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Shinji ITO and Kazuo MUROTA

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

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Abstract

We deal with the generalized eigenvalue problem $A\boldsymbol{x} = \lambda B\boldsymbol{x}$ for nonsquare matrix pencils $A - \lambda B$, where $A, B \in \mathbb{C}^{m\times n}$ and $m > n$. A major difficulty inherent in this problem is that perturbation to inputs may cause eigenvalues to fail to exist even if eigenvalues are known to exist in the noiseless case. To cope with this situation, Boutry et al. have proposed a novel approach that searches for the minimal perturbation to the pencil such that the perturbed pencil has eigenpairs. Boutry et al. first aimed to find the minimal perturbation such that the perturbed pencil has n eigenpairs, but they settled for a simplified version that guarantees at least one eigenpair. The aim of this paper is to present an algorithm for the original version of the problem with n eigenpairs. The proposed algorithm is based on the total least squares problem (TLS) introduced by Golub and Van Loan. The algorithm is much simpler and runs faster than Boutry et al.'s algorithm. It is confirmed numerically that the proposed algorithm is robuster against data noise than Boutry et al.'s algorithm.

1 Introduction

We deal with the generalized eigenvalue problem $A\boldsymbol{x} = \lambda B\boldsymbol{x}$ for nonsquare matrix pencils $A - \lambda B$, where $A, B \in \mathbb{C}^{m\times n}$ and $m > n$. Recently, such problems, with more rows than columns, have appeared in several different fields of research [1, 6, 11] and attracted much attention from theoretical and numerical points of view [2, 4, 5, 7, 10, 12, 13]. This is because, in many applications, the more measurements we have, the more rows of matrix pencils we can construct and the better estimation can be expected.

However, the generalized eigenvalue problem for nonsquare pencils has a major difficulty in many real situations: Even if it is known that there

*Department of Mathematical Informatics, University of Tokyo, Tokyo 113-8656, Japan. {shinji_ito, murota}@mist.i.u-tokyo.ac.jp

exist n linearly independent eigenpairs $(\lambda, \boldsymbol{x})$ in the noiseless case, these eigenpairs may fail to exist in real situations. To cope with this difficulty, Boutry, Elad, Golub and Milanfar [2] considered an optimization problem that finds the minimum perturbation of the given pair of matrices (A, B) such that the perturbed pair $(\hat{A}, \hat{B})$ has n linearly independent eigenvectors:

$$\begin{cases} \text{min.} & \|\hat{A} - A\|_{\mathrm{F}}^2 + \|\hat{B} - B\|_{\mathrm{F}}^2 \\ \text{s.t.} & \hat{A}, \hat{B} \in \mathbb{C}^{m \times n}, \quad \{(\lambda_k, \boldsymbol{v}_k)\}_{k=1}^n \subseteq \mathbb{C} \times \mathbb{C}^n, \\ & \hat{A}\boldsymbol{v}_k = \lambda_k \hat{B}\boldsymbol{v}_k, \quad k = 1, 2, \ldots, n, \\ & \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}\text{: linearly independent,} \end{cases} \tag{1}$$

where, for a matrix M in general, $\|M\|_{\mathrm{F}}$ denotes the Frobenius norm of M, i.e., $\|M\|_{\mathrm{F}}^2$ is equal to the sum of squares of the entries of M. This formulation is reasonable in many applications since it coincides with the maximum-likelihood description under the assumption that the perturbation to (A, B) is an additive white Gaussian noise with zero mean. It is noted, however, that this formulation suffers from a mathematical difficulty as an optimization problem since the infimum of $\|\hat{A} - A\|_{\mathrm{F}}^2 + \|\hat{B} - B\|_{\mathrm{F}}^2$ may or may not be attained, and hence the optimal solution may or may not exist.

The objective of this paper is to show that the problem (1) can be solved numerically via the total least squares problem considered by Golub and Van Loan [8]. Given two matrices A and B with the same number of rows, the total least squares problem (TLS) is to find perturbation matrices E and R that minimize the sum of their squared Frobenius norms $\|E\|_{\mathrm{F}}^2 + \|R\|_{\mathrm{F}}^2$ subject to the condition that[1]

$$\mathrm{range}(A + R) \subseteq \mathrm{range}(B + E), \tag{2}$$

where, for a matrix M in general, $\mathrm{range}(M)$ denotes the subspace spanned by the column vectors of M. TLS also suffers from a mathematical difficulty of the same kind: the infimum of $\|E\|_{\mathrm{F}}^2 + \|R\|_{\mathrm{F}}^2$ may or may not be attained. By using the singular value decomposition (SVD), Golub and Van Loan [8] gave a sufficient condition for the minimum to be attained, and proposed an algorithm for TLS under that sufficient condition.

The main contribution of this paper consists of the following:

- To establish a close relationship between (1) and TLS.
- To give a sufficient condition for the existence of an optimal solution in (1).
- To give a solution procedure for (1) in terms of TLS and SVD.

[1]Usually, the condition (2) reads "range$(B + R) \subseteq$ range$(A + E)$" with A and B interchanged. Our choice here is for the consistency with the problem (1), to be made clear in Section 2.

These results are presented as Theorem 2 and Algorithm 1 in Section 2.

It is worth mentioning that Boutry et al. [2] considered the following problem

$$\begin{cases} \text{min.} & \|\hat{A}-A\|_{\mathrm{F}}^2+\|\hat{B}-B\|_{\mathrm{F}}^2 \\ \text{s.t.} & \hat{A},\hat{B}\in\mathbb{C}^{m\times n}, \quad \lambda\in\mathbb{C}, \quad \boldsymbol{v}\in\mathbb{C}^n, \\ & \hat{A}\boldsymbol{v}=\lambda\hat{B}\boldsymbol{v}, \quad \boldsymbol{v}\neq\mathbf{0} \end{cases} \tag{3}$$

as a simplified variant of (1). This version imposes a weaker condition that the perturbed pair $(\hat{A},\hat{B})$ admits at least one eigenpair. As the imposed constraint is weaker, the problem (3) can be understood as a relaxation of the original problem (1). An algorithm for solving (3) is given in [2], whereas algorithms for (1) were left open in [2].

A number of related works can be found in the literature. Chu and Golub [3] considered a family of problems parameterized by a positive integer l:

$$\begin{cases} \text{min.} & \|\hat{A}-A\|_{\mathrm{F}}^2+\|\hat{B}-B\|_{\mathrm{F}}^2 \\ \text{s.t.} & \hat{A},\hat{B}\in\mathbb{C}^{m\times n}, \quad \{(\lambda_k,\boldsymbol{v}_k)\}_{k=1}^{l}\subseteq\mathbb{C}\times\mathbb{C}^n, \\ & \hat{A}\boldsymbol{v}_k=\lambda_k\hat{B}\boldsymbol{v}_k, \quad k=1,2,\ldots,l, \\ & \{\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_l\}\text{: linearly independent.} \end{cases} \tag{4}$$

The case of $l=n$ is (1) and that of $l=1$ is (3). Although it is shown in [3] that the problem (4) can be reduced to a certain optimization problem over a compact set of matrices, no algorithm for solving that optimization problem was given in [3]. Lecumberri et al. [10] dealt with a special case of (1), in which the input matrices A and B are structured in the sense that there exists a matrix $G\in\mathbb{C}^{(m+1)\times n}$ such that A consists of the last m rows of G and B consists of the first m rows of G.

The rest of this paper is organized as follows. Section 2 constitutes the main body of this paper, giving a sufficient condition for the existence of an optimal solution in (1) and an algorithm for the problem (1) based on TLS and SVD. Section 3 shows numerical results, which demonstrate the effectiveness of the proposed algorithm in comparison with Boutry et al.'s algorithm for the simpler problem (3), which is a relaxation of (1).

2 Proposed algorithm

In this section, we will show that the problem (1) can be reduced to TLS. Since $\mathrm{range}(A+R)\subseteq\mathrm{range}(B+E)$ in (2) holds if and only if there exists a matrix Z satisfying $(B+E)Z=A+R$, TLS can be rewritten as follows:

$$\begin{cases} \text{min.} & \|\,[E,R]\,\|_{\mathrm{F}}^2 \\ \text{s.t.} & E\in\mathbb{C}^{m\times n}, R\in\mathbb{C}^{m\times k}, Z\in\mathbb{C}^{n\times k}, \\ & (B+E)Z=A+R. \end{cases} \tag{5}$$

Throughout this paper, the following notation for matrices $M \in \mathbb{C}^{m\times n}, M' \in \mathbb{C}^{m\times k}$ is used:

- $\sigma_i(M)$: the ith largest singular value of M,
- M^*: the conjugate transpose of M,
- $[M, M']$: the $m \times (n+k)$ matrix consisting of M and M'.

The following theorem gives a solution to the problem (5) via the singular value decomposition.

Theorem 1 (Golub, Van Loan [8])**.** *Let $B \in \mathbb{C}^{m\times n}$, $A \in \mathbb{C}^{m\times k}$ and assume $m \geq n+k$. Let the singular value decomposition of $[B, A]$ be given by $U^*[B, A]V = \mathrm{diag}(\sigma_1, \ldots, \sigma_{n+k}) = \Sigma$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n+k} \geq 0$. Let U, V and Σ be partitioned as follows:*

$$U = [U_1\ U_2], \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_1 & O \\ O & \Sigma_2 \end{bmatrix}, \tag{6}$$

where $U_1 \in \mathbb{C}^{m\times n}$, $U_2 \in \mathbb{C}^{m\times k}$, $V_{11} \in \mathbb{C}^{n\times n}$, $\Sigma_1 \in \mathbb{C}^{n\times n}$, $V_{22} \in \mathbb{C}^{k\times k}$ and $\Sigma_2 \in \mathbb{C}^{k\times k}$. If $\sigma_n(B) > \sigma_{n+1}([B, A])$, then V_{22} is nonsigular and $(E, R, Z) = (E_0, R_0, Z_0)$ given by

$$[E_0, R_0] = -U_2\Sigma_2[V_{12}^*, V_{22}^*], \quad Z_0 = -V_{12}V_{22}^{-1} \tag{7}$$

is the unique optimal solution to the problem (5)*.*

Proof. See, for example, [9]. □

Our main result of this paper is that the problem (1) can be reduced to the problem (5), which is stated in the following theorem.

Theorem 2. *Let $A, B \in \mathbb{C}^{m\times n}$ and assume $m \geq 2n$, $\sigma_n(B) > \sigma_{n+1}([B, A])$. Let (E_0, R_0, Z_0) be the optimal solution of* (5)*, given by* (7)*. The optimal (infimum) value of* (1) *is equal to the optimal value $\|\ [E_0, R_0]\ \|_{\mathrm{F}}^2$ of* (5)*. If Z_0 is diagonalizable, then for eigenpairs $(\eta_1, \boldsymbol{w}_1), \ldots, (\eta_n, \boldsymbol{w}_n)$ of Z_0 with $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ linearly independent, $(\hat{A}, \hat{B}, \{(\lambda_k, \boldsymbol{x}_k)\}_{k=1}^n) = (A + R_0, B + E_0, \{(\eta_k, \boldsymbol{w}_k)\}_{k=1}^n)$ is an optimal solution to* (1)*. Otherwise,* (1) *has no solution attaining the optimal (infimum) value $\|\ [E_0, R_0]\ \|_{\mathrm{F}}^2$.*

Proof. Let $\hat{A}$, $\hat{B}$ and $\{(\lambda_k, \boldsymbol{x}_k)\}_{k=1}^n$ be a feasible solution to the problem (1), and define matrices $X, \Lambda \in \mathbb{C}^{n\times n}$ by $X = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$, $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$. Then the constraint of (1) is equivalent to that $\hat{A} = \hat{B}X\Lambda X^{-1}$. Hence, the set of all feasible $(\hat{A}, \hat{B})$ in (1), denoted by P_1, coincides with the set of all $(A', B') \in \mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n}$ satisfying $A' = B'Z$ for a diagonalizable matrix $Z \in \mathbb{C}^{n\times n}$. Define P_2 to be the set of $(A', B') \in \mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n}$ satisfying $A' = B'Z$ for a matrix $Z \in \mathbb{C}^{n\times n}$ that is not necessarily diagonalizable.

Then, P_2 contains P_1, and is contained in the closure $\overline{P_1}$ of P_1 since for an arbitrary square matrix $Z \in \mathbb{C}^{n\times n}$ and an arbitrary positive real number ϵ, there exists a diagonalizable Z' with $\|Z' - Z\|_{\mathrm{F}} \leq \epsilon$. Since $P_1 \subseteq P_2 \subseteq \overline{P_1}$ and $\|A' - A\|_{\mathrm{F}}^2 + \|B' - B\|_{\mathrm{F}}^2$ is a continuous function in (A', B'), we have

$$\inf_{(A',B')\in P_1} \{\|A' - A\|_{\mathrm{F}}^2 + \|B' - B\|_{\mathrm{F}}^2\} = \inf_{(A',B')\in P_2} \{\|A' - A\|_{\mathrm{F}}^2 + \|B' - B\|_{\mathrm{F}}^2\}$$
$$= \inf_{\mathrm{range}(A+R)\subseteq \mathrm{range}(B+E)} \|\ [E, R]\ \|_{\mathrm{F}}^2, \tag{8}$$

which means that (1) has the same optimal (infimum) value as (5).

Let (E_0, R_0, Z_0) be the optimal solution of (5), given by (7). If Z_0 is diagonalizable ($Z_0 = WHW^{-1}$, $W = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n]$, $H = \mathrm{Diag}(\eta_1, \ldots, \eta_n)$), then $(\hat{A}, \hat{B}, \{(\lambda_k, \boldsymbol{x}_k)\}_{k=1}^n) = (A + R_0, B + E_0, \{(\eta_k, \boldsymbol{w}_k)\}_{k=1}^n)$ is a feasible solution attaining the optimal (infimum) value (8), i.e., an optimal solution. Conversely, if (1) has an optimal solution $(\hat{A}, \hat{B}, \{(\lambda_k, \boldsymbol{x}_k)\}_{k=1}^n)$, then $(R, E, Z) = (\hat{A} - A, \hat{B} - B, [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]\mathrm{Diag}(\eta_1, \ldots, \eta_n)[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]^{-1})$ is the unique optimal solution to (5), which implies that Z_0 defined by (7) is diagonalizable. Hence, if Z_0 is not diagonalizable, (1) has no solution attaining the optimal value. □

We can compute eigenpairs of Z_0 by solving a generalized eigenvalue problem without computing Z_0. This is because the eigenpairs of Z_0 coincide with those of the square matrix pencil $V_{21}^* - \lambda V_{11}^*$, which is proved as follows. We use the well known fact that for an arbitrary nonsingular 2-by-2 block matrix $M = (M_{ij})_{j=1,2}^{i=1,2}$ with M_{22} also nonsingular, the Schur complement $S := M_{11} - M_{12}M_{22}^{-1}M_{21}$ is nonsingular and

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}^{-1} = \begin{bmatrix} S^{-1} & -S^{-1}M_{12}M_{22}^{-1} \\ -M_{22}^{-1}M_{21}S^{-1} & M_{22}^{-1} + M_{22}^{-1}M_{21}S^{-1}M_{12}M_{22}^{-1} \end{bmatrix}.$$

From this fact, nonsingularity of V_{22} implies that V_{11} is also nonsingular. Furthermore, from $V^*V = I_{2n}$, we have

$$V_{21}^*V_{22} + V_{11}^*V_{12} = O_n.$$

Multiplying $(V_{11}^*)^{-1}$ from the left and V_{22}^{-1} from the right to both sides of this equation, we obtain $-V_{12}V_{22}^{-1} = (V_{11}^*)^{-1}V_{21}^*$, and therefore Z_0 defined by (7) is equal to $(V_{11}^*)^{-1}V_{21}^*$. Hence, $(Z_0 - \lambda I_n)\boldsymbol{x} = \boldsymbol{0}$ if and only if $V_{11}^*(Z_0 - \lambda I_n)\boldsymbol{x} = (V_{21}^* - \lambda V_{11}^*)\boldsymbol{x} = \boldsymbol{0}$ for arbitrary $\lambda \in \mathbb{C}$ and $\boldsymbol{x} \in \mathbb{C}^n$. This means that we can obtain desired eigenpairs by solving the generalized eigenvalue problem $V_{21}^*\boldsymbol{x} = \lambda V_{11}^*\boldsymbol{x}$.

From the above arguments, we propose Algorithm 1. Theorem 2 guarantees that Algorithm 1 outputs the optimal solution to the problem (1) under the assumptions that $\sigma_n(B) > \sigma_{n+1}([B, A])$ and that the problem (1) has an optimal solution.

Algorithm 1 Algorithm of solving the problem (1) via TLS

Input: $A, B \in \mathbb{C}^{m \times n}$

1: Compute the singular value decomposition of $[B, A]$ to obtain U, Σ, V such that $[B, A] = U\Sigma V^*$.

2: Solve the generalized eigenvalue problem $V_{21}^* \boldsymbol{x} = \lambda V_{11}^* \boldsymbol{x}$ and output the eigenpairs, where V_{11} and V_{21} are submatrices given as (6).

3: If needed, compute E_0 and R_0 by (7) and output $\hat{A} = A + R_0$ and $\hat{B} = B + E_0$.

3 Numerical examples

In this section, we shall numerically compare Algorithm 1 with Boutry et al.'s algorithm [2], which we denote by BEGM algorithm. We used Matlab(R2008b) for numerical computations.

We mention some relevant properties of BEGM algorithm [2]. BEGM algorithm is designed to solve not (1) but (3). In many cases, this algorithm computes all local optimal solutions to (3). Note that one local optimal solution to (3) corresponds to one eigenvalue of the input pencil. In the noiseless case where $A - \lambda B$ has eigenvalues, (3) has local optimal solutions corresponding to each eigenvalue of the input pencil $A - \lambda B$. However, in the case of noisy data, optimal solutions will perturb and may end up ceasing to exist, which means that BEGM algorithm fails to catch some eigenvalues. The following numerical results show that such difficulty for BEGM algorithm is not imaginary. The proposed Algorithm 1, on the other hand, is free from this difficulty.

We create matrices $A_0, B_0 \in \mathbb{C}^{300 \times 5}$ ($m = 300$, $n = 5$) so that $A_0 - \lambda B_0$ has 5 eigenpairs and add noises to them by the following procedure [2]:

- Choose random matrices $\tilde{A}, \tilde{B} \in \mathbb{C}^{5 \times 5}$ and $\tilde{Q} \in \mathbb{C}^{300 \times 5}$ whose entries' real and imaginary parts are drawn from the zero mean independent Gaussian distribution with standard deviation 1.
- Compute the QR decomposition of $\tilde{Q}$ to define Q_0 as its Q part.
- Define A_0 and B_0 by $A_0 = Q_0\tilde{A}, B_0 = Q_0\tilde{B}$.
- Define A and B by $A = A_0 + N_A$ and $B = B_0 + N_B$, where N_A and N_B are matrices of random noise.

Real and imaginary parts of the entries of N_A and N_B are drawn from zero mean independent Gaussian distribution with standard deviation σ. Note that $A_0 - \lambda B_0$ has the same eigenpairs as the square matrix pencil $\tilde{A} - \lambda\tilde{B}$. We used matrices A_0, B_0 with condition numbers $20.0, 4.98$, respectively. The matrix pencil $A_0 - \lambda B_0$ has five eigenvalues $-0.49 - 2.59\mathrm{i}, -1.45 + 1.69\mathrm{i}, -1.78 - 0.24\mathrm{i}, -0.17 - 1.01\mathrm{i}, 0.53 + 0.35\mathrm{i}$. We prepared 10 data sets with different noise (N_A, N_B) with $\sigma = 0.25, 0.50, 0.75, 1.0, 1.25, 1.50$, respectively.

Figure 1 shows numerically computed eigenvalues by Algorithm 1 and BEGM algorithm [2]. "exact solution" means eigenvalues of the noiseless pencil $A_0 - \lambda B_0$, which is not necessarily equal to the optimal solution to the problem (1) with noisy input. Since we used 10 data sets for each σ, each figure has many points representing numerical solutions.

Our computational results show that, in the noiseless case, i.e., when $\sigma = 0$, the two algorithms compute "exact solution", i.e., eigenvalues of $A_0 - \lambda B_0$, accurately. As the noise level σ increases, discrepancy between numerical solutions and "exact solution" increases. Furthermore, we can see that the proposed algorithm is less affected by, i.e., robuster against, the data noise than BEGM algorithm. In fact, the influence of the data noise for BEGM algorithm is profound. In the case of $\sigma \geq 0.75$, BEGM algorithm misses a solution close to $\lambda_0 := -1.78 - 0.24\mathrm{i}$, one of the eigenvalues of $A_0 - \lambda B_0$, probably because (3) has no local optimal solution corresponding to λ_0. On the other hand, the proposed algorithm always gives five solutions even for noisy data, and gives better estimation for the eigenvalues of the noiseless pencil.

4 Conclusion

In this paper, we have considered the generalized eigenvalue problem for nonsquare pencils. We have focused on the minimal perturbation approach (1) of Boutry et al. and proposed an algorithm for the problem (1) which Boutry et al. first aimed to solve. The proposed algorithm is simple and gives good solutions according to our numerical results.

A challenging future work is to extend our results to the problem (4) containing a parameter l, which seems to be much more complicated than (1).

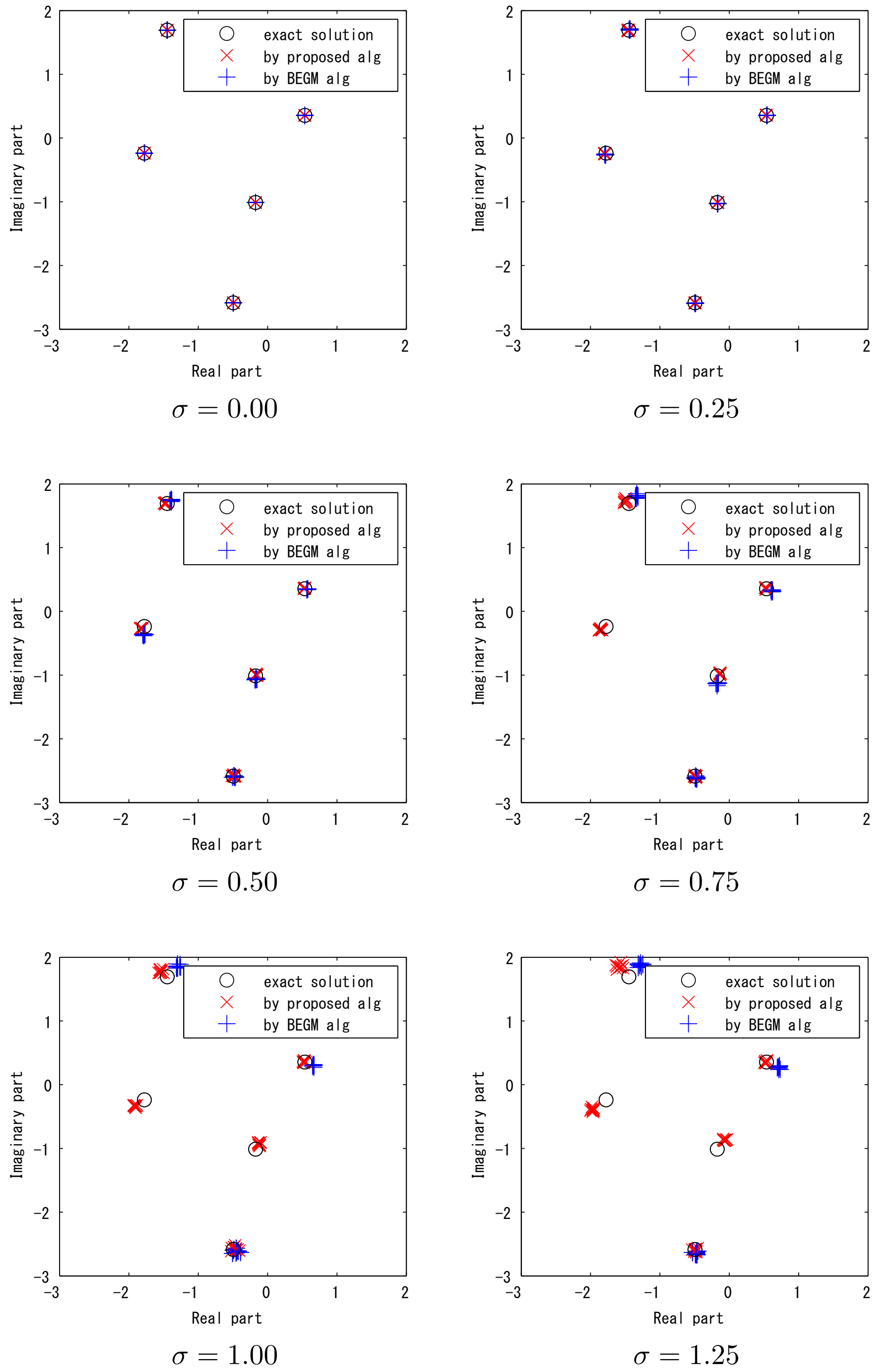

Figure 1: Eigenvalues computed by the two algorithms. "exact solution" means the eigenvalues of the noiseless pencil. σ represents the level of noise in inputs.

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