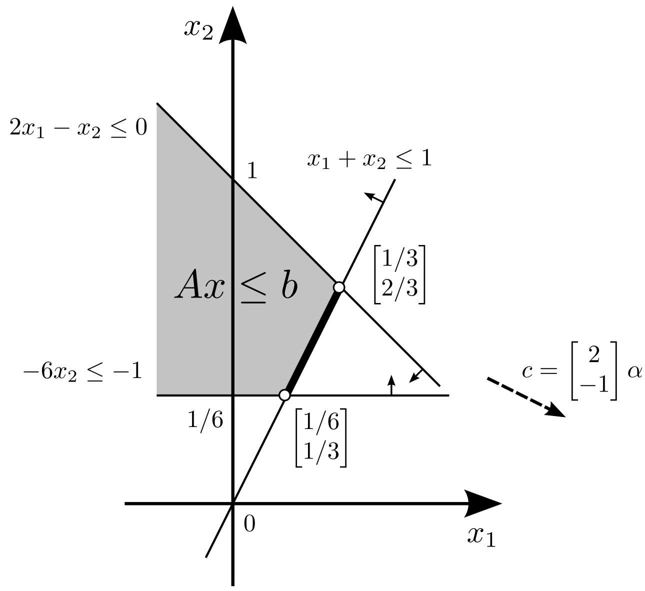

Figure 1: An example for Proposition 4.5.

This implies that $Mz + q \ge 0,\ z \ge 0$ is an S-dual integral system. □

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Assume that LCP (M, q) is totally dual integral. Take an integral vector r such that LCP(M, q, r) has a solution. Let z^* be a basic solution to LCP(M, q, r) with respect to some B, that is, z^* satisfies that $(Mz^* + q)_B = 0,\ z^*_{\overline{B}} = 0$, and M_{BB} is nonsingular. Since z^* is also a basic solution to LCP (M, q) with respect to B, $P_B(M, q)$ is a vertex of polyhedron $P(M, q)$ such that $P_B(M, q) = \{z^*\}$.

Let $A = -\begin{bmatrix}M^\top & I\end{bmatrix}^\top$ and $b = \begin{bmatrix}q^\top & 0\end{bmatrix}^\top$. We also define a matrix A' and a vector b' by $A' = -C_{-M^\top}(B)^\top$, $b'_B = q_B$ and $b'_{\overline{B}} = 0$. Then we can write $P(M, q) = \{z \mid Az \le b\}$ and $P_B(M, q) = \{z \mid A'z = b'\}$, where $A'z \le b'$ is a subsystem of $Az \le b$.

Since LCP (M, q) is totally dual integral, the system $Mz + q \ge 0,\ z \ge 0$ is a pos $(A')^\top$-dual integral system from Lemma 4.10. Therefore, since $P_B(M, q) = \{z^*\}$, it follows from Proposition 4.5 that $P_B(M, q)$ contains an integral vector, and thus z^* is integral. □

5 Sufficient and rank-symmetric matrices

In this section, we restrict our attention to (column) sufficient matrices and rank-symmetric matrices, and show that total dual integrality of LCP (M, q) characterizes that LCP (M, q) is integral.

5.1 Sufficient matrices

A square matrix M is called *column sufficient* if for all vectors x, it holds that

$$[x_i(Mx)_i \leq 0 \ (\forall i)] \Rightarrow [x_i(Mx)_i = 0 \ (\forall i)].$$

A matrix M is *row sufficient* if $M^\top$ is column sufficient, and *sufficient* if it is both column and row sufficient. The class of (column) sufficient matrices contains positive semi-definite matrices and P-matrices (i.e., all principal minors are positive). It is not difficult to see that column sufficiency of a matrix M implies the one of principal submatrices. The LCP with sufficient matrices possesses several important properties. For instance, any such LCP instances $\mathrm{LCP}\,(M,q)$ have a (possibly empty) convex solution set, and have a solution whenever $P(M,q)$ is not empty [8].

Chandrasekaran, Kabadi and Sridhar [3] showed that principal unimodular matrices are crucial for integral solutions to the LCP with column sufficient matrices.

Theorem 5.1 ([3])**.** *Let M be an integral, column sufficient matrix. Then M is principally unimodular if and only if* $\mathrm{LCP}\,(M,q)$ *has an integral solution for each integral vector q such that* $\mathrm{LCP}\,(M,q)$ *has a solution.*

In this section, we show that the total dual integrality implies integrality for the LCP with column sufficient matrices, which provides an alternative proof of Theorem 5.1 in the case of sufficient matrices as Theorem 5.5 below.

Our proof uses the following fundamental property for column sufficient matrices. For a matrix A, let $\mathrm{rank}\,A$ denote the rank of A.

Lemma 5.2 ([21])**.** *Let A be a column sufficient matrix of order n, and let $R \subseteq [n]$ be an index set such that $|R| = \mathrm{rank}\,A_{\cdot R} = \mathrm{rank}\,A$. Then A_{RR} is nonsingular.*

Lemma 5.3. *Let M be a column sufficient matrix of order n. For each $B \subseteq [n]$ such that $P_B(M,q)$ is nonempty, $P_B(M,q)$ contains a basic solution to* $\mathrm{LCP}\,(M,q)$ *with respect to some $B' \subseteq [n]$.*

Proof. Assume that $P_B(M,q)$ is nonempty. If M_{BB} is nonsingular, then $P_B(M,q)$ consists of a basic solution with respect to B. On the other hand, if M_{BB} is singular, then let z be a vector in $P_B(M,q)$ with the smallest $|\{i \mid z_i > 0\}| + |\{i \mid (Mz+q)_i > 0\}|$. Define $S = \{i \mid z_i > 0\}$ and $T = \{i \mid (Mz+q)_i = 0\}$. We note that $S \subseteq B \subseteq T$. We claim that M_{TS} has full column rank.

Assume a contrary that there exists a nonzero vector x in $\mathbb{R}^S$ such that $M_{TS}x = 0$. We may suppose that x has a negative element by multiplying -1 if necessary. Define x' to be an vector in $\mathbb{R}^n$ such that $x'_S = x$ and $x'_{\overline{S}} = 0$. Then consider the maximum number δ such that $z + \delta x'$ is in $P_B(M,q)$. By definition of x', we have $\delta > 0$, and at least one element of $z_S + \delta x$ and $M_{\overline{T}S}(z_S + \delta x) + q_{\overline{T}}$ is zero. This contradicts the minimality of $|S| + |\overline{T}|$.

Since M_{TS} has full column rank, we can choose a column index set R such that $S \subseteq R \subseteq T$, and $|R| = \mathrm{rank}\,M_{TR} = \mathrm{rank}\,M_{TT}$. Then by applying Lemma 5.2 to M_{TT} and R, we see that M_{RR} is nonsingular. Therefore, z is a basic solution to $\mathrm{LCP}\,(M,q)$ with respect to R. □

Theorem 5.4. *Let M be an integral, column sufficient matrix, and let q be an integral vector. If* $\mathrm{LCP}\,(M,q)$ *is totally dual integral, then* $\mathrm{LCP}\,(M,q)$ *is integral.*

Proof. By Lemma 5.3, each nonempty set $P_B(M, q)$ contains some basic solution z. This together with Theorem 4.2 completes the proof. □

Let us remark that the statement in Lemma 5.3 holds also for the negation of column sufficient matrices M, that is, for each $B \subseteq [n]$ such that $P_B(-M, q)$ is nonempty, $P_B(-M, q)$ contains some basic solution to LCP $(-M, q)$. Using this observation, together with Theorem 2.5, we obtain the following necessary and sufficient condition for a principally unimodular matrix, provided that the coefficient matrix is sufficient (cf. Corollary 2.4).

Theorem 5.5. *Let M be an integral sufficient matrix. Then the following three statements are equivalent:*

(a) *M is principally unimodular.*
(b) LCP (M, q) *is totally dual integral for each integral vector q.*
(c) LCP (M, q) *is integral for each integral vector q.*

Proof. (a) ⇒ (b): Assume that LCP$(-M^\top, r, q)$ has a solution. Let B be a subset in $[n]$ with $q \in \operatorname{pos} C_M(B)$ and $r \in \operatorname{pos} C_{-M^\top}(B)$. Then $P_B(-M^\top, r)$ is nonempty. Since $M^\top$ is column sufficient, Lemma 5.3 and the discussion before the theorem imply that $P_B(-M^\top, r)$ contains some basic solution y to LCP $\left(-M^\top, r\right)$. This y is integral by Theorem 2.5, and is also a solution to LCP$(-M^\top, r, q)$.

(b) ⇒ (c): This follows from Theorem 5.4.

(c) ⇒ (a): Since a basic solution to LCP (M, q) with respect to $B \subseteq [n]$ corresponds to a zero-dimensional face P_B, every basic solution is integral by (c). Thus by Theorem 2.5, M is principally unimodular. □

5.2 Rank-symmetric matrices

A square matrix M of order n is called *rank-symmetric* if $\operatorname{rank} M_{JK} = \operatorname{rank} M_{KJ}$ for all $K, J \subseteq [n]$. For a rank-symmetric matrix M, any principal submatrix M_{BB} is rank-symmetric from the definition. Symmetric and skew-symmetric matrices are examples of rank-symmetric matrices. Rank-symmetric matrices appear in the LCP associated with the linear programming, the convex quadratic programming, and the market equilibrium [7, 18]. Cunningham and Geelen [9] showed that for each vector q, if M is rank-symmetric and LCP (M, q) has a solution, then LCP (M, q) always has a basic solution. By combining the fact and Theorem 2.5, the following theorem holds.

Theorem 5.6 ([9])**.** *Let M be an integral rank-symmetric matrix. If M is principally unimodular, then* LCP (M, q) *has an integral solution for each integral vector q such that* LCP (M, q) *has a solution.*

The converse of Theorem 5.6 does not necessarily hold. For example, consider a rank-symmetric matrix

$$M = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}.$$

It is observed that $K(M) = \bigcup_{B\subseteq[n]} \operatorname{pos} C_M(B)$ coincides with $\operatorname{pos} C_M(\{2\}) \cup \operatorname{pos} C_M(\{1,2\})$. We also see that $\det C_M(\{2\}) = 1$ and $\det C_M(\{1,2\}) = -1$. Then LCP (M, q) has an integral solution for each integral vector q in $K(M)$. However, M is not principally unimodular.

For rank-symmetric matrices, we obtain similar results to (column) sufficient matrices in Section 5.1 by a similar argument to the proof of Theorem 5.4. For the proof, we use the following lemmas.

Lemma 5.7 ([9]). *Let A be a rank-symmetric matrix of order n, and $S \subseteq [n]$ be an index set. If the principal submatrix A_{SS} of A is nonsingular, then the Schur complement of A_{SS} in A, i.e.,*

$$A_{\overline{S}\overline{S}} - A_{\overline{S}S} A_{SS}^{-1} A_{S\overline{S}}$$

is a rank-symmetric matrix whose rank is $\operatorname{rank} A - |S|$.

By using Lemma 5.7, we show that rank-symmetric matrices have a similar property to column sufficient matrices stated in Lemma 5.2, which implies Lemma 5.9.

Lemma 5.8. *Let A be a rank-symmetric matrix of order n, and let $R \subseteq [n]$ be an index set such that $|R| = \operatorname{rank} A_{\cdot R} = \operatorname{rank} A$. Then A_{RR} is nonsingular.*

Proof. We show the lemma by induction on $\operatorname{rank} A$. For a matrix A with $\operatorname{rank} A = 1$, let $R = \{j\}$. Then $A_{ij} \neq 0$ holds for some i. If $i \neq j$, then $A_{ji} \neq 0$ by rank-symmetry, and consequently $\operatorname{rank} A > 1$, which is a contradiction. Thus we have $A_{jj} \neq 0$, meaning that A_{jj} is nonsingular.

We assume that the lemma holds for matrices of rank $< r$. Let A be a rank-symmetric matrix of rank r, and $R \subseteq [n]$ be an index set such that $|R| = \operatorname{rank} A_{\cdot R} = r$. Suppose to the contrary that $\operatorname{rank} A_{RR} = k < r$. Let $S \subseteq R$ be an index set such that $|S| = \operatorname{rank} A_{RS} = k$. We assume without loss of generality that $R = \{1, \dots, r\}$ and $S = \{1, \dots, k\}$. By the induction hypothesis, A_{SS} is nonsingular, and thus $C_A(S)$ is nonsingular. Consider a matrix

$$\hat{A} = C_A(S)^{-1} A = \begin{array}{c} \\ S \\ \overline{S} \end{array} \!\!\begin{array}{c} \begin{array}{cc} S & \overline{S} \end{array} \\ \begin{bmatrix} -I_k & -A_{SS}^{-1} A_{S\overline{S}} \\ O & A_{\overline{S}\overline{S}} - A_{\overline{S}S} A_{SS}^{-1} A_{S\overline{S}} \end{bmatrix} \end{array},$$

where I_k denotes the identity matrix of order k. Let $R' = R \setminus S$. By applying Lemma 5.7 to A_{RR} and S, we have $\operatorname{rank}(A_{R'R'} - A_{R'S} A_{SS}^{-1} A_{SR'}) = \operatorname{rank} A_{RR} - k = 0$. Since $\hat{A}_{R'R'} = A_{R'R'} - A_{R'S} A_{SS}^{-1} A_{SR'}$, $\hat{A}$ is of the form

$$\hat{A} = \begin{array}{c} \\ S \\ R' \\ \overline{R} \end{array} \!\!\begin{array}{c} \begin{array}{ccc} S & R' & \overline{R} \end{array} \\ \begin{bmatrix} -I_k & * & * \\ O & O & \hat{A}_{R'\overline{R}} \\ O & \hat{A}_{\overline{R}R'} & \hat{A}_{\overline{R}\overline{R}} \end{bmatrix} \end{array}.$$

Since $\operatorname{rank} \hat{A}_{\cdot R} = \operatorname{rank} A_{\cdot R} = r$, we have $\operatorname{rank} \hat{A}_{\overline{R}R'} = r - k$. Moreover, by Lemma 5.7, $\hat{A}_{\overline{S}\overline{S}}$ is rank-symmetric, and hence $\operatorname{rank} \hat{A}_{R'\overline{R}} = r - k > 0$. Therefore, we have a contradiction

$$\operatorname{rank} A = \operatorname{rank} \hat{A} = k + \operatorname{rank} \hat{A}_{\overline{S}\overline{S}} \geq k + \operatorname{rank} \hat{A}_{\overline{R}R'} + \operatorname{rank} \hat{A}_{R'\overline{R}} = r + r - k > r.$$

□

Lemma 5.9. *Let M be a rank-symmetric matrix. For each $B \subseteq [n]$ such that $P_B(M, q)$ is nonempty, $P_B(M, q)$ contains a basic solution to* LCP (M, q) *with respect to some $B' \subseteq [n]$.*

Therefore, we have the following two theorems, where the proofs are almost same as the ones for sufficient matrices.

Theorem 5.10. *Let M be an integral rank-symmetric matrix, and let q be an integral vector. If* LCP (M, q) *is totally dual integral, then* LCP (M, q) *is integral.*

Theorem 5.11. *Let M be an integral rank-symmetric matrix. Then the following three statements are equivalent:*

(a) *M is principally unimodular.*
(b) LCP (M, q) *is totally dual integral for each integral vector q.*
(c) LCP (M, q) *is integral for each integral vector q.*

Proof. (a) $\Rightarrow$ (b): Assume that LCP$(-M^\top, r, q)$ has a solution. Let B be a subset in $[n]$ with $q \in \operatorname{pos} C_M(B)$ and $r \in \operatorname{pos} C_{-M^\top}(B)$. Then $P_B(-M^\top, r)$ is nonempty. Since $-M^\top$ is rank-symmetric, Lemma 5.9 implies that $P_B(-M^\top, r)$ contains some basic solution y to LCP $\left(-M^\top, r\right)$. This y is integral by Theorem 2.5, and is also a solution to LCP$(-M^\top, r, q)$.

(b) $\Rightarrow$ (c): This follows from Theorem 5.10.

(c) $\Rightarrow$ (a): Since a basic solution to LCP (M, q) with respect to $B \subseteq [n]$ corresponds to a zero-dimensional face P_B, every basic solution is integral by (c). Thus by Theorem 2.5, M is principally unimodular. □

6 Hardness of recognizing the total dual integrality

In this section, we show that it is coNP-hard to recognize that a given LCP instance is totally dual integral. This is proved by reduction from coNP-completeness of recognizing quasi-bipartite graphs.

Let $G = (V, E)$ be an undirected graph. We denote by A_G the vertex-edge incidence matrix of G, i.e., A_G in $\{0, 1\}^{V \times E}$ such that the column vector $A_{\cdot e}$ for $e = (u, v)$ in E satisfies $A_{ue} = A_{ve} = 1$ and $A_{we} = 0$ for any vertex $w \neq u, v$. An undirected graph G is called *quasi-bipartite* if for any odd cycle C in G, deleting all vertices in C from G results in at least one isolated vertex. Ding, Feng and Zang [10] showed that it is coNP-complete to decide whether a given connected simple graph is quasi-bipartite. It is known that total dual integrality of linear systems associated with incidence matrices is characterized by quasi-bipartite graphs.

Lemma 6.1 ([10])**.** *Let G be a connected simple undirected graph. The linear system $A_G x \geq 1$, $x \geq 0$ is totally dual integral if and only if G is a quasi-bipartite graph that is not K_4 (i.e., the complete graph with four vertices).*

By Lemma 6.1, together with the fact that recognizing a quasi-bipartite graph is coNP-complete, we have Theorem 6.2 below.

Theorem 6.2 ([10])**.** *It is coNP-complete to decide if the linear system $A_G x \geq 1,\ x \geq 0$ is totally dual integral for a given connected simple undirected graph G.*

For an undirected graph $G = (V, E)$, we define a square matrix M_G in $\mathbb{R}^{(E\cup V)\times(E\cup V)}$ and a vector q_G in $\mathbb{R}^{E\cup V}$ as

$$M_G = \begin{array}{c} \\ E \\ V \end{array}\!\!\begin{array}{c} \begin{array}{cc} E & V \end{array} \\ \begin{bmatrix} O & O \\ A_G & O \end{bmatrix} \end{array}, \quad q_G = \begin{array}{c} E \\ V \end{array}\!\!\begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{15}$$

Then we will show the following theorem in the subsequent subsection. It should be noted that the equivalence in Lemma 6.3 may not hold if the coefficient matrix is not an incidence matrix.

Lemma 6.3. *Let G be a connected simple undirected graph. The linear system $A_G x \geq 1,\ x \geq 0$ is totally dual integral if and only if* LCP (M_G, q_G) *is totally dual integral.*

By Theorem 6.2 and Lemma 6.3, we have the following result.

Theorem 6.4. *The problem of deciding if a given LCP instance is totally dual integral is coNP-hard.*

6.1 Proof of Theorem 6.3

To prove Theorem 6.3, we use the following characterization for quasi-bipartite graphs. We denote by $d(v)$ the degree of a vertex v.

Lemma 6.5 ([10])**.** *A connected simple graph $G = (V, E)$ is quasi-bipartite if and only if it is K_4 or there exists a partition $(X_1, X_2, Y, Z_1, \dots, Z_k)$ (possibly $X_1 \cup X_2 = \emptyset$ or $k = 0$) of V such that*

- (a) *for each vertex v in X_1, it holds that $d(v) = 1$ and the unique neighbor of v is in X_2,*
- (b) *each vertex in X_2 is adjacent to at least one vertex in X_1, and there is no edge between X_2 and $Z_1 \cup \cdots \cup Z_k$,*
- (c) *there exist k distinct unordered pairs $\{y_1^1, y_1^2\}, \dots, \{y_k^1, y_k^2\}$ of vertices in Y such that*
  - (i) *$y_i^1 \neq y_i^2$ for $i = 1, \dots, k$,*
  - (ii) *both y_i^1 and y_i^2 are adjacent to all vertices in Z_i for $i = 1, \dots, k$, and*
  - (iii) *each odd cycle in G with no vertex in $X_1 \cup X_2$ contains both y_i^1 and y_i^2 for some i with $1 \leq i \leq k$,*
- (d) *$|Z_i| \geq 2$ for $i = 1, \dots, k$, and $d(v) = 2$ for all vertices v in $Z_1 \cup \cdots \cup Z_k$.*

For notational convenience, let $A = A_G$, $M = M_G$, and $q = q_G$.
We first claim the following fact.

Claim 6.6. *Let r be a vector in $\mathbb{R}^{E\cup V}$ with nonnegative r_V.*

- (a) *For any optimal solution $\hat{y} \in \mathbb{R}^V$ to*

$$\min\{-1^\top y \mid -A^\top y + r_E \geq 0,\ y \geq 0\}, \tag{16}$$

  $\hat{t} = \begin{bmatrix} 0 & \hat{y}^\top \end{bmatrix}^\top \in \mathbb{R}^{E\cup V}$ is an optimal solution to

$$\min\{q^\top t \mid -M^\top t + r \geq 0,\ t \geq 0\}. \tag{17}$$

(b) *For any optimal solution* $\hat{t} \in \mathbb{R}^{E \cup V}$ *to* (17), $\hat{t}_V$ *is an optimal solution to* (16).

(c) *For any optimal solution* $\hat{x} \in \mathbb{R}^E$ *to*

$$\max\{-r_E^\top x \mid Ax - 1 \geq 0,\ x \geq 0\}, \tag{18}$$

$\hat{z} = \begin{bmatrix}\hat{x}^\top & 0\end{bmatrix}^\top \in \mathbb{R}^{E \cup V}$ *is an optimal solution to*

$$\max\{-r^\top z \mid Mz + q \geq 0,\ z \geq 0\}. \tag{19}$$

(d) *For any optimal solution* $\hat{z} \in \mathbb{R}^{E \cup V}$ *to* (19), $\hat{z}_E$ *is an optimal solution to* (18).

Proof of Claim 6.6. We only show (a) and (b), since (c) and (d) are proven similarly. In (17), we have $q^\top t = \begin{bmatrix}0 & -1^\top\end{bmatrix}\begin{bmatrix}t_E \\ t_V\end{bmatrix} = -1^\top t_V$, and the constraints can be decomposed into $-A^\top t_V + r_E \geq 0$, $r_V \geq 0$, and $t_E, t_V \geq 0$. Since r_V is nonnegative, (17) is equivalent to

$$\min\{-1^\top t_V \mid -A^\top t_V + r_E \geq 0,\ t_E, t_V \geq 0\}. \tag{20}$$

Since t_E is redundant, it is equivalent to (16). Thus we have (a) and (b). □

We now prove Lemma 6.3.

For the if part, assume that LCP (M, q) is totally dual integral. To show that $Ax \geq 1,\ x \geq 0$ is totally dual integral, choose arbitrarily an integral vector b such that $\max\{-b^\top x \mid Ax \geq 1,\ x \geq 0\}$ and its dual $\min\{-1^\top y \mid A^\top y \leq b,\ y \geq 0\}$ have optimal solutions $\hat{x}$ and $\hat{y}$, respectively. Define a vector $r \in \mathbb{R}^{E \cup V}$ to be $r_E = b$ and $r_V = 0$. We first show that LCP$(-M^\top, r, q)$ has a solution.

Let $\hat{t}$ be a vector in $\mathbb{R}^{E \cup V}$ defined by $\hat{t}_E = 0$ and $\hat{t}_V = \hat{y}$. By Claim 6.6 (a), $\hat{t}$ is an optimal solution to (17). In addition, $\hat{t}$ satisfies that $\hat{t}^\top(-M^\top \hat{t} + r) = 0^\top \cdot (-A^\top \hat{y} + b) + \hat{y}^\top \cdot 0 = 0$. Also define a vector $\hat{z}$ in $\mathbb{R}^{E \cup V}$ to be $\hat{z}_E = \hat{x}$ and $\hat{z}_V = 0$. Then $\hat{z}$ is an optimal solution to (19) by Claim 6.6 (c). Moreover, it holds that $\hat{z}^\top(M\hat{z} + q) = \hat{x}^\top \cdot 0 + 0^\top \cdot (A\hat{x} - 1) = 0$. Therefore, it follows from Lemma 3.2 that $(\hat{t}, \hat{z})$ is a solution pair to LCP$(-M^\top, r, q)$.

Since LCP (M, q) is totally dual integral, LCP$(-M^\top, r, q)$ has an integral solution t^*. This t^* is an optimal solution to (17) by Lemma 3.2, and hence t^*_V is an integral optimal solution to (16) by Claim 6.6 (b). Thus $Ax \geq 1,\ x \geq 0$ is totally dual integral.

For the only-if part, assume that $Ax \geq 1,\ x \geq 0$ is totally dual integral. Let r be an integral vector such that LCP$(-M^\top, r, q)$ has a solution pair (t, z). Note that r is nonnegative, since r can be expressed as a nonnegative linear combination of column vectors of two nonnegative matrices $M^\top$ and I. By Lemma 3.2, t is an optimal solution to (17), and z is an optimal solution to (19). Moreover, we have $t_V^\top r_V = 0$. By Claim 6.6 (b) and (d), t_V and z_E are optimal solutions to (16) and (18), respectively.

We show that

$$\text{(16) has an integral optimal solution } y^* \text{ which satisfies } {y^*}^\top r_V = 0 \tag{21}$$

by proving the following three claims for partitions $(X_1, X_2, Y, Z_1, \ldots, Z_k)$ of V satisfying the four conditions in Lemma 6.5, since total dual integrality implies existence of such a partition by Lemmas 6.1 and 6.5. Then (21) and Claim 6.6 (a) imply that $t^* = \begin{bmatrix}0 & {y^*}^\top\end{bmatrix}^\top$ is an integral optimal solution to (17). This together with Lemma 3.2 implies that t^* is an integral solution to LCP$(-M^\top, r, q)$, which completes the proof.

Claim 6.7. *There exists an optimal solution y' to* (16) *such that $y'^{\top} r_V = 0$ and $y'_{X_1 \cup X_2}$ is integral.*

Proof of Claim 6.7. Let $y = t_V$. Then y is an optimal solution to (16) that satisfies $y^{\top} r_V = 0$. We define a vector $y' \in \mathbb{R}^V$ from y as follows. Let $y'_u = y_u$ if $u \notin X_1 \cup X_2$. To define y'_u ($u \in X_1 \cup X_2$), let E_1 be the set of edges incident to X_1. Since $d(v) = 1$ for any vertex v in X_1, it is observed that $A_{X_1 E_1}$ becomes the identity matrix by appropriately rearranging rows and columns, and $A_{X_1 \overline{E}_1} = O$. Let x be an optimal solution to the dual problem of (16), that is, (18). $A_{X_1 \cdot} x \geq 1$ implies $x_{E_1} > 0$. By the complementary slackness of (16) and (18), for each $e = (u, v)$ in E_1, we have

$$y_u + y_v = r_e. \tag{22}$$

Let v be a vertex in X_2, and let $u_1, \ldots, u_p$ denote neighbors of v in X_1. By Lemma 6.5 (b), we have $p \geq 1$. If $r_{u_i} > 0$ for some i, then we define $y'_v = y_v$ and $y'_{u_j} = y_{u_j}$ for $j = 1, \ldots, p$. Note that $y_{u_i} = 0$, $y_v = r_{(u_i, v)}$, and $y_{u_j} = r_{(u_j, v)} - r_{(u_i, v)}$ for $j \neq i$, by $y_{u_i} r_{u_i} = 0$ and (22). Hence these y'_v and y'_{u_i} ($i = 1, \ldots, p$) are all integers. On the other hand, if $r_{u_i} = 0$ for all i, then we define $y'_v = 0$ and $y'_{u_i} = r_{(u_i, v)}$ for all i. Note that $-1^{\top} y' \leq -1^{\top} y$, and y' is feasible to (16). This y' is an optimal solution to (16). Since $y'_{X_1 \cup X_2}$ is integral, the claim is proven. □

Claim 6.8. *There exists an optimal solution y to* (16) *such that $y^{\top} r_V = 0$, $y_{X_1 \cup X_2}$ is integral, and each odd cycle C in G contains a vertex v in C for which y_v is integral.*

Proof of Claim 6.8. Let y be a vector such that $y^{\top} r_V = 0$ and $y_{X_1 \cup X_2}$ is integral. We denote $Z = Z_1 \cup \cdots \cup Z_k$, and let $U = \{v \in V \mid y_v \text{ is integral}\} \setminus Z$. Let C be an odd cycle with no vertex in U. Since $y_{X_1 \cup X_2}$ is integral, C has no vertex in $X_1 \cup X_2$. By Lemma 6.5, C contains two vertices v_1 and v_2 in Y (which correspond to y_i^1 and y_i^2 in Lemma 6.5) such that any vertex in Z_i is adjacent only to v_1 and v_2. Assume that $y_{v_1} \leq y_{v_2}$ without loss of generality, and we arbitrarily choose vertices u_1 and u_2 in Z_i, since $|Z_i| \geq 2$.

We first observe that $r_v = 0$ for any vertex v in $C \setminus Z$, because otherwise $r_v > 0$ for some vertex v in $C \setminus Z$, implying $y_v = 0$, that is, v is in U. Moreover, it holds that $r_{u_1} = r_{u_2} = 0$. Indeed, suppose to the contrary that at least one of r_{u_1} and r_{u_2}, say r_{u_1}, is positive. Since $y_{u_1} r_{u_1} = 0$, we have $y_{u_1} = 0$. Let $e_1 = (v_1, u_1)$ and $e_2 = (v_2, u_1)$. Since $d(u_1) = 2$, the row vector $A_{u_1 \cdot}$ has ones at position e_1 and e_2, and zeros at the other positions. Take an optimal solution x to (18) (i.e., the dual of (16)). Then x satisfies $(Ax)_{u_1} = x_{e_1} + x_{e_2} \geq 1$, and hence at least one of x_{e_1} and x_{e_2} is positive. By complementarity slackness, at least one of $y_{v_1} + y_{u_1} \leq r_{e_1}$ and $y_{v_2} + y_{u_1} \leq r_{e_2}$, is satisfied with equality. By $y_{u_1} = 0$, we have $y_{v_1} = r_{e_1}$ or $y_{v_2} = r_{e_2}$. At least one of y_{v_1} and y_{v_2} is integer, which contradicts that C has no vertex in U.

We then modify y so that

$$y_{u_1} := y_{u_1} + y_{v_1}, \quad y_{u_2} := y_{u_2} + y_{v_1}, \quad y_{v_2} := y_{v_2} - y_{v_1}, \quad y_{v_1} := 0.$$

Since y_{v_1} is zero, we can replace U with $U \cup \{v_1\}$. Note that the resulting y remains optimal to (16), and satisfies $y^{\top} r_V = 0$.

By repeatedly applying the above modification, we obtain a desired y of the claim. □

Claim 6.9. *Let y be an optimal solution to* (16) *that satisfies three conditions in Claim 6.8. Let $U = \{i \in V \mid y_i \text{ is integral}\}$. Then there exists an integral optimal solution y^* to* (16) *such that ${y^*}^{\top} r_V = 0$ and $y^*_U = y_U$.*

Proof of Claim 6.9. Let $d = r_E - (A^\top)_{\cdot U} y_U$. Note that d is an integral vector. We consider a linear programming problem

$$\min\{-1^\top \xi \mid (A^\top)_{\cdot \overline{U}} \xi \le d,\ \xi \ge 0\}, \tag{23}$$

where ξ represents the vectors of variables in $\mathbb{R}^{\overline{U}}$. We first show that (23) has an integral optimal solution ξ.

We rewrite $A_{\overline{U}\cdot}$ as $[A'\ E]$, where A' has two 1's in each column and E has at most one 1 in each column. Let G' be the subgraph of G whose vertex-edge incidence matrix is A'. Then G' has the vertex set $\overline{U}$. Since any odd cycle in G contains a vertex in U by Claim 6.8, G' has no odd cycle, i.e., G' is bipartite. Hence A' is totally unimodular, and so is $A_{\overline{U}\cdot}$. Therefore, (23) has an integral optimal solution ξ.

Let y^* be an integral vector with $y^*_U = y_U$ and $y^*_{\overline{U}} = \xi$. Note that y^* is an optimal solution to (16). In addition, it holds that ${y^*}^\top r_V = 0$. Indeed, if $r_v > 0$ for some v in V, then we have $y_v = 0$ by $y_v r_v = 0$. Hence, for any i not in U, we have $r_i = 0$. This implies that ${y^*}^\top r_V = y_U^\top r_U + \xi^\top r_{\overline{U}} = 0$. □

7 Matrix classes

Before concluding this paper, we discuss matrix classes studied in the LCP literature, where Figure 2 shows their relationship.

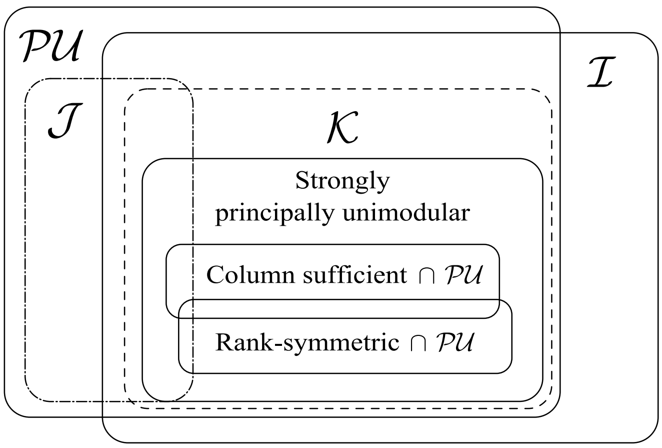

Figure 2: Matrix classes studied in the integral LCP.

Let $\mathcal{PU}$ be the set of principally unimodular matrices, and let $\mathcal{I}$ be the set of integral matrices M such that LCP(M, q) has an integral solution for each integral vector q such that it has a solution. We define $\mathcal{J}$ as the set of integral matrices M such that LCP(M, q) is totally dual integral for each integral vector q such that it has a solution. Let also $\mathcal{K}$ be the set of integral matrices M such that LCP(M, q) is integral for each integral vector q such that it has a solution.

Matrix class $\mathcal{I}$ is defined by Chandrasekaran, Kabadi and Sridhar [3]. It is known [3] that if we restrict matrices M to be column sufficient, then $\mathcal{I}$ coincides with $\mathcal{PU}$. By definition, $\mathcal{K}$ is contained in $\mathcal{I}$.

Chandrasekaran, Kabadi and Sridhar introduced strongly principally unimodular matrices as matrices in $\mathcal{I}$. An integral matrix M is called *strongly principally unimodular* if for each submatrix M_{KJ} of full column rank where $\emptyset \neq J \subseteq K$, the greatest common divisor (g.c.d) of the determinants of all $|J| \times |J|$ submatrices of M_{KJ} is one. The concept of strong principal unimodularity is based on the following lemma.

Lemma 7.1 ([19, Chapter 4])**.** *Let A be an $n \times k$ integral matrix of full column rank. Then the g.c.d of $k \times k$ subdeterminants of A is one if and only if for any vector x such that Ax is an integral vector, x is integral.*

By definition, any strongly principally unimodular matrix is principal unimodular. Column sufficient, principally unimodular matrices are known to be strongly principally unimodular [3].

In addition to these inclusion relationships among matrix classes, we have the following proposition.

Proposition 7.2. *For matrix classes defined above, we have the following statement.*

(a) *$\mathcal{J}$ is contained in $\mathcal{PU}$.*
(b) *$\mathcal{K}$ is contained in $\mathcal{I} \cap \mathcal{PU}$.*
(c) *Strongly principally unimodular matrices are contained in $\mathcal{K}$.*
(d) *Integral rank-symmetric, principally unimodular matrices are strongly principally unimodular.*

Proof. (a): Let M be a matrix in $\mathcal{J}$. By Corollary 4.3, any basic solution to LCP (M, q) is integral for any integral vector q. Theorem 2.5 implies that this is equivalent to $M \in \mathcal{PU}$.

(b): Let M be a matrix in $\mathcal{K}$. Since M is clearly contained in $\mathcal{I}$, we only show that M is principally unimodular. Choose arbitrarily an index set B such that M_{BB} is nonsingular. For any integral vector q, if $P_B(M, q)$ is nonempty, then $C_M(B)x = q$ has a (unique) integral solution. This and Lemma 7.1 imply that $\det M_{BB} = \pm 1$. Thus M is principally unimodular.

(c): Let M be a strongly principally unimodular matrix. By Proposition 3.1, a matrix M belongs to $\mathcal{K}$ if and only if LCP(M, q, r) has an integral solution for each integral vectors q, r such that it has a solution. We show that a strongly principally unimodular matrix M satisfies the latter condition.

Take arbitrarily integral vectors q and r such that LCP(M, q, r) has a solution. We choose a solution z with the smallest $|\{i \mid z_i > 0\}| + |\{i \mid (Mz + q)_i > 0\}|$. Let $S = \{i \mid z_i > 0\}$ and $T = \{i \mid (Mz + q)_i = 0\}$. We note that $z_{\overline{S}} = 0$. Then we can see that M_{TS} has full column rank as in the proof of Lemma 5.3. Thus z_S is a unique solution to the linear system $M_{TS} z_S + q_T = 0$. Moreover, since the g.c.d of the determinants of all $|S| \times |S|$ submatrices of M_{TS} is one, Lemma 7.1 implies that z_S is integral.

(d): Let M be a rank-symmetric and principally unimodular matrix. Then every basic solution to LCP (M, q) is integral for any integral vector q by Theorem 2.5.

Suppose to the contrary that M is not strongly principally unimodular. Then there exists a submatrix M_{KJ} where $\emptyset \neq J \subseteq K$ such that M_{KJ} has full column rank, and the g.c.d α of the determinants of all $|J| \times |J|$ submatrices of M_{KJ} is more than one. We note that M_{JJ} is singular, since otherwise, the g.c.d α becomes one by principal unimodularity. By Lemma 7.1, there exists a non-integral vector $x \in \mathbb{R}^J$ such that $M_{KJ}x$ is integral. By adding a positive integer to each element of x, we may suppose that $x > 0$.

Define two vectors z and q by $z_J = x$, $z_{\overline{J}} = 0$, $q_K = -M_{KJ}x$, and $q_{\overline{K}} = -\lfloor M_{\overline{K}J}x \rfloor$. Then q is an integral vector, and z is a solution for LCP (M, q), since it satisfies that $Mz + q \geq 0$, $z \geq 0$, $(Mz + q)_K = 0$, and $z_{\overline{J}} = 0$.

Let $T = \{i \mid (Mz+q)_i = 0\}$. We note that $K \subseteq T$. Since M_{KJ} has full column rank, M_{TJ} also has full column rank. If M_{TT} is nonsingular, then z is a basic solution with respect to T, and this contradicts Theorem 2.5 that every basic solution to LCP (M, q) is integral. On the other hand, if M_{TT} is singular, then by Lemma 5.8, there exists a set R such that $J \subseteq R \subseteq T$ and M_{RR} is nonsingular. This again implies that z is a basic solution (with respect to R), which contradicts Theorem 2.5. □

We remark that $\mathcal{K}$ is a proper subclass of $\mathcal{PU} \cap \mathcal{I}$. For example, let us consider a matrix

$$M = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}.$$

This M is clearly principally unimodular. For any integral vector q, it is observed that LCP (M, q) has a solution if and only if $q \geq 0$, which implies that LCP (M, q) has an integral solution $z = 0$. Thus M belongs to $\mathcal{I}$. On the other hand, if $q = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$ and $B = \{1, 2\}$, then $P_B(M, q)$ contains only one vector $\begin{bmatrix} 1/2 & 0 \end{bmatrix}^\top$, which implies $M \notin \mathcal{K}$. In addition, by the above argument, we can see that $-M^\top$ belongs to $\mathcal{PU} \setminus \mathcal{J}$.

In addition, $\mathcal{J}$ and $\mathcal{K}$ have no inclusion relationship. Let us consider a matrix

$$M = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -3 & 0 \end{bmatrix}.$$

This M is strongly principally unimodular, and hence by Proposition 7.2 (c), M belongs to $\mathcal{K}$. On the other hand, M does not belong to $\mathcal{J}$. To see this, let $q = \begin{bmatrix} -3 & -1 & 6 \end{bmatrix}^\top$ and let $r = \begin{bmatrix} -1 & -1 & 0 \end{bmatrix}^\top$. It is observed that q is contained in $\operatorname{pos} C_M(\{1, 2\})$ and $\operatorname{pos} C_M(\{1, 2, 3\})$. Then we see that $\operatorname{pos} C_{-M^\top}(\{1, 2, 3\})$ contains r, and $\operatorname{pos} C_{-M^\top}(\{1, 2, 3\})$ does not contain r. Since $Mz + q = 0$, $z \geq 0$ has only one solution $\begin{bmatrix} 0 & 0 & 1/3 \end{bmatrix}^\top$, LCP$(-M^\top, r, q)$ has no integral solution. Thus M is contained in $\mathcal{K} \setminus \mathcal{J}$. This also implies that $-M^\top$ is contained in $\mathcal{J} \setminus \mathcal{K}$.

We note that $-M^\top$ belongs to $\mathcal{I}$. It is observed that $K(-M^\top)$ coincides with $\operatorname{pos} C_{-M^\top}(\emptyset) \cup \operatorname{pos} C_{-M^\top}(\{3\})$. The determinant of $C_{-M^\top}(\emptyset)$ is one, and linear system $C_{-M^\top}(\{3\})y = r$, $y \geq 0$ has an integral solution for each integral vector r such that the system has a solution. Hence, LCP $\left(-M^\top, r\right)$ has an integral solution for any integral vector r such that it has a solution.

There also exists a matrix in $\mathcal{K}$ which is not strongly principally unimodular. For example, let us consider a matrix

$$M = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}.$$

By letting $J = \{1\}$ and $K = \{1, 2\}$, we can see that M is not strongly principally unimodular. On the other hand, for any index set $B \subseteq \{1, 2\}$ and any integral vector q, if linear system

$C_M(B)x = q,\ x \geq 0$ has a solution, then the system has an integral solution. Thus LCP (M, q) is integral for any integral vector q, which means that $M \in \mathcal{K}$.

Let us remark that although rank-symmetric matrices and (column) sufficient matrices possess a similar property, the associated LCP instances have no inclusion relationship.

In the remainder of this paper, we show that the weaker variants of $\mathcal{J}$ coincide with $\mathcal{PU}$. Recall that $\mathcal{J}$ is the set of integral matrices M such that LCP$(-M^\top, r, q)$ has an integral solution for each integral vectors q and r such that it has a solution.

Proposition 7.3. *Let M be an integral matrix. Then the following are equivalent.*

(a) *M belongs to $\mathcal{PU}$.*
(b) *any basic solution to* LCP$(-M^\top, r, q)$ *is integral for each integral vectors q and r such that it has a basic solution.*
(c) *some basic solution to* LCP$(-M^\top, r, q)$ *is integral for each integral vectors q and r such that it has a basic solution.*
(d) LCP$(-M^\top, r, q)$ *has an integral solution for each integral vectors q and r such that it has a basic solution.*

Proof. By definition, it is not difficult to see that (b) $\Rightarrow$ (c) $\Rightarrow$ (d). If M is a principally unimodular matrix, then since $-M^\top$ is also principally unimodular, Theorem 2.5 implies that M satisfies (b). We show that (d) $\Rightarrow$ (a).

Assume that M satisfies (d), and let q be an integral vector. Then M and q satisfy that LCP$(-M^\top, r, q)$ has an integral solution for each integral vector r such that it has a basic solution. By a similar proof to Lemma 4.10, we can show that for any index set B such that $q \in \operatorname{pos} C_M(B)$ and M_{BB} is nonsingular, the linear system $Mz + q \geq 0,\ z \geq 0$ is $\operatorname{pos}(-C_{-M^\top}(B))$-dual integral. By using the proof of Theorem 4.2 with this fact instead of Lemma 4.10, we can show that any basic solution to LCP(M, q, r) is integral for any integral vector r. Note that the proof of Theorem 4.2 considers only bases B such that M_{BB} is nonsingular. Therefore, any basic solution to LCP (M, q) is integral, which means that M belongs to $\mathcal{PU}$ by Theorem 2.5. □

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