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Abstract

Choice functions are common tools in many-to-one or many-to-many matching models. They can represent more general preferences of agents than the classical form that consists of a list and a quota. Choice functions are usually assumed to satisfy the substitutability, which is an essential condition for the existence of stable matchings.

In this paper, we introduce "matroidal choice functions" as a class of choice functions which satisfy a kind of matroid constraints in addition to the substitutability. We show that matroidal choice functions admit succinct representations, with which one can find a stable matching efficiently utilizing a greedy algorithm for matroids.

Furthermore, we show that matroidal choice functions afford nice properties of stable matchings such as the strategy-proofness of the deferred acceptance algorithm, and the distributive lattice structure of the set of stable matchings.

1 Introduction

In the classical college admissions model of Gale–Shapley [18], the preference of a college is represented by a list and a quota. That is, a college has a complete ordering on individuals, and a fixed number $q \in \mathbf{N}$. Among available applicants, the college takes top q applicants if there are more than q applicants, and takes all of them if there are less than the quota.

To represent more general preferences, choice functions are introduced to the matching literature by Roth [35]. He found that if each agent's choice function enjoys the "substitutability" (or the "substitutes condition"), then the set of stable matchings is nonempty and the so-called deferred acceptance algorithm finds a stable matching. This class of choice functions is so general that it does not necessarily refer to any ordering on individuals nor quota. Since then, the substitutability has been a common assumption on choice functions in the matching literature.

However, with only the substitutability, many distinctive properties of the classical model [3, 8] cannot be extended. For example, the set of stable matchings is not necessarily a distributive lattice, and the lattice operations are complicated [9]. This implies the difficulty of extending algorithms to enumerate or optimize stable matchings [19]. Also, the strategy-proofness of the deferred acceptance algorithm [1, 13, 34] cannot be guaranteed only by the substitutability.

Furthermore, the choice function approach has another problem from the viewpoint of implementation. Since they are functions defined on powersets, naive representations of them need exponential space complexity. This makes it hard to deal with general choice functions in implementation.

There have been proposed intermediate choice function classes between the classical choices in the Gale-Shapley model and the substitutable choice functions as follows.

Quotafilling preferences In [4], Alkan generalized classical choices by removing orderings on individuals. Here, choice functions are imposed to be substitutable and "quotafilling."[1] Alkan showed that, for such choice functions, stable matchings form a distributive lattice and many other structural properties of the classical model can be extended.

Here is an example given in [6] to represent the difference between this class and the classical choice: The college can admit two applicants. The applicants are men m, m' and women w, w'. The college prefers $m(w)$ to $m'(w')$, and also the college admits members of both sexes as equally as possible. Hence, college's first choice is mw. However if the available set is

[1] The property such that "A college takes q applicants if there are more than q applicants, otherwise takes all of them."

$m'ww'(mm'w')$, then the choice is $m'w(mw')$. In the case of $m'ww'$, man m' is preferred to woman w', contrary to the case of $mm'w'$, in which w' is preferred to m'. Thus we cannot define an ordering on individuals.

Totally Ordered Matroids The choices in Fleiner's model [14] can be obtained by relaxing quota restrictions of the classical model to matroid constraints. That is, a preference of a college is represented by a pair of a total order on individuals and a matroid on them. Among available applicants, the college takes ones from the highest to the lowest preserving the matroid constraint.

Fleiner showed that for this class of choice functions, the set of stable matchings retains properties of the classical model such as the distributive lattice structure. Also, he exhibited a polyhedral description for the set of stable matchings [14].

Aside from the classical model, this choice function class can represent laminar classified model, for example. For this model with additional lower quotas, Huang [23] gave a polynomial time algorithm to find a stable matching (or report the nonexistence). Fleiner and Kamiyama [16] solved a generalized version of Huang's model by a matroid approach.

In this paper, we introduce a new class of choice functions, "matroidal choice functions," as a common generalization of the above two classes. A choice function in this class is imposed to be substitutable and to satisfy a matroid constraint. There is no ordering on individuals nor quota. As will be shown in Sections 4 and 5, this class has the following properties.

1. Matroidal choice functions include both of the above two classes of choice functions. Furthermore, there exist matroidal choice functions contained in neither of them (Examples 4.4 and 7.8).

2. Matroidal choice functions satisfy the "size-monotonicity" (or the "law of aggregate demand"), investigated in [5, 15, 22]. Using their results, we can assert that the many-to-one matching model with matroidal choice functions retains the following properties: the deferred acceptance algorithm is strategy-proof for agents of one side, the set of stable matchings is a distributive lattice, and the "rural hospital theorem"[2] holds (Propositions 5.8, 5.15, and 5.16, respectively).

3. Matroidal choice functions are preferable for implementation. We show that each matroidal choice function admits a succinct representation with the aid of the "de-cycle function," from which we can efficiently reconstruct the original choice function by a greedy algorithm for matroids

[2]This states that each hospital is assigned to the same number of contracts across all stable matchings. Also, if a hospital is assigned to less contracts than its quota, then it is assigned to the same set across all stable matchings [22, 26, 36].

(Propositions 4.6 and 4.8). Moreover, a stable matching can be found efficiently by using de-cycle functions instead of choice functions (Theorem 5.10). These properties help us to reduce the storage size in implementation.

De-cycle functions, introduced in this paper, are functions defined on circuit families of matroids. For each circuit, a de-cycle function returns one of its element. Assume that a de-cycle function indicates the worst element of each circuit, and consider the situation where we are given a subset of the ground set as a sequence of elements. To obtain an acceptable independent set, we can naturally conceive the following greedy algorithm: Start with the emptyset and add elements of the sequence from the first to the last, but whenever some circuit comes up, eliminate the element indicated by the de-cycle function.

In fact, this algorithm works well for some kind of de-cycle functions. For example, if there exist positive weights on elements and the de-cycle function returns the minimum weight element in each circuit, then the output of the above algorithm is the maximum weight independent subset of the given set no matter what the order of the sequence is. For general de-cycle functions, however, the output of the algorithm differs depending on the order of the sequence.

The "coherency" is the key property of de-cycle functions to characterize the independence of outputs from orders. See Section 6.2 for the precise definition. We show that the algorithm returns the unique output for each subset regardless of the order of the sequence if and only if the de-cycle function is coherent (Lemma 6.9). Furthermore, we show that a de-cycle function is coherent if and only if it gives a succinct representation of a matroidal choice function (Theorem 6.7). Equivalently, a choice function is matroidal if and only if it can be represented by outputs of the above algorithm for a certain coherent de-cycle function (Theorem 6.10). As a result, we obtain a one-to-one correspondence between matroidal choice functions and coherent de-cycle functions.

The above greedy algorithm shows a marked similarity to the known maximization algorithm for valuated matroids. Section 7 describes the relationship between matroidal choice functions and valuated matroids. Valuated matroids, introduced by Dress and Wenzel [10, 12], are matroids accompanied with valuations satisfying a special axiom. They are an extension of weighted matroids, but valuations are not limited to be modular. We show that the choice function defined by the maximizers of a valuated matroid is a matroidal choice function (Proposition 7.3). Then, we can realize that some known facts of valuated matroids, such as the validity of a greedy algorithm and the equivalence between local and global optimality, are obtained as special cases of our results in Section 4 (Remark 7.5).

The rest of this paper is organized as follows. In Sections 2 and 3, we provide preliminaries on choice functions and matroids, respectively. Our contribution starts with Section 4, in which we introduce matroidal choice functions and show their expressiveness as well as succinct representations with "de-cycle functions." In Section 5, we formulate a matching model with matroidal choice functions based on the model of Hatfield and Milgrom [22]. We give a variant of the deferred acceptance algorithm which uses de-cycle functions instead of choice functions, and show its strategy-proofness and efficiency. Section 6 is devoted to the characterization of matroidal choice functions via de-cycle functions. We show that matroidal choice functions can be characterized as choice functions which can be calculated validly by a natural greedy algorithm for matroids. In Section 7, we investigate the relationships between matroidal choice functions and valuated matroids.

2 Choice Functions

A *choice function* on a finite set E is a function $F : 2^E \to 2^E$ such that $F(X) \subseteq X$ for any $X \subseteq E$. We interpret $F(X)$ as the most preferred subset of X. A choice function F is said to be *substitutable* if it satisfies

(Sub) $X \subseteq Y \implies F(Y) \cap X \subseteq F(X)$.

This property was introduced to the matching literature[3] by Kelso–Crawford [24] and Roth [35]. It is now known as an essential condition for the existence of a stable matching [37]. One can easily confirm that the condition (Sub) is equivalent to

(Sub*) $X \subseteq Y \implies X \setminus F(X) \subseteq Y \setminus F(Y)$.

This says that an item rejected in some set will also be rejected if the set is expanded. We refer to both (Sub) and (Sub*) as the substitutability. It is easily seen that the substitutability implies the idempotence, i.e., a substitutable choice function F satisfies

$$F(F(X)) = F(X) \quad (\forall X \subseteq E).$$

A choice function F is said to be *size-monotone* if it satisfies

(Size) $X \subseteq Y \implies |F(X)| \leq |F(Y)|$,

which says that the number of chosen items does not decrease when available items increase. This property is also called "increasing property" or "law of aggregate demand." Its importance has been emphasized in several works such as [5, 15, 22]. The size-monotonicity yields, in conjunction with the substitutability, some favorable properties of stable matchings such as the strategy-proofness of the deferred acceptance algorithm [22], and the distributive lattice structure of the set of stable matchings [5, 15].

A substitutable and size-monotone choice function F is *consistent*,[4] i.e.,

$$F(X) \subseteq Y \subseteq X \implies F(Y) = F(X),$$

and *path-independent*,[5] i.e.,

$$F(F(X) \cup F(Y)) = F(X \cup Y) \quad (\forall X, Y \subseteq E).$$

[3]This property was originally studied outside the matching literature, and known as Chernoff's condition or Sen's α [28].

[4]See [7]. The consistency is also called "irrelevance of rejected contracts."

[5]The equivalence between the path-independence and the combination of the substitutability and the consistency was first noted in [2].

3 Matroids

This section provides preliminaries on matroids. See Oxley [33] for more information.

Let E be a finite set. For a family $\mathcal{I} \subseteq 2^E$, a pair $\mathbf{M} = (E, \mathcal{I})$ is called a *matroid* if it satisfies the following (I1)–(I3):

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I_1 \subseteq I_2 \in \mathcal{I}$, then $I_1 \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists an element $e \in I_2 \setminus I_1$ such that $I_1 + e \in \mathcal{I}$.

We call the family $\mathcal{I}$ the *independent set family* of $\mathbf{M}$. We say that a set $X \subseteq E$ is independent if $X \in \mathcal{I}$. Otherwise, we say that X is dependent.

For a set $X \subseteq E$, a maximal independent subset of X is called a *base* of X. We denote by $\mathcal{B}(X)$ the set of bases of X, i.e.,

$$\mathcal{B}(X) = \{ B \subseteq X \mid B \in \mathcal{I},\ B + e \notin \mathcal{I}\ (\forall e \in X \setminus B) \}.$$

In particular, we write $\mathcal{B}$ for $\mathcal{B}(E)$.

The *circuit family* $\mathcal{C} \subseteq 2^E$ of $\mathbf{M}$ is defined by

$$\mathcal{C} = \{ C \subseteq E \mid C \notin \mathcal{I},\ C - e \in \mathcal{I}\ (\forall e \in C) \},$$

and each member is called a circuit. That is, a circuit is a minimal dependent set. It is known that $\mathcal{C}$ satisfies the following (C1)–(C3):

(C1) $\emptyset \notin \mathcal{C}$.

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \neq C_2$, then for all $e \in C_1 \cap C_2$ there exists $C_e \in \mathcal{C}$ such that $C_e \subseteq (C_1 \cup C_2) - e$.

The *rank function* of $\mathbf{M}$ is a function $r : 2^E \to \mathbf{Z}$ defined by

$$r(X) = \max\{\, |I| : I \in \mathcal{I},\ \ I \subseteq X\} \quad (X \subseteq E).$$

It is known that r satisfies the following (R1)–(R3):

(R1) $\forall X \subseteq E, \quad 0 \leq r(X) \leq |X|$.

(R2) If $X \subseteq Y$, then $r(X) \leq r(Y)$.

(R3) $\forall X, Y \subseteq E, \quad r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$.

We call $r(E)$ the rank of $\mathbf{M}$, and write it $r(\mathbf{M})$.

In this paper, we often use the following properties of matroids.

Proposition 3.1. For an independent set I and an element $e \in E \setminus I$, the subset $I + e$ contains a unique circuit if $I + e \notin \mathcal{I}$. ∎

We write $C(I|e)$ for such a unique circuit contained in $I + e$.

Proposition 3.2. For any $X \subseteq E$, every $B \in \mathcal{B}(X)$ satisfies $|B| = r(X)$. ∎

4 Matroidal Choice Functions

In this section, we introduce a new class of choice functions as a common generalization of Alkan's quotafilling preferences [4] and the choices in Fleiner's totally ordered matroid model [14].

Alkan's quotafilling preference is a substitutable choice function which chooses as many elements as possible up to a quota $q \in \mathbf{N}$. Note that this quota restriction can be regarded as the matroid constraint with a uniform matroid of rank q.

Fleiner's totally ordered matroid model represents a preference of an agent by a matroid $\mathbf{M} = (E, \mathcal{I})$ and a total order $\succ$ on E. Given a subset $X \subseteq E$, the agent chooses a base of X by the greedy algorithm with respect to $\succ$. One can confirm that such a choice function satisfies the substitutability.

4.1 Definition

Here is the definition of "matroidal choice functions."

Definition 4.1. For a matroid $\mathbf{M} = (E, \mathcal{I})$, a function $F : 2^E \to 2^E$ is said to be a *matroidal choice function* on $\mathbf{M}$ if it satisfies the substitutability and $F(X) \in \mathcal{B}(X)$ for each $X \subseteq E$.

We simply say F is a matroidal choice function if there is such a matroid $\mathbf{M}$, which is called the *underlying matroid* of F.

The condition $F(X) \in \mathcal{B}(X)$ says that the function chooses a maximal independent set included in the given set X. Since this implies $|F(X)| = r(X)$, the axiom of rank function (R2) yields the following.

Observation 4.2. A matroidal choice function satisfies the size-monotonicity in addition to the substitutability. ■

For any $X \subseteq E$, we see that $X \in \mathcal{I}$ implies $\mathcal{B}(X) = \{X\}$ and that $X \notin \mathcal{I}$ implies $\mathcal{B}(X) \not\ni X$. This leads to the following.

Observation 4.3. For a matroidal choice function F on $\mathbf{M} = (E, \mathcal{I})$, a subset $X \subseteq E$ satisfies $F(X) = X$ if and only if $X \in \mathcal{I}$.

Given a matroidal choice function F, we can obtain its underlying matroid by setting $\mathcal{I} = \{\, X \subseteq E \mid F(X) = X \,\}$. ■

We see that both of quotafilling preferences and the choice functions defined by totally ordered matroids are subsumed in matroidal choice functions. Moreover, this new class is properly broader than the union of these two classes. The following example gives a matroidal choice function which is contained in neither of them. (Example 7.8 gives another example of such a matroidal choice function.)

Example 4.4. We give a choice function which represents a criterion of selecting students of some college.

The college has two courses, "architecture" and "business." Let A and B denote the disjoint sets of possible applicants for these courses, respectively, and E denote the union of them (i.e., $E = A \cup B$ and $A \cap B = \emptyset$). The capacities of the courses are q_A and q_B, respectively. Also, the college can accept at most q_E students in total. Here, q_E naturally satisfies $q_E \leq q_A + q_B$. The college also has the ideal proportion, i.e., the numbers p_A and p_B with

$$p_A + p_B = q_E, \quad p_A \leq q_A, \quad p_B \leq q_B.$$

We write M and W for the sets of men and women (i.e., $E = M \cup W$ and $M \cap W = \emptyset$). Each course wants to accept members of both sexes as equally as possible. Also each course has a total ordering $\succ_A$ on A and $\succ_B$ on B, respectively.

Assume that the applicants in $X \subseteq E$ apply for the college. Denote $X_A = X \cap A$ and $X_B = X \cap B$, respectively. Then the college decides whom to accept according to the following two steps.

Step 1. First, the college determines $n_A(X)$ and $n_B(X)$, the numbers of students the college will accept for each course, respectively. Let $N_A(X) = \min\{|X_A|, q_A\}$ and $N_B(X) = \min\{|X_B|, q_B\}$. If $N_A(X) + N_B(X) \leq q_E$, then let $n_A(X) = N_A(X)$ and $n_B(X) = N_B(X)$. Otherwise, let $(n_A(X), n_B(X))$ be the optimal solution of the following problem, which minimizes the gap from the ideal proportion keeping the sum of two variables being q_E:

$$\begin{aligned} \text{minimize} \quad & |n_A(X) - p_A| + |n_B(X) - p_B| \\ \text{subject to} \quad & n_A(X) \leq N_A(X), \\ & n_B(X) \leq N_B(X), \\ & n_A(X) + n_B(X) = q_E. \end{aligned} \tag{1}$$

Note that an optimal solution of this problem is unique.

Step 2. Next, two courses choose applicants from X_A and X_B, respectively. Here is a criterion of the architecture course which chooses $n_A(X)$ applicants from X_A. (Similar for business course.) Let $k = \lfloor \frac{1}{2} \cdot n_A(X) \rfloor$.

1. The case $|X_A \cap M| \geq k + 1$, $|X_A \cap W| \geq k + 1$. Take top k men and top k women with respect to $\succ_A$. When $n_A(X)$ is odd, i.e., $n_A(X) = 2k + 1$, add the better of the $(k + 1)$-th man and the $(k + 1)$-th woman with respect to $\succ_A$.
2. The case $|X_A \cap M| \leq k$. Take all men and top $n_A(X) - |X_A \cap M|$ women with respect to $\succ_A$.
3. The case $|X_A \cap W| \leq k$. Take all women and top $n_A(X) - |X_A \cap W|$ men with respect to $\succ_A$.

Let $F(X)$ be the union of the choices of two courses. By the definition, $F(X) \subseteq X$ holds for any $X \subseteq E$ obviously, so F is a choice function. As is shown in Appendix A, this F is actually a matroidal choice function. ■

4.2 Representation and construction

In this section, we show that each matroidal choice function has a kind of compressed representation, which we call a "de-cycle function," and that one can reconstruct the choice function from the de-cycle function efficiently.

Consider a matroidal choice function $F : 2^E \to 2^E$ on a matroid $\mathbf{M} = (E, \mathcal{I})$ and let $\mathcal{C}$ denote the circuit family of $\mathbf{M}$.

Definition 4.5. A function $\delta : \mathcal{C} \to E$ is called a *de-cycle function* if it satisfies $\delta(C) \in C$ for each $C \in \mathcal{C}$. ■

Since F is a matroidal choice function, we have $F(C) \in \mathcal{B}(C)$ for each $C \in \mathcal{C}$, and hence $|F(C)| = r(C) = |C| - 1$. Then, the set $C \setminus F(C)$ is a singleton. Define a de-cycle function $\delta_F : \mathcal{C} \to E$ by letting $\delta_F(C)$ be the only element in $C \setminus F(C)$ for each $C \in \mathcal{C}$. That is, identifying a singleton with its element, a de-cycle function δ_F is defined by

$$\delta_F(C) = C \setminus F(C) \quad (C \in \mathcal{C}).$$

We call δ_F the *associated* de-cycle function of F.

Proposition 4.6. For any $X \subseteq E$, we have

$$F(X) = X \setminus \{\, \delta_F(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}. \tag{2}$$

Proof. Let $R(X) := \{\, \delta_F(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}$. For any $e \in R(X)$, there exists $C \in \mathcal{C}$ such that $\delta_F(C) = e$ and $C \subseteq X$. Then $\{\delta_F(C)\} = C \setminus F(C)$ holds by the definition of δ_F. By (Sub*), this implies $e = \delta_F(C) \in X \setminus F(X)$. Therefore, we have $R(X) \subseteq X \setminus F(X)$ which implies $F(X) \subseteq X \setminus R(X)$.

Since $R(X)$ contains at least one element of each circuit, the set $X \setminus R(X)$ does not include any circuit, and so is independent. Then $|X \setminus R(X)| \leq r(X) = |F(X)|$ holds by $F(X) \in \mathcal{B}(X)$. With $F(X) \subseteq X \setminus R(X)$, this implies $F(X) = X \setminus R(X)$ which means (2). □

Note that a circuit C is a minimal subset from which F discards some element, and $\delta_F(C)$ is the unique discarded element. The formula (2) says that $F(X)$ can be obtained by eliminating all such elements from X.

Proposition 4.7. Let Y be an arbitrary subset of E. A subset $X \subseteq Y$ satisfies $X = F(Y)$ if and only if the following two conditions hold:
(i) $X \in \mathcal{I}$, (ii) $\forall e \in Y \setminus X,\ [\ X + e \notin \mathcal{I},\ \delta_F(C(X|e)) = e\ ]$.

Proof. Since $F(Y) \in \mathcal{B}(Y)$, the condition $F(Y) = X$ implies $X \in \mathcal{I}$ and $X + e \notin I$ ($\forall e \in Y \setminus X$). Then, the "only if" part is implied by Proposition 4.6. We now show the "if" part. By (ii), every $e \in Y \setminus X$ satisfies $e = \delta_F(C(X|e))$. As $C(X|e) \subseteq Y$, this implies $e \notin F(Y)$ by Proposition 4.6. Hence, we obtain $Y \setminus X \subseteq Y \setminus F(Y)$, and so $X \supseteq F(Y)$. Since $F(Y)$ is a maximal independent set included in Y, (i) implies $X = F(Y)$. □

We design an algorithm MCF(δ) as follows for any de-cycle function $\delta : \mathcal{C} \to E$. Here, S_k means the set of all permutations on $\{1, 2, \ldots, k\}$.

Algorithm: MCF(δ)
Input: $X = \{e_1, e_2, \ldots, e_k\} \subseteq E$ and $\pi \in S_k$.

1. $J \leftarrow \emptyset$.
2. For $i = 1$ to k, do:
   (a) If $J + e_{\pi(i)} \in \mathcal{I}$, then $J \leftarrow J + e_{\pi(i)}$.
   (b) Otherwise, $J \leftarrow J + e_{\pi(i)} - \delta(C(J|e_{\pi(i)}))$.
3. Return J.

This algorithm simulates the matroidal choice function F when the associated de-cycle function δ_F is used.

Proposition 4.8. The algorithm MCF(δ_F) returns $F(X)$ for any $X \subseteq E$ and any $\pi \in S_k$.

Proof. Let us denote J_i for J in the algorithm just after i-th Step 2. Note that the output is J_k. By the algorithm, we see that J_i is independent for any $i \in \{0, 1, \ldots, k\}$. This implies $|J_k| \leq r(X) = |F(X)|$ since $F(X) \in \mathcal{B}(X)$.

By the algorithm, $e \in X \setminus J_k$ means that there is some $i \in \{1, 2, \ldots, k\}$ such that $\delta_F(C(J_{i-1}|e_{\pi(i)})) = e$. Then, $e \in X \setminus F(X)$ by Proposition 4.6. Thus we have $X \setminus J_k \subseteq X \setminus F(X)$, and so $J_k \supseteq F(X)$. With $|J_k| \leq |F(X)|$, this implies $J_k = F(X)$. □

5 Matching model with matroidal choice functions

Here we introduce a many-to-one matching model with matroidal choice functions based on the model of Hatfield and Milgrom [22].

In our model, there are two types of agents, "doctors" and "hospitals" (they correspond to "students" and "colleges" of the classical model [18]), and preferences of hospitals are represented by matroidal choice functions.

We give an algorithm which finds the so-called doctor-optimal stable matching using de-cycle functions. We also present results on the strategy-proofness and the lattice structure of the set of stable matchings.

5.1 Matching model

An instance of our matching model is provided as a tuple $(D, H, E, \{\succ_d\}_{d \in D}, \{F_h\}_{h \in H})$ with finite sets D and H which represent sets of doctors and hospitals, respectively. A finite set E denotes a set of contracts. Each contract $e \in E$ is bilateral, i.e., it is associated with one doctor $e_D \in D$ and one hospital $e_H \in H$. We write $E_d = \{\, e \in E \mid e_D = d \,\}$ for each $d \in D$ and $E_h = \{\, e \in E \mid e_H = h \,\}$ for each $h \in H$. Also, for any $X \subseteq E$, we use the notation $X_d = X \cap E_d$ and $X_h = X \cap E_h$ for each agent.

Each doctor $d \in D$ is assigned to at most one contract, and so his preference is represented by a total order $\succ_d$ on $E_d \cup \{\bot\}$. The symbol $\bot$ represents unemployment and is placed at the bottom of the order[6]. For example, a preference of d is like $e \succ_d e' \succ_d e'' \succ_d \bot$, and $e \succ_d e'$ means that d prefers e to e'. We use the notation $e \succeq_d e'$ to mean $e \succ_d e'$ or $e = e'$.

Each hospital $h \in H$ can be assigned to multiple contracts, but to at most one with each doctor. His preference is represented by a matroidal choice function $F_h : 2^{E_h} \to 2^{E_h}$ satisfying $|F_h(X_h) \cap E_d| \leq 1$ for each $d \in D$ and $X_h \subseteq E_h$.

Definition 5.1. A set of contracts $X \subseteq E$ is said to be *doctor-feasible* if $|X_d| \leq 1$ $(\forall d \in D)$, and *hospital-feasible* if $F_h(X_h) = X_h$ $(\forall h \in H)$. We call X a *matching* if it is both doctor-feasible and hospital-feasible. ■

For a doctor-feasible set $X \subseteq E$ and a doctor $d \in D$, we write x_d for the unique element in X_d if $X_d \neq \emptyset$, and otherwise we let $x_d = \bot$.

Definition 5.2. A matching $X \subseteq E$ is said to be (pairwise[7]) *stable* if there is no contract $e \in E \setminus X$ such that

$$e \succ_d x_d \quad \text{and} \quad e \in F_h(X_h + e) \tag{3}$$

with $d = e_D$ and $h = e_H$. ■

[6] In the model of [22], there may exist contracts which are worse than unemployment. However, we can remove such contracts without changing the stable matchings.

[7] There is also the notion of "setwise stability," however, it is equivalent to the pairwise stability in this setting. See [22, 38].

Recall that F_h is a matroidal choice function for each $h \in H$. Let $\mathbf{M}_h$ be the underlying matroid and δ_h be the associated de-cycle function of F_h. For any doctor-feasible set $X \subseteq E$, let ${}^{D}\overline{X}$ be the set of contracts which are preferable to the current ones for doctors, i.e.,

$$ {}^{D}\overline{X} = \{\, e \in E \mid e \succ_d x_d \ \text{ for } \ d = e_D \,\} . $$

Then, the stability defined above can be rephrased as follows.

Lemma 5.3. A doctor-feasible set $X \subseteq E$ is a stable matching if and only if it satisfies the following two conditions for each $h \in H$:

(i) $X_h \in \mathcal{I}_h$,

(ii) $\forall e \in E_h \setminus X_h, \quad e \in {}^{D}\overline{X_h} \implies [\ X_h + e \notin \mathcal{I}_h,\ \delta_h(C(X_h|e)) = e\ ]$.

Also, a combination of (i) and (ii) is equivalent to $F_h(X_h \cup {}^{D}\overline{X_h}) = X_h$.

Proof. By Observation 4.3, $F_h(X_h) = X_h$ is equivalent to $X_h \in \mathcal{I}_h$. Hence X is hospital-feasible if and only if (i) holds for any $h \in H$. Also, for $X_h \in \mathcal{I}_h$ and $e \in E_h \setminus X_h$, the condition $F_h(X_h + e) \not\ni e$ is equivalent to $[X_h + e \notin \mathcal{I}_h,\ \delta_h(C(X_h|e)) = e]$ by Proposition 4.6. Hence, the negation of (3) is equivalent to the condition in (ii), and the first claim is shown.

The second claim follows from Proposition 4.7. □

5.2 Matroidal deferred acceptance algorithm

One can find a stable matching by using $\{(\mathbf{M}_h, \delta_h)\}_{h \in H}$ instead of $\{F_h\}_{h \in H}$. The following algorithm is a variant of the deferred acceptance algorithm [18]. In our algorithm, $X \subseteq E$ represents a temporary matching and $R \subseteq E$ represents the set of contracts rejected by hospitals until then.

Algorithm: MDA

Input: $(D, H, E, \{\succ_d\}_{d \in D}, \{(\mathbf{M}_h, \delta_h)\}_{h \in H})$.

1. $X \leftarrow \emptyset,\ R \leftarrow \emptyset$.
2. While there exists $d \in D$ such that $x_d = \bot$ and $E_d - R_d \neq \emptyset$, repeat the following:
   (a) Take such d and $e \leftarrow \max_{\succ_d}(E_d - R_d)$, $h \leftarrow e_H$.
   (b) If $X_h + e \in \mathcal{I}_h$, then $X \leftarrow X + e$.
   (c) Otherwise, $X \leftarrow X + e - \delta_h(C(X_h|e))$ and $R \leftarrow R + \delta_h(C(X_h|e))$.
3. Return X.

Remark 5.4. This algorithm is a generalization of the "recursive algorithm" of McVitie–Wilson [27], and also a variant of the "cumulative offer process" of Hatfield–Kojima [21]. Similarly to these algorithms, our algorithm lets only one doctor offer to a hospital in each step.

Let X^* and R^* denote X and R, respectively, at the termination of the algorithm.

Claim 5.5. ${}^{D}\overline{X^*} = R^*$.

Proof. By the algorithm, there is no doctor $d \in D$ which satisfies both $x_d^* = \bot$ and $E_d - R_d^* \neq \emptyset$. Hence, $x_d^* = \bot$ holds only when $E_d = R_d^*$. Also, observe that the algorithm takes contracts of E_d from the top to the bottom. Then, for any $e \in E_d$, the condition $e \succ_d x_d^*$ is equivalent to $e \in R^*$. Thus, we have ${}^{D}\overline{X_d^*} = R_d^*$ for any $d \in D$, and hence ${}^{D}\overline{X^*} = R^*$. □

Theorem 5.6. The output X^* is a stable matching.

Proof. We can observe that X is doctor-feasible throughout the algorithm. Hence, by Lemma 5.3, it suffices to show that $F_h(X_h^* \cup {}^{D}\overline{X_h^*}) = X_h^*$ for any $h \in H$. Note that the way to update X_h in Step 2 can be identified with Step 2 of MCF(δ_h). Hence we have $F_h(X_h^* \cup R_h^*) = X_h^*$ by Proposition 4.8. By Claim 5.5, this means $F_h(X_h^* \cup {}^{D}\overline{X^*}_h) = X_h^*$. □

Define a relation $\succeq_D$ on matchings by $X \succeq_D Y \iff x_d \succeq_d y_d$ ($\forall d \in D$). Then, it is clearly a partial order.

Theorem 5.7. The output X^* is a doctor-optimal stable matching. That is, X^* satisfies $X^* \succeq_D Y$ for every stable matching Y.

Proof. Let Y be an arbitrary stable matching. Note that $X^* \succeq_D Y$ is equivalent to ${}^{D}\overline{X^*} \cap Y = \emptyset$, and also this is equivalent to $R^* \cap Y = \emptyset$ by Claim 5.5. Hence it suffices to show that the condition ($\star$) $R \cap Y = \emptyset$ holds at any time of the algorithm. We prove this by induction on R. Assume that ($\star$) holds for the current R. To prove that the next updated R still satisfies ($\star$), we show that $X_h + e \notin \mathcal{I}_h$ implies $\delta_h(C(X_h|e)) \notin Y_h$ for every $h \in H$ and $e \in E_h \setminus R_h$.

By the algorithm, any $d \in D$ and any $e', e'' \in E_d$ satisfy

$$[\ e' \in X + e, \quad e'' \succ_d e'\ ] \implies e'' \in R. \tag{4}$$

Then, any $e' \in (X + e) \cap E_d$ satisfies $e' \succeq_d y_d$, because otherwise (4) implies $y_d \in R$ which contradicts ($\star$). Thus, we obtain $X + e \subseteq Y \cup {}^{D}\overline{Y}$. Then, $X_h + e \notin \mathcal{I}_h$ implies $C(X_h|e) \subseteq Y_h \cup {}^{D}\overline{Y_h}$. By Proposition 4.6, this implies

$$\delta_h(C(X_h|e)) \notin F_h(Y_h \cup {}^{D}\overline{Y}_h). \tag{5}$$

On the other hand, since Y is a stable matching, we have $F_h(Y_h \cup {}^{D}\overline{Y_h}) = Y_h$ by Lemma 5.3. Then, (5) means $\delta_h(C(X_h|e)) \notin Y_h$. □

For an algorithm which returns a matching for any instance, we call it *strategy-proof for doctors* if no doctor can improve his assignment by reporting a false preference. That is, for a strategy-proof algorithm, there is no doctor $d \in D$ and preference $\succ'_d$ such that the algorithm assigns a better contract (w.r.t. $\succ_d$) to d if the instance is modified by replacing $\succ_d$ with $\succ'_d$.

Proposition 5.8. The algorithm MDA is strategy-proof for doctors.

Proof. This is shown by an adaptation of the results of Hatfield and Milgrom [22]. For any algorithm which returns the doctor-optimal stable matching, they proved that it is strategy-proof if choice functions are substitutable and size-monotone. Since matroidal choice functions satisfy these properties by Observation 4.2, the proof is completed by Theorems 5.6 and 5.7. □

Remark 5.9. There is also the notion of "group strategy-proofness" for matching algorithms. This represents the nonexistence of a group of doctors such that each of them can improve their assignment by jointly misreporting their preferences. By [20], we can show that the algorithm MDA is also group strategy-proof.

5.3 Time complexity

Assume that matroidal choice functions $\{F_h\}_{h \in H}$ are represented by corresponding $\{(\mathbf{M}_h, \delta_h)\}_{h \in H}$ and that each $\mathbf{M}_h$ is given by an independence oracle $\mathcal{O}_h$. Given a subset $X \subseteq E_h$, the oracle $\mathcal{O}_h$ returns "yes" if X is independent, and otherwise it returns some circuit included in X as a certificate of the dependency.

Theorem 5.10. The algorithm MDA finds a doctor-optimal stable matching in $O(|E|)$ time, provided that each oracle call takes constant time.

Proof. In the algorithm, each $e \in E$ is chosen in Step 2 at most once, and so Step 2 is repeated at most $|E|$ times. Also, Step 2 needs only constant time. Hence the algorithm needs $O(|E|)$ in total. Note that $\mathcal{O}_h(X_h + e)$ is either "yes" or the circuit $C(X_h|e)$, since $X_h + e$ includes at most one circuit. □

Theorem 5.11. For a subset $X \subseteq E$, one can determine whether X is a stable matching or not in $O(|E|)$ time.

Proof. To determine the stability of X, it suffices to check doctor-feasibility and whether (i) and (ii) of Lemma 5.3 hold for each $h \in H$. These need $O(|E|)$ time in total. □

Remark 5.12. It may be more common that an independence oracle returns only "no" for a dependent set without giving a certificate circuit. Even with such an oracle, MDA needs only $O(|E| \cdot\ r_{\max})$ time, where $r_{\max} := \max_{h \in H} r(\mathbf{M}_h)$. This is because $C(X_h|e) = \{\, e' \in X_h + e \mid X_h + e - e' \in \mathcal{I}_h \,\}$ which can be obtained by at most $r_{\max} + 1$ oracle calls. ■

Remark 5.13. We have presented the algorithm to find a stable matching using independence oracles and de-cycle functions, and shown its efficiency. Using de-cycle functions is also superior in terms of space complexity since the domain of a de-cycle function (i.e., the circuit family) is substantially smaller than that of the choice function (i.e., the powerset of the ground set). ■

5.4 Structure of the set of stable matchings

Alkan [5] and Fleiner [14, 15] studied the structure of the set of stable matchings for models with substitutable and size-monotone choice functions. Since matroidal choice functions satisfy these two properties (Observation 4.2), the following three propositions immediately follow from their results (see [5, 15] for the proofs).

Proposition 5.14. Stable matchings X and Y satisfy $X \succeq_D Y$ if and only if they satisfy $F_h(X_h \cup Y_h) = Y_h$ for each $h \in H$. ■

Proposition 5.15. The set of all stable matchings forms a distributive lattice under the order $\succeq_D$. Moreover, for stable matchings X and Y, their join $X \vee_D Y$ and meet $X \wedge_D Y$ can be obtained by

$$(X \vee_D Y)_d = \max_{\succeq_d}\{x_d, y_d\}, \quad (X \wedge_D Y)_d = \min_{\succeq_d}\{x_d, y_d\} \quad (d \in D).$$

■

Proposition 5.16. Stable matchings X and Y satisfy $x_d = \bot \iff y_d = \bot$ for each $d \in D$ and $|X_h| = |Y_h|$ for each $h \in H$. ■

These propositions imply the following corollary.

Corollary 5.17. Let U be the union of all stable matchings. Then, every stable matching X satisfies $X_h \in \mathcal{B}_h(U_h)$ for each $h \in H$, where $\mathcal{B}_h(U_h)$ denotes the set of bases of U_h in $\mathbf{M}_h$.

Proof. By Proposition 5.15, there is a minimum stable matching with respect to $\succeq_D$. Let us denote it by V. Take $h \in H$ arbitrarily and let $\mathcal{I}_h$ and r_h respectively denote the independent set family and the rank function of $\mathbf{M}_h$. For any stable matching X, since it satisfies $X \succeq_D V$, we have $F_h(X_h \cup V_h) = V_h$ by Proposition 5.14. Then, the substitutability implies

$$X_h \setminus V_h = (X_h \cup V_h) \setminus F_h(X_h \cup V_h) \subseteq U_h \setminus F_h(U_h).$$

Since this holds for every stable matching, we obtain $U_h \setminus V_h \subseteq U_h \setminus F_h(U_h)$, and hence $V_h \supseteq F_h(U_h)$. By $V_h \in \mathcal{I}_h$, $V_h \subseteq U_h$ and $F_h(U_h) \in \mathcal{B}_h(U_h)$, this implies $V_h = F_h(U_h)$. Thus, we have $|V_h| = r_h(U_h)$.

For every stable matching X and every $h \in H$, we have $X_h \subseteq U_h$ and $X_h \in \mathcal{I}_h$ by the definition, and $|X_h| = |V_h| = r_h(U_h)$ by Proposition 5.16. These imply $X_h \in \mathcal{B}_h(U_h)$. □

If $\mathbf{M}_h$ is a uniform matroid of rank $q \in \mathbf{N}$, Corollary 5.17 implies that: If some stable matching X satisfies $|X_h| < q$, then every stable matching Y satisfies $X_h = Y_h$. Combined with Proposition 5.16, this means the so-called "rural hospital theorem."

6 Characterization

In this section, we introduce an axiom of de-cycle functions called "coherency" with which we can characterize matroidal choice functions. We give a one-to-one correspondence between coherent de-cycle functions and matroidal choice functions. We also characterize matroidal choice functions via the algorithm MCF described in Section 4.

6.1 Minimal Pair of Circuits

We first introduce the concept of "minimal pair of circuits."[8] Let $\mathbf{M} = (E, \mathcal{I})$ be a matroid and $\mathcal{C}$ be its circuit family.

Definition 6.1. A pair of circuits $(C_1, C_2) \in \mathcal{C} \times \mathcal{C}$ is said to be *minimal* if it satisfies the following two conditions:

1. $C_1 \neq C_2$.
2. There exists no pair of circuits $(C'_1, C'_2) \in \mathcal{C} \times \mathcal{C}$ such that $C'_1 \neq C'_2$ and $(C'_1 \cup C'_2) \subsetneq (C_1 \cup C_2)$.

■

For a uniform matroid, a pair of circuits (C_1, C_2) is minimal if and only if $|C_1 \setminus C_2| (= |C_2 \setminus C_1|) = 1$. For a graphic matroid, a pair of circuits (i.e., cycles) (C_1, C_2) is minimal if and only if it satisfies one of the following two conditions: (a) C_1 and C_2 are disjoint; (b) $C_1 \cup C_2$ forms a theta graph, i.e., a graph which consists of three internally disjoint simple paths that have the same two distinct end vertices.

In Proposition 6.3, we show that this minimality can also be represented by the rank function using the following lemma.

Lemma 6.2. A subset $X \subseteq E$ includes two or more circuits if and only if it satisfies $r(X) \leq |X| - 2$.

Proof. The "if" part is clear since we have $r(X) = |X|$ if X includes no circuit, and $r(X) = |X| - 1$ if X includes only one circuit.

For the"only if" part, assume X includes two distinct circuits C_1 and C_2. We prove $r(X) \leq |X| - 2$ by showing that $X - e$ includes a circuit for any $e \in X$. The case $e \notin C_1$ or $e \notin C_2$ is trivial. If $e \in C_1 \cap C_2$, then there is a circuit $C \subseteq (C_1 \cup C_2) - e \subseteq X - e$ by the axiom (C3) of circuits. □

Proposition 6.3. Any two distinct circuits C_1 and C_2 satisfy

$$r(C_1 \cup C_2) \leq |C_1 \cup C_2| - 2, \tag{6}$$

and the equality holds if and only if the pair (C_1, C_2) is minimal.

[8] A similar concept is used in [11, 31] to characterize valuated matroids.

Proof. Lemma 6.2 immediately implies (6). Then we show that the equality of (6) holds if and only if (C_1, C_2) is minimal.

The "if" part: If (C_1, C_2) is minimal, then $C_1 \neq C_2$. Take $e_1 \in C_1 \setminus C_2$ and $e_2 \in C_2 \setminus C_1$ arbitrarily. Then, there is no circuit C such that $C \subseteq (C_1 \cup C_2) \setminus \{ e_1, e_2 \}$ since such C satisfies $C_1 \neq C$ and $(C_1 \cup C) \subsetneq (C_1 \cup C_2)$ which contradicts the minimality of (C_1, C_2). Hence $(C_1 \cup C_2) \setminus \{ e_1, e_2 \}$ is independent, so $r(C_1 \cup C_2) \geq |C_1 \cup C_2| - 2$. With (6), the equality holds.

The "only if" part: Assume $r(C_1 \cup C_2) = |C_1 \cup C_2| - 2$. For any $e \in C_1 \cup C_2$, we have $r((C_1 \cup C_2) - e) = r(C_1 \cup C_2)$, and hence $r((C_1 \cup C_2) - e) = |(C_1 \cup C_2) - e| - 1$. By Lemma 6.2, this implies that $(C_1 \cup C_2) - e$ cannot include two distinct circuits. Therefore (C_1, C_2) is minimal. □

The following proposition is used in the subsequent section.

Proposition 6.4. For two distinct circuits C_1 and C_0, there exists a circuit $C_2 \subseteq C_1 \cup C_0$ such that (C_1, C_2) is minimal. In addition, for any $e \in C_1$, there exists such a circuit C_2 which satisfies $e \notin C_2$.

Proof. Among all circuits which are included in $C_1 \cup C_0$ and distinct from C_1, let C_2 be the one that minimizes $|C_1 \cup C_2|$.

First, we show that (C_1, C_2) is minimal. Suppose, to the contrary, it is not minimal. Then, $r(C_1 \cup C_2) \leq |C_1 \cup C_2| - 3$ by Proposition 6.3. Take $f \in C_2 \setminus C_1$ arbitrarily. Then $r(C_1 \cup C_2 - f) = r(C_1 \cup C_2) \leq |C_1 \cup C_2| - 3 = |C_1 \cup C_2 - f| - 2$. Hence, there is a circuit $C_2' \subseteq C_1 \cup C_2 - f$ with $C_2' \neq C_1$ by Lemma 6.2. Such C_2' satisfies $|C_1 \cup C_2'| \leq |C_1 \cup C_2 - f| < |C_1 \cup C_2|$, a contradiction.

Next, we show that, for each $e \in C_1$, there is a circuit $C_e \subseteq C_1 \cup C_0$ such that (C_1, C_e) is minimal and $e \notin C_e$. This is obvious if the above C_2 satisfies $e \notin C_2$. If not, then we have $e \in C_1 \cap C_2$, and hence $(C_1 \cup C_2) - e$ includes a circuit by the axiom (C3). Let C_e be such a circuit. Then, the pair (C_1, C_e) satisfies $C_1 \cup C_e \subseteq C_1 \cup C_2$, and hence it is minimal, because otherwise the minimality of (C_1, C_2) fails. □

6.2 Characterization via de-cycle functions

We now introduce an axiom of de-cycle functions which characterizes matroidal choice functions.

Definition 6.5. A de-cycle function $\delta : \mathcal{C} \to E$ is said to be *coherent* if it satisfies the following condition:

(D) For a minimal pair $(C_1, C_2) \in \mathcal{C} \times \mathcal{C}$ and $C_3 \in \mathcal{C}$ with $C_3 \subseteq C_1 \cup C_2$,

$$|\{\delta(C_1), \delta(C_2), \delta(C_3)\}| \leq 2.$$

■

The inequality in (D) says that, among $\delta(C_1)$, $\delta(C_2)$, and $\delta(C_3)$, at least two of them coincide. Since the minimality of a pair of circuits represents a kind of closeness, this axiom can be translated as follows: If three circuits are close to each other, then the same element is selected from at least two of them.

Lemma 6.6. For a matroidal choice function $F : 2^E \to 2^E$ on $\mathbf{M}$, its associated de-cycle function $\delta_F : \mathcal{C} \to E$ is coherent.

Proof. Suppose, to the contrary, that there is a minimal pair (C_1, C_2) and a circuit $C_3 \subseteq C_1 \cup C_2$ such that $\delta(C_1)$, $\delta(C_2)$, and $\delta(C_3)$ are all distinct. By Proposition 4.6, this implies $|F(C_1 \cup C_2)| \leq |C_1 \cup C_2| - 3$. However, we have $F(C_1 \cup C_2) \in \mathcal{B}(C_1 \cup C_2)$ and this implies $|F(C_1 \cup C_2)| = r(C_1 \cup C_2) = |C_1 \cup C_2| - 2$ by Proposition 6.3, a contradiction. □

By Lemma 6.6 and Proposition 4.6, we see that matroidal choice functions can be represented by coherent de-cycle functions. Actually, this property characterizes matroidal choice functions.

Theorem 6.7. A function $F : 2^E \to 2^E$ is a matroidal choice function on $\mathbf{M}$ if and only if it can be represented as

$$F(X) = X \setminus \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\} \quad (X \subseteq E) \tag{7}$$

by some coherent de-cycle function $\delta : \mathcal{C} \to E$. In particular, for a given matroidal choice function F, such a coherent de-cycle function δ is unique.

Proof. The"only if" part: For a matroidal choice function F, the associated de-cycle function δ_F satisfies (7) as δ by Proposition 4.6 and it is coherent by Lemma 6.6. Also, δ_F is the only de-cycle function which satisfies (7) since (7) forces $\delta(C)$ to be the unique element in $C \setminus F(C)$ for each $C \in \mathcal{C}$.

The "if" part: We show that if d is coherent, then the function F defined by (7) satisfies (Sub*) and $F(X) \in \mathcal{B}(X)$ $(\forall X \subseteq E)$. The condition (Sub*), i.e., $X \subseteq Y \implies X \setminus F(X) \subseteq Y \setminus F(Y)$ immediately follows from the form of (7), and the rest is shown by Lemma 6.8 below. □

Lemma 6.8. For a coherent de-cycle function $\delta : \mathcal{C} \to E$ and any $X \subseteq E$, the subset $F(X)$ defined by (7) satisfies $F(X) \in \mathcal{B}(X)$.

Proof. By (7), at least one element of each circuit is eliminated from $F(X)$, and hence $F(X)$ is independent. We now show $|F(X)| = r(X)$ $(\forall X \subseteq E)$ which completes the proof. We use an induction. Clearly $|F(\emptyset)| = r(\emptyset) = 0$. Then it suffices to show that

$$|F(X)| = r(X) \tag{8}$$

implies

$$|F(X + e)| = r(X + e) \tag{9}$$

for each $e \in E \setminus X$.

There are two cases: (i) $r(X + e) = r(X) + 1$, and (ii) $r(X + e) = r(X)$.

Case (i) As $r(X+e) = r(X)+1$, there is no circuit C such that $e \in C \subseteq X+e$. Hence, we have $\{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X+e \,\} = \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}$ which implies $F(X+e) = F(X)+e$ by (7). As we have (8), this yields $|F(X+e)| = |F(X)|+1 = r(X)+1 = r(X+1)$.

Case (ii) When $r(X+e) = r(X)$, the equation (9) is equivalent to

$$|\{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X+e \,\}| = |\{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}| + 1.$$

by (7) and (8). Hence, it suffices to show that the set

$$D := \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X+e \,\} \setminus \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}$$

is a singleton. As $F(X) \in \mathcal{B}(X)$ and $r(X) = r(X+e)$, the set $F(X)+e$ includes the unique circuit $C(F(X)|e) \in \mathcal{C}$. Let us denote it by C_0. Then,

$$\delta(C_0) \in \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X+e \,\}. \tag{10}$$

Also, by $C_0 \subseteq F(X)+e = X \setminus \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\} + e$ and $e \in E \setminus X$,

$$C_0 \cap \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\} = \emptyset \tag{11}$$

holds, which implies $\delta(C_0) \notin \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}$. With (10), this yields $\delta(C_0) \in D$.

Next, we show that there is no other element in D. Suppose, to the contrary, that there is a circuit C_1 such that

$$\delta(C_1) \in D, \tag{12}$$

$$\delta(C_1) \neq \delta(C_0). \tag{13}$$

Note that (12) implies $e \in C_1$ by the definition of D. If multiple circuits satisfy (12) and (13), let C_1 be the one which minimizes

$$|C_1 \cap \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}|. \tag{14}$$

In what follows, we show that when δ satisfies the axiom (D), there is another circuit which satisfies (12) and (13) in place of C_1 and makes (14) strictly smaller, a contradiction.

Because of (13), C_1 and C_0 are distinct. Hence, we can apply Proposition 6.4 to C_1, C_0 and $e \in C_1$. Then there exists a circuit $C_2 \subseteq C_1 \cup C_0$ such that (C_1, C_2) is minimal and $e \notin C_2$. In addition, since $C_2 \subseteq C_1 \cup C_0 \subseteq X+e$, we have $C_2 \subseteq X$, which implies

$$\delta(C_2) \in \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}. \tag{15}$$

Also, since (12) implies $\delta(C_1) \notin \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}$, we have

$$\delta(C_1) \neq \delta(C_2). \tag{16}$$

By (11) and (15), we obtain $\delta(C_2) \notin C_0$. As $\delta(C_2) \in C_2 \subseteq C_1 \cup C_0$, this implies

$$\delta(C_2) \in C_1. \tag{17}$$

Thus $\delta(C_2) \in C_1 \cap C_2$ holds, and then, by the axiom of circuit family (C3), there is a circuit C_3 such that

$$\delta(C_2) \notin C_3 \subseteq C_1 \cup C_2. \tag{18}$$

Since (C_1, C_2) is minimal and $C_3 \subseteq C_1 \cup C_2$, the axiom (D) yields

$$|\{\delta(C_1), \delta(C_2), \delta(C_3)\}| \leq 2.$$

Then, by (16) and (18), we obtain $\delta(C_3) = \delta(C_1)$. Therefore, (12) and (13) hold with C_1 replaced by C_3. Also, $|C_3 \cap \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}|$ is strictly smaller than (14) as below: By $C_3 \subseteq C_1 \cup C_2$ and $C_2 \subseteq C_1 \cup C_0$, we have $C_3 \subseteq C_1 \cup C_0$. Then, (11) implies

$$C_3 \cap \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\} \subseteq C_1 \cap \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}. \tag{19}$$

Also, as we have (15), (17) and (18), the element $\delta(C_2)$ is contained only in the right-hand side of (19). Thus, $|C_3 \cap \{\, \delta(C) \mid C \in \mathcal{C},\ C \subseteq X \,\}|$ is strictly smaller than (14). □

6.3 Characterization via the greedy algorithm

In Section 4, we have shown that a matroidal choice function can be calculated by the algorithm MCF(δ) if δ is its corresponding de-cycle function. In fact, this property characterizes matroidal choice functions.

The following lemma clarifies the equivalence between the coherency and the independence of outputs from permutations.

Lemma 6.9. A de-cycle function $\delta : \mathcal{C} \to E$ is coherent if and only if the output of MCF(δ) for (X, π) does not depend on π for each $X \subseteq E$.

For a given coherent de-cycle function δ, the output of MCF(δ) for (X, π) coincides with $F(X)$, which is the subset defined by (7).

Proof. The "only if" part and the second claim: If δ is coherent, then Theorem 6.7 implies that F defined by (7) is a matroidal choice function, and we can see that its associated de-cycle function δ_F coincides with δ. Then, by Proposition 4.8, the output of MCF(δ) for (X, π) is $F(X)$ regardless of the permutation π for each $X \subseteq E$.

The "if" part: Assume that the output of MCF(δ) does not depend on the permutation for each input. Take a minimal pair (C_1, C_2) and a circuit $C_3 \subseteq C_1 \cup C_2$ arbitrarily. Set $X = C_1 \cup C_2 = \{e_1, e_2, \ldots, e_{|X|}\}$. For each $i \in \{1, 2, 3\}$, let $\pi_i \in S_{|X|}$ be a permutation such that

$$\{\, e_k \mid k = \pi_i(1), \pi_i(2), \ldots, \pi_i(|C_i|) \,\} = C_i.$$

Let J be the common output of MCF(δ) for (X, π_i) $(i \in \{1, 2, 3\})$. For each $i \in \{1, 2, 3\}$, the element $\delta(C_i)$ is eliminated at the $|C_i|$-th Step 2 of the algorithm, and hence $J \not\ni \delta(C_i)$. Then we have $J \subseteq X - \{\delta(C_1), \delta(C_2), \delta(C_3)\}$.

On the other hand, as (C_1, C_2) is minimal, $X - \delta(C_i) = C_1 \cup C_2 - \delta(C_i)$ contains only one circuit, and hence only one element is eliminated after $|C_i|$-th Step 2. Thus, we have $|J| = |X| - 2$. Therefore, we obtain $\{\delta(C_1), \delta(C_2), \delta(C_3)\} \leq 2$. □

The following theorem easily follows from Theorem 6.7 and Lemma 6.9.

Theorem 6.10. A function $F : 2^E \to 2^E$ is a matroidal choice function if and only if there is a coherent choice function such that the output of MCF(δ) for X is $F(X)$ for each $X \subseteq E$. In particular, for a given matroidal choice function, such a coherent de-cycle function δ is unique.

Remark 6.11. By Theorem 6.7 (or Theorem 6.10), we see that, for any matroid $\mathbf{M}$, there is a one-to-one correspondence between matroidal choice functions on $\mathbf{M}$ and coherent de-cycle functions on the circuit family of $\mathbf{M}$.

7 Relationships with valuated matroids

Valuated matroids are matroids accompanied with valuations satisfying the "exchange axiom" (defined below) [10, 12]. On this structure, many combinatorial properties of matroids can be generalized naturally. For example, we can maximize a valuation by the greedy algorithm. (Their properties and applications are discussed in detail by Murota [29].)

In this section, we show that the choice function defined by a valuated matroid in a reasonable way is a matroidal choice function. Then, we see that some known facts about valuated matroids (the validity of a greedy algorithm and the equivalence between local and global optimality) follow from our results in Section 4.

7.1 Inducing matroidal choice functions from valuations

Let $\mathbf{M} = (E, \mathcal{I})$ be a matroid and $\mathcal{B}$ be its base set. A pair $(\mathcal{B}, \omega)$ is called a *valuated matroid* on $\mathbf{M}$ if $\omega : \mathcal{B} \to \mathbf{R}$ satisfies the following exchange axiom:

(VM) For any $B_1, B_2 \in \mathcal{B}$ and $e_2 \in B_2 \setminus B_1$, there exists $e_1 \in B_1 \setminus B_2$ such that $B_1 - e_1 + e_2 \in \mathcal{B}$, $B_2 + e_1 - e_2 \in \mathcal{B}$ and

$$\omega(B_1) + \omega(B_2) \leq \omega(B_1 - e_1 + e_2) + \omega(B_2 + e_1 - e_2).$$

We call such a function ω a *valuation* on $\mathcal{B}$.

Fact 7.1. For a set $X \subseteq E$, take a set $I \subseteq E \setminus X$ such that

$$|I| = r(E) - r(X), \quad r(X \cup I) = r(E). \tag{20}$$

Then, $B \subseteq X$ satisfies $I \cup B \in \mathcal{B}$ if and only if $B \in \mathcal{B}(X)$. ■

We denote by $\mathcal{B}/X$ the family of subsets $I \subseteq E \setminus X$ that satisfies (20). Then $\mathcal{B}/X$ forms the base family of the contraction of $\mathbf{M}$ by X. With Fact 7.1, one can define a restriction of valuated matroid as follows.

Fact 7.2. (Dress and Wenzel [12]) Let ω be a valuation on $\mathcal{B}$. For a set $X \subseteq E$ and $I \in \mathcal{B}/X$, define a function $\omega_I : \mathcal{B}(X) \to \mathbf{R}$ by

$$\omega_I(B) = \omega(B \cup I) \quad (B \in \mathcal{B}(X)).$$

Then, $(\mathcal{B}(X), \omega_I)$ is a valuated matroid. Also, for any $J \in \mathcal{B}/X$, the valuation ω_J is equal to ω_I, up to addition by a constant (i.e., there is $\alpha \in \mathbf{R}$ such that $\omega_I(B) = \omega_J(B) + \alpha$ for every $B \in \mathcal{B}(X)$). ■

By virtue of the last claim of Fact 7.2, the set of maximizers of ω_I in $\mathcal{B}(X)$, which we denote by $\mathcal{F}(X)$, is determined independently of the choice of $I \in \mathcal{B}/X$. Then, it depends only on $X \subseteq E$.

Proposition 7.3. For a valuated matroid $(\mathcal{B}, \omega)$ on $\mathbf{M}$ such that $\mathcal{F}(X)$ is a singleton for each $X \subseteq E$, we let $F(X)$ be the only member of $\mathcal{F}(X)$. Then $F : 2^E \to 2^E$ forms a matroidal choice function on $\mathbf{M}$.

Proof. By the definition, $F(X) \in \mathcal{B}(X)$ clearly holds for any $X \subseteq E$. Then, it suffices to show that F satisfies (Sub).

For $X, Y \subseteq E$ with $X \subseteq Y$, take $I_X \in \mathcal{B}/X$ and $I_Y \in \mathcal{B}/Y$. By the definition of F, we have

$$\omega(F(X) \cup I_X) > \omega(B \cup I_X) \quad (\forall B \in \mathcal{B}(X) \setminus \{F(X)\}), \tag{21}$$
$$\omega(F(Y) \cup I_Y) > \omega(B \cup I_Y) \quad (\forall B \in \mathcal{B}(Y) \setminus \{F(Y)\}). \tag{22}$$

Suppose, to the contrary, that we have $F(Y) \cap X \not\subseteq F(X)$. Then, there is some $e_Y \in (F(Y) \setminus F(X)) \cap X$, and such e_Y satisfies $e_Y \in B_Y \setminus B_X$ where $B_X = F(X) \cup I_X$ and $B_Y = F(Y) \cup I_Y$. Let us apply the exchange axiom (VM) to B_X, B_Y and e_Y. Then, there is some $e_X \in B_X \setminus B_Y$ such that

$$B_X - e_X + e_Y \in \mathcal{B}, \tag{23}$$
$$B_Y + e_X - e_Y \in \mathcal{B}, \tag{24}$$
$$\omega(B_X) + \omega(B_Y) \leq \omega(B_X - e_X + e_Y) + \omega(B_Y + e_X - e_Y). \tag{25}$$

Because $F(X) \in \mathcal{B}(X)$ and $e_Y \in X \setminus F(X)$, the condition (23) implies $e_X \in F(X)$ and $F(X) - e_X + e_Y \in \mathcal{B}(X)$. Hence (21) yields

$$\omega(B_X) = \omega(F(X) \cup I_X) > \omega((F(X) - e_X + e_Y) \cup I_X) = \omega(B_X - e_X + e_Y).$$

By a similar argument, (22) and (24) imply

$$\omega(B_Y) = \omega(F(Y) \cup I_Y) > \omega((F(Y) + e_X - e_Y) \cup I_Y) = \omega(B_Y + e_X - e_Y).$$

These two inequalities contradict (25). □

We say that F is *induced* from $(\mathcal{B}, \omega)$ for such F and $(\mathcal{B}, \omega)$.

Proposition 7.4. For a choice function F induced from a valuated matroid $(\mathcal{B}, \omega)$, its associated de-cycle function δ_F satisfies

$$\omega(B + e - \delta_F(C)) > \omega(B + e - e') \quad \big(\, \forall e' \in C - \delta(C)\, \big)\,. \tag{26}$$

for every $B \in \mathcal{B}$ and $e \in E \setminus B$ with $C = C(B|e)$.

Proof. For $B \in \mathcal{B}$ and $e \in E \setminus B$, let $C = C(B|e)$. Since $B \setminus C \in \mathcal{B}/C$, the choice function F induced from $(\mathcal{B}, \omega)$ satisfies

$$\omega(F(C) \cup (B \setminus C)) > \omega(B' \cup (B \setminus C)) \quad (\forall B' \in \mathcal{B}(C) \setminus \{F(C)\}).$$

Because $F(C) = C - \delta_F(C)$ and $\mathcal{B}(C) = \{\, C - e' \mid e' \in C\,\}$, this means the condition (26). □

Remark 7.5. It is written in [10] that the following algorithm finds the maximizer $B \in \mathcal{B}$ of a valuation ω: Let B be an arbitrary base $B_0 \in \mathcal{B}$, and then for each $e \in E \setminus B_0$, update B by swapping e with $e' \in B + e$ which maximizes $w(B + e - e')$.

By Proposition 7.4, this algorithm can be identified with MCF(δ_F) for the input E and π such that $\{ e_{\pi(i)} \mid i = 1, 2, \ldots, |B_0| \} = B_0$. The validity of above algorithm follows from Proposition 4.8. Even if $\mathcal{F}(X)$ is not a singleton for some $X \subseteq E$, we can apply this argument through appropriate perturbations of ω.

The equivalence between the local optimality and the global optimality [10, 12] of valuated matroids also follows from Proposition 4.7.

Remark 7.6. Murota and Tamura [31] characterized valuated matroids in terms of axioms of real vectors defined on circuit families. From these axioms, we can show the coherency of a de-cycle function which satisfies (26) for some valuated matroid. This gives another proof of Proposition 7.3 through Theorem 6.7.

7.2 Matroidal choice function uninducible from valuations

As shown in Proposition 7.3, valuated matroids induce matroidal choice functions. One may wonder if every matroidal choice function is obtained in such a way. This is not true in general. We give a counter example using the following proposition.

Proposition 7.7. (Dress and Wenzel [12]) For a valuated matroid $(\mathcal{B}, \omega)$ on $\mathbf{M}$, if $\mathbf{M}$ is a binary matroid, then ω can be represented as

$$\omega(B) = \alpha + \sum_{e \in B} \eta(e) \quad (B \in \mathcal{B})$$

for some $\alpha \in \mathbf{R}$ and $\eta : E \to \mathbf{R}$. ■

Example 7.8. We give an example of matroidal choice function which cannot be induced from any valuated matroid. Let $\mathbf{M} = (E, \mathcal{I})$ be a matroid such that $E = \{e_1, e_2, \ldots, e_6\}$ and $\mathcal{C}$ consists of the following seven circuits:

$$C_1 = \{e_1, e_2, e_3, e_4\}, \quad C_2 = \{e_1, e_3, e_5, e_6\}, \quad C_3 = \{e_2, e_4, e_5, e_6\},$$
$$C_4 = \{e_1, e_2, e_6\}, \quad C_5 = \{e_2, e_3, e_5\}, \quad C_6 = \{e_3, e_4, e_6\}, \quad C_7 = \{e_1, e_4, e_5\}.$$

This is a graphic matroid. (Figure 1 shows its graphical representation.) Define a de-cycle function $\delta : \mathcal{C} \to E$ as

$$\begin{aligned} \delta(C_1) &= e_4, \\ \delta(C_2) = \delta(C_4) = \delta(C_6) &= e_6, \\ \delta(C_3) = \delta(C_5) = \delta(C_7) &= e_5. \end{aligned}$$

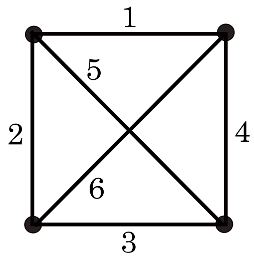

Figure 1: A graphical representation of $\mathbf{M}$

Then, we can see that d is coherent. Hence, $F : 2^E \to 2^E$ defined by δ via (7) is a matroidal choice function on $\mathbf{M}$. We now show that there is no valuation on $\mathcal{B}$ which induces this F.

Suppose, to the contrary, that there is a valuation ω such that $(\mathcal{B}, \omega)$ induces F. Apply Proposition 7.4 to $B := \{e_1, e_3, e_5\}$ and $e := e_6$. Then, by $\delta(C(B|e)) = \delta(C_2) = e_6$, we obtain

$$\omega(\{e_1, e_3, e_5\}) > \omega(\{e_1, e_3, e_6\}). \tag{27}$$

Similarly, applying Proposition 7.4 to $B := \{e_2, e_4, e_6\}$, $e := e_5$, we obtain

$$\omega(\{e_2, e_4, e_6\}) > \omega(\{e_2, e_4, e_5\}). \tag{28}$$

Since $\mathbf{M}$ is graphic, and hence binary, by Proposition 7.7, there exists $\alpha \in R$ and $\eta : E \to \mathbf{R}$ such that $\omega(B) = \alpha + \sum_{e \in B} \eta(e)$. Then, (27) implies $\eta(e_5) > \eta(e_6)$ while (28) implies $\eta(e_6) > \eta(e_5)$, a contradiction. ■

We can observe that the choice function in Example 7.8 is neither quotafilling nor representable by a totally ordered matroid. Note that we cannot define the order of priority between e_5 and e_6.

Remark 7.9. Recently, the stable matching model with $\mathrm{M}^\natural$-concave [30] value functions has been investigated [17, 25, 32]. It has been shown that choice functions induced from $\mathrm{M}^\natural$-concave functions satisfy the substitutability and the size-monotonicity. However, such choice functions are not necessarily matroidal. Conversely, not all matroidal choice functions can be induced from $\mathrm{M}^\natural$-concave functions.

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A Complement to Example 4.4

We prove that the choice function $F : 2^E \to 2^E$ described in Example 4.4 is in fact matroidal. Let $\mathcal{I} = \{X \subseteq E : \ |X \cap A| \leq q_A, \ |X \cap B| \leq q_B, \ |X| \leq q_S\}$. Then, $\mathbf{M} = (E, \mathcal{I})$ is a laminar matroid. For any $X \subseteq E$, by Step 1 of the process, the chosen set does not exceed any quota, i.e., $F(X) \in \mathcal{I}$. Also, $F(X) + e$ exceeds some quota for every $e \in X \setminus F(X)$, and hence $F(X) \in \mathcal{B}(X)$. Then, it suffices to show the substitutability of F. We use the following claim.

Claim A.1. If $(X \setminus F(X)) \cap A \neq \emptyset$ and $X \subseteq Y$, then $n_A(X) \geq n_A(Y)$.

Proof. The condition $(X \setminus F(X)) \cap A \neq \emptyset$ implies that someone in X_A is rejected, i.e., $n_A(X) < |X_A|$. Since the claim is obvious if $n_A(X) = q_A$, assume $n_A(X) < q_A$. Then, $n_A(X) < N_A(X)$ follows from $N_A(X) = \min\{|X_A|, q_A\}$. By Step 1 of the process, this means that $(n_A(X), n_B(X))$ is determined as the optimal solution of (1). Because $n_A(X) < N_A(X)$, the optimality of $(n_A(X), n_B(X))$ implies that one of the following two conditions holds:

$$(n_A(X), n_B(X)) = (p_A, p_B),$$

$$N_A(X) > n_A(X) > p_A \quad \text{and} \quad N_B(X) = n_B(X) < p_B.$$

Note that $X \subseteq Y$ implies $N_A(X) \leq N_A(Y)$ and $N_B(X) \leq N_B(Y)$. Then, the feasible domain of (1) is enlarged when $(n_A(Y), n_B(Y))$ is determined. Hence $(n_A(Y), n_B(Y))$ is not farther from (p_A, p_B) than $(n_A(X), n_B(X))$. Therefore, we have $n_A(X) \geq n_A(Y) \geq p_A$. $\square$

To show the substitutability, it suffices to show that $X \subseteq Y$ implies

$$(X \setminus F(X)) \cap A \cap M \subseteq (Y \setminus F(Y)) \cap A \cap M. \tag{29}$$

Let us write $n_{A,M}(X)$ for $|F(X) \cap A \cap M|$. Then, it is easy to check that $X \subseteq Y$ along with $n_{A,M}(X) \geq n_{A,M}(Y)$ implies (29). Hence, in what follows we consider only the case $n_{A,M}(X) < n_{A,M}(Y)$.

Assume $(X \setminus F(X)) \cap A \cap M \neq \emptyset$, since otherwise (29) is obvious. Then, we have $n_A(X) \geq n_A(Y)$ by Claim A.1. With $X_A \subseteq Y_A$, this implies

$$\max\{\ \lfloor n_A(X)/2 \rfloor,\ n_A(X) - |X_A \cap W|\ \} \geq \max\{\ \lfloor n_A(Y)/2 \rfloor,\ n_A(Y) - |Y_A \cap W|\ \}.$$

By Step 2, this implies that $n_{A,M}(X) < n_{A,M}(Y)$ holds only if

$$n_A(X) = n_A(Y) = 2k + 1, \quad n_{A,M}(X) = k, \quad n_{A,M}(Y) = k + 1$$

where $k = \lfloor n_A(X)/2 \rfloor$.

This means that the $(k+1)$-th man is worse than the $(k+1)$-th woman in X_A, but it is the other way round in Y_A. This implies that some man in $Y_A \setminus X_A$ precedes the man who is $(k+1)$-th in X_A. Then, each man in $(X \setminus F(X)) \cap A \cap M$ moves down on the list, and hence he cannot be contained in top $k+1$ men in Y_A. Thus we obtain (29).