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A Lyapunov-Type Theorem for Dissipative Numerical Integrators with Adaptive Time-Stepping

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Abstract

The asymptotic behavior of continuous dissipative systems and dissipative numerical integrators with fixed time-stepping can be fully investigated by Lyapunov-type theorem on continuous and discrete dynamical systems, respectively. However, once adaptive time-stepping is involved, such theories cease to work, and usually the dynamics should be investigated in a backward way, such as in terms of pullback attractors. In this paper, we present a different approach—we stick to a forward definition of limit sets, and show that still we can establish a Lyapunov-type theorem, which reveals the precise asymptotic behavior of adaptive time-stepping integrators in the presence of a discrete Lyapunov functional.

1 Introduction

In this paper, we consider the numerical integration of the evolutionary differential equation on a Banach space X:

$$\frac{\mathrm{d}}{\mathrm{d}t}x = f(x), \qquad x(0) = x_0, \tag{1}$$

where $x_0 \in X$, $x : \mathbb{R}^+ \to X$, $f : X \to X$, and we assume it has the following "dissipation" property with respect to a functional $G : X \to \mathbb{R}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}G(x(t)) \leq 0. \tag{2}$$

In this paper, such systems are refered to as "dissipative systems." This is a subclass of general dissipative systems in dynamical systems theory; see, for example, [15, 32]. Still it includes wide variety of practical applications. The Cahn–Hilliard equation describing phase separation problems [2] (see also [37, §4.2], [33, §5.5]), the time-dependent Ginzburg–Landau equation describing superconductivity [24] (see also [37, §5]), and the Swift–Hohenberg equation describing thermal convection [36], among others. Also, their appropriate spatial discretizations would yield corresponding finite-dimensional, dissipative ordinary differential equations.

The system (1) can be treated as a continuous dynamical system, which is a pair (X, S), where X is a Banach space and S is a family of continuous operators (we state the complete definition later). We can treat its asymptotic behavior via its ω-limit set defined by

$$\omega(x_0) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)x_0} \qquad (x_0 \in X). \tag{3}$$

Roughly speaking, it is a set of such points that are visited infinitely many times (see, e.g., [15, 32]). For the system (1) with (2), the energy function G can serve as a Lyapunov functional, and the Lyapunov-type theorem reveals the following fact: if the level set of G is compact, then the ω-limit set of an arbitrary chosen $x_0 \in X$ is a subset of the fixed points. In other words, for any bounded initial value x_0, the solution of (1) eventually tends to the fixed points.

In view of this significant nature, a numerical integrator for such a dissipative system should keep the same property as exactly as possible. Let us consider an abstract one step numerical integrator of (1) in the following form:

$$\frac{x^{(n+1)} - x^{(n)}}{\Delta t} = \hat{f}\left(x^{(n+1)}, x^{(n)}\right), \tag{4}$$

where $\Delta t > 0$ is a time-stepping width, and $x^{(n)} \simeq x(n\Delta t)$ $(n = 0, 1, \ldots)$ is the approximate solution. The function $\hat{f} : X \times X \to X$, which represents a numerical integrator, is assumed to satisfy $\hat{f}(x, x) = f(x)$ for any $x \in X$. We also introduce a continuous operator $F(\Delta t) : X \to X$ that gives another expression of the integrator: $x^{(n+1)} = F(\Delta t)x^{(n)}$. Let $\Delta T \subseteq (0, +\infty)$ denote the set of feasible time-steppings. In view of the continuous dissipation property (2), a numerical integrator in the form (4) is called a *dissipative integrator with respect to* ΔT if $G(F(\Delta t)x) \le G(x)$ holds for any $x \in X$ and $\Delta t \in \Delta T$.

As far as the time step size Δt is fixed, we can regard the one-step numerical integrator (4) as a discrete dynamical system, which is a pair $(X, F(\Delta t))$. For this system, we can analyze its asymptotic behavior by the concept of ω-limit set, which is, in this case, defined by

$$\omega(x_0) = \bigcap_{n \ge 0} \overline{\bigcup_{m \ge n} F(\Delta t)^m x_0} \qquad (x_0 \in X). \tag{5}$$

Again, the function G can serve as a Lyapunov functional, and Lyapunov-type theorem on discrete dynamical systems (see, e.g., Humphries–Stuart [16]; see also [34, 35], and the seminal paper Kloeden–Lorenz [18] which discussed the asymptotic behavior of numerical integrators) reveals a similar fact regarding the asymptotic behavior: if the level set of G is compact, then the ω-limit set of an arbitrarily chosen point $x_0 \in X$ is a subset of the fixed points.

For example, let us consider the case that $X = \mathbb{R}^d$ and the equation (1) is in the linear-gradient form, i.e., $f(z) = L(z)\nabla G(z)$ $(z \in \mathbb{R}^d)$, where $L(z) \in \mathbb{R}^{d \times d}$ is such an operator that $\frac{\mathrm{d}}{\mathrm{d}t}G(z) < 0$ holds for any $z \in \mathbb{R}^d$ except for the fixed points of (1); for example, it is sufficient if $L(z)$ is negative definite. (In this paper, we try to distinguish general Banach space setting on X and concrete numerical integrator setting on $\mathbb{R}^d$; "x" denotes the former, while "z" implies the latter.) Note that a dissipative system can be rewritten in a linear-gradient form (McLachlan–Quispel–Robidoux [27, 28]). In this case, there is a well known class of dissipative integrators, "discrete gradient method" (Quispel–Capel [29], Quispel–Turner[31], Gonzalez [13], see also recent developments in [30, 4]). For an energy function G, its discrete gradient $\nabla_{\mathrm{d}} G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a map satisfying the following properties:

1. $G(z_1) - G(z_2) = \nabla_{\mathrm{d}} G(z_1, z_2) \cdot (z_1 - z_2)$ for any $z_1, z_2 \in \mathbb{R}^d$,
2. $\nabla_{\mathrm{d}} G(z, z) = \nabla G(z)$ for any $z \in \mathbb{R}^d$,

where the symbol '$\cdot$' denotes the standard inner product in $\mathbb{R}^d$. The former property is called the discrete chain rule, which is the essential property of discrete gradients. The latter demands that the discrete gradient is actually an approximation of the original gradient. The discrete gradient for a specified function is not necessarily unique. If we once have a $\nabla_{\mathrm{d}} G$, we can systematically construct a numerical integrator that strictly replicates the dissipation property of (1):

$$\frac{z^{(n+1)} - z^{(n)}}{\Delta t} = \hat{L}\left(z^{(n+1)}, z^{(n)}\right) \nabla_{\mathrm{d}} G\left(z^{(n+1)}, z^{(n)}\right). \tag{6}$$

We assume $\hat{L}$ is such an approximation that $\hat{L}(z,z)=L(z)$ holds for all $z\in\mathbb{R}^d$. Furthermore, since

$$\begin{aligned} G\left(z^{(n+1)}\right)-G\left(z^{(n)}\right) &= \nabla_{\mathrm{d}}G\left(z^{(n+1)},z^{(n)}\right)\cdot\left(z^{(n+1)}-z^{(n)}\right) \\ &= \Delta t\left(\nabla_{\mathrm{d}}G\right)^{\top}\hat{L}\left(\nabla_{\mathrm{d}}G\right) \end{aligned}$$

holds, we assume that $\left(\nabla_{\mathrm{d}}G(z,\zeta)\right)^{\top}\hat{L}(z,\zeta)\nabla_{\mathrm{d}}G(z,\zeta)<0$ holds except for the fixed points of (6). The latter assumption implies the dissipation property.

Now let us turn our attention to an important fact that the dissipative integrator (4), and its special case (6), can make sense, even if we employ some adaptive time-stepping technique; the integrator is a one-step method, and the energy dissipation property holds even if we change $\Delta t\in\Delta T$ in each time step. In this case as well, we naturally expect that the energy function G would serve as a Lyapunov functional, and thus it would tell us the asymptotic behavior. This, however, turns out not so simple unfortunately. If we execute a dissipative integrator of the form (4) with adaptive time-stepping, it cannot be viewed as an either discrete or continuous dynamical system. This issue has already been noticed in the literature—see Kloeden–Schmalfuß [20] and its followers, for example, [22, 21, 12, 6, 23, 17, 19].

Let us be more specific about the difficulty. The continuous dynamical systems have the following semigroup properties (see, e.g., [15, 32]):

1. $S(0)$ is an identity operator;
2. $S(t)S(s)=S(t+s)=S(s)S(t)$ holds for all $s,t\geq 0$.

The second property is crucial in the analysis of asymptotic behavior. For example, this property allows the definition of ω-limit set to make sense. Recall the definition (3), which refers to a set of all the points visited after some time t; but for this being able to make sense, the state at time t, $x(t)=S(t)x_0$, should be uniquely defined for any initial value $x_0\in X$, so that we can consider the position at time s (after time t) by $S(s-t)S(t)x_0$. The same applies to the discrete case with the constant time step Δt; recall (5). This will be, however, destroyed as soon as we employ adaptive time-stepping. The position at time t should vary depending on the time-stepping employed until t, and accordingly, the definition of ω-limit set itself becomes a mathematical challenge.

Let us illustrate this through several examples. First, we consider the following (reduced form of the) Lotka–Volterra equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}p\\q\end{pmatrix}=J\nabla H(p,q);\qquad J=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\ H(p,q)=pq(p+q-1).$$

Although this is a conservative system, which is outside the direct scope of this paper, it is helpful to understand the strange behavior of ω-limit set under adaptive time-stepping. We can construct a discrete gradient numerical integrator:

$$\frac{z^{(n+1)}-z^{(n)}}{\Delta t}=J\nabla_{\mathrm{d}}H\left(z^{(n+1)},z^{(n)}\right),\quad z^{(n)}=\begin{pmatrix}p^{(n)}\\q^{(n)}\end{pmatrix},\tag{7}$$

which strictly preserves the Hamiltonian H. The discrete gradient $\nabla_{\mathrm{d}}H\left(z^{(n+1)},z^{(n)}\right)$ is given by

$$\nabla_{\mathrm{d}}H\left(z^{(n+1)},z^{(n)}\right)=\begin{pmatrix}\frac{\left(q^{(n+1)}\right)^2+\left(q^{(n)}\right)^2}{2}+\left(\frac{q^{(n+1)}+q^{(n)}}{2}\right)\left(p^{(n+1)}+p^{(n)}-1\right)\\ \frac{\left(p^{(n+1)}\right)^2+\left(p^{(n)}\right)^2}{2}+\left(\frac{p^{(n+1)}+p^{(n)}}{2}\right)\left(q^{(n+1)}+q^{(n)}-1\right)\end{pmatrix}.$$

We denote this integrator again by $F(\Delta t)$. Let us first consider to choose a special set of time steppings $\Delta t_1,\Delta t_2,\Delta t_3$ such that $z^{(0)}=F(\Delta t_3)F(\Delta t_2)F(\Delta t_1)z^{(0)}$ holds. In this case, obviously the behavior is

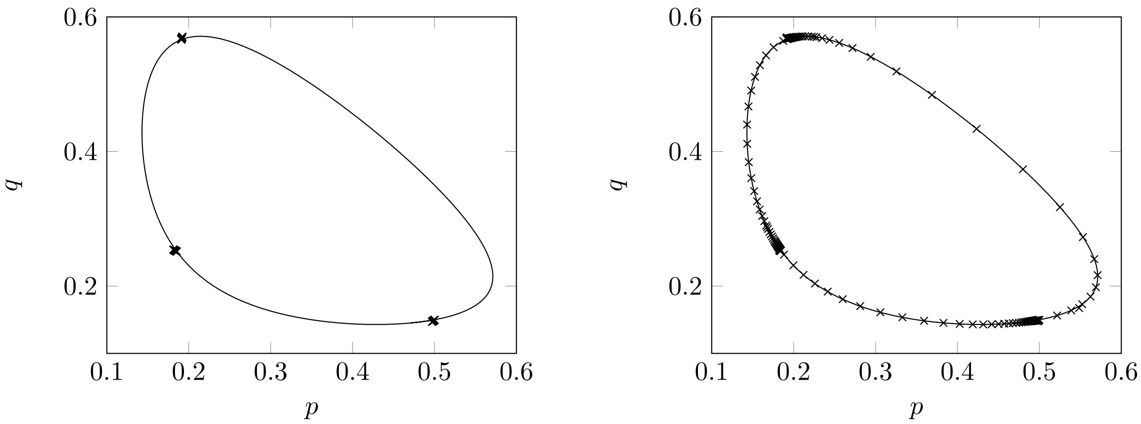

Figure 1: Numerical solutions: the solid line denotes the true solution of the Lotka–Volterra equation. The marks denote the solution of the numerical integrator (7) with each time-stepping (the left panel corresponds to the first choice, and the right to the second). The initial point is $z^{(0)} = (0.5, 0.15)$, and the number of steps is 300.

cyclic (the left panel of Fig. 1) and the ω-limit set should consist only of the three points. Now let us change the schedule and execute the integrator (7) with the time-steppings $\Delta t_3, \Delta t_1, \Delta t_2, \Delta t_3, \Delta t_1, \Delta t_2, \Delta t_3, \ldots$. Although this is just a reordering and still keeps the time length $\Delta t_1 + \Delta t_2 + \Delta t_3$, due to the lack of the semigroup property, $F(\Delta t_2)F(\Delta t_1)F(\Delta t_3)z^{(0)}$ does not return to $z^{(0)}$ in general (the right panel of Fig. 1). In such a case, we expect that the ω-limit set is the entire closed orbit. This example clearly shows that the asymptotic behavior should be discussed depending on the employed time-steppings.

Next, let us consider a dissipative case, with the toy problem:

$$\frac{\mathrm{d}}{\mathrm{d}t} z = -\nabla G(z), \quad G(z) = \frac{1}{2} z^2.$$

One might expect in such a simple system, the solution should tend to the only fixed point $z = 0$ (which is also the global attractor) whatever the time-stepping schedule is. With the discrete gradient $\nabla_{\mathrm{d}} G(x, y) = (x + y)/2$, let us consider the dissipative numerical integrator

$$\frac{z^{(n+1)} - z^{(n)}}{\Delta t_n} = -\nabla_{\mathrm{d}} G\left(z^{(n+1)}, z^{(n)}\right).$$

If we execute the integrator with

$$\Delta t_n = 2 \coth\left(\frac{1}{(n+1)^2}\right),$$

we obtain

$$z^{(n+1)} = (-1)^{n+1} \exp\left(-\sum_{i=0}^{n} \frac{2}{(i+1)^2}\right) z^{(0)}.$$

Obviously, this sequence asymptotically oscillates: $z^* = \pm \exp\left(-\pi^2/3\right) z^{(0)}$, whereas the numerical integrator with a fixed time-step width Δt converges to the correct fixed point 0 for all $\Delta t > 0$. This, again, claims that things are not so straightforward when we employ a varying time-stepping.

In this paper, to circumvent this difficulty, we consider a class of dynamical systems that no longer fully enjoy the semigroup properties. We call them "non-semigroup dynamical systems" here. It is a generalization of discrete dynamical systems, and suitable for analyzing adaptive time-stepping numerical integrators. Then we will show that a Lyapunov-type theorem can be successfully established on non-semigroup dynamical systems. As its consequence, we will be able to discuss the behaviors of the dissipative numerical integrators.

Before getting to the main part, let us mention closely related studies. As described above, the issue of adaptive time-stepping has already been considered in several studies. It seems such investigations had begun in the seminal papers by Kloeden–Schmalfuß [20, 22, 21], which introduced the concept of cocycles. Roughly speaking, cocycle is a weaker concept of semigroup, in the sense that it only assumes the weaker action: $C(t,s) = C(t,\tau)C(\tau,s)$ for all $t \geq \tau \geq s$, where $C(t,s)$ denotes the map that moves $x(s)$ to $x(t)$. Of course a semigroup $S(t)$ (or an integrator $F(\Delta t)$ with constant time-stepping) satisfies this with $C(t,s) = S(t-s)$ (or $C(m\Delta t, n\Delta t) = (F(\Delta t))^{(m-n)}$, where $m > n$). Notice that adaptive time-steppings do not form semigroups, but they do cocycles for each time-stepping schedule: $C(t_m,t_n) = C(t_m,t_\mu)C(t_\mu,t_n)$ for all $m \geq \mu \geq n$, where t_n's denote the time grids. The concept of cocycles has been also employed in random dynamical systems [1, 7]. Similar studies can be found in the area of nonautonomous dynamical systems, which deals with the same topic in a different language; see, for example, [3, 5, 14, 19].

Most of the above researches have focused on attractors. The difficulty there is that for cocycles (or nonautonomous systems), the standard definition of attractors in terms of the behaviors in $t \to \infty$, which we call here the "forward" definition, is no longer valid, since even if we can find an attracting set, it is generally not invariant. A better solution for a cocycle $C(t,s)$ is to consider the limit $s \to -\infty$ instead. This gives a way to consider time-dependent asymptotic behaviors, and it has been proved that in this sense invariant attractors can be defined (see, for example, [3, §1.4]). They are called "pullback" attractors. The studies along this line include, in addition to the references above, [12, 6].

Unfortunately, the concept of pulling back seems not quite useful for the present study; we hope to establish a Lyapunov-type theorem, which demands the forward definition along orbits (see the remark after Lemma 1). Thus in this paper we take the completely opposite approach—we consider a forward definition of ω-limit sets, giving up its invariance (which is generally not available). Then we will show that still a weaker invariance can be established, and with its aid, a desired Lyapunov-type theorem can be constructed. A similar forward approach can be found in [23]. But it considered attractors for sufficiently small time-stepping widths, and the goal is different from ours.

The remainder of this paper is organized as follows: In §2, we review the Lyapunov theory on continuous and discrete dynamical systems. §3 is devoted to the main results including several definitions and the main theorem. There the discussion is done in general Banach space setting. In §4, the implications of the main theorem in terms of numerical integrators are discussed. §5 is for conclusion.

2 Preliminaries

The description of this section is based on Hale [15] and Robinson [32].

Definition 1 (Continuous dynamical systems)**.** *Let X be a Banach space, and $S(t): X \to X$ $(t \geq 0)$ be a C^0-semigroup defined as*

1. *$S(0)$ is an identity operator;*
2. *$S(t)x$ is continuous in t and x;*
3. *$S(t)S(s) = S(t+s) = S(s)S(t)$ holds for all $s,t \geq 0$.*

Then, the pair (X,S) is called a continuous dynamical system.

In this definition, the third property of S is quite essential for the proof of the Lyapunov theorem below. For any $x \in X$, the *positive orbit* $\gamma^+(x)$ through x is defined as $\gamma^+(x) = \{S(t)x \mid t \geq 0\}$. Note that, thanks to the semigroup property, this definition makes sense.

The ω-limit set of $B \subseteq X$ is defined as follows:

$$\omega(B) := \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(t)B}. \tag{8}$$

This $\omega(B)$ can be characterized as

$$\omega(B) = \{y \in X \mid \exists t_n \to \infty, x_n \in B \text{ s.t. } S(t_n)x_n \to y\}. \tag{9}$$

In the continuous dynamical systems, ω-limit set is invariant under certain conditions. A subset $B \subseteq X$ is called *invariant* if $S(t)B = B$ holds for all $t \geq 0$, and called *positively invariant* if $S(t)B \subseteq B$ holds for all $t \geq 0$.

Proposition 1. *Let $B \subseteq X$ be a compact and positively invariant set of S. For all $x_0 \in B$, $\omega(x_0)$ is a nonempty and invariant set.*

The semigroup property of S plays a crucial role in (the proof of) Proposition 1: For any $x \in \omega(B)$, from the characterization (9), we obtain

$$S(t)x = S(t) \lim_{n\to\infty} S(t_n)x_n = \lim_{n\to\infty} S(t)S(t_n)x_n = \lim_{n\to\infty} S(t + t_n)x_n \in \omega(B)$$

for any $t \geq 0$. That means $S(t)\omega(B) \subseteq \omega(B)$ holds. The converse can be also proved by the semigroup property (see, e.g., [32, Proposition 10.3]).

Here, we define fixed points and Lyapunov functionals to state the Lyapunov theorem below.

Definition 2 (Fixed point)**.** *$x \in X$ is called a fixed point of S if $S(t)x = x$ holds for all $t \geq 0$. $\mathcal{E}(S)$ denotes the set of all the fixed points of S.*

Definition 3 (Lyapunov functional)**.** *Let B be a positively invariant subset of X with respect to S. A continuous map $\Phi : B \to \mathbb{R}$ is called a Lyapunov functional for S if the following conditions are satisfied.*

1. *$\Phi(S(t)x_0) \leq \Phi(x_0)$ holds for all $x_0 \in B$ and $t \geq 0$;*
2. *If there exists $t > 0$ and $x \in B$ such that $\Phi(S(t)x) = \Phi(x)$ holds, then $x \in \mathcal{E}(S)$.*

Proposition 2 below can be shown using Proposition 1.

Proposition 2 (Lyapunov theorem)**.** *Let $B \subseteq X$ be a compact and positively invariant set of S, such that there exists a Lyapunov functional on B. Then for every $x_0 \in B$, $\omega(x_0) \subseteq \mathcal{E}(S)$. If $\mathcal{E}(S)$ is discrete, then $\omega(x_0) \in \mathcal{E}$.*

For discrete dynamical systems (see, e.g., [15, 32]), there are similar definitions and propositions (Humphries–Stuart [16]). The key for this is the fact the discrete dynamical systems keep similar semigroup property as the continuous systems, and accordingly the same line of discussion holds for the corresponding Lyapunov-type theorems.

3 Main Results

We define the concept of *non-semigroup dynamical systems*, which does not suppose the semigroup property. More precisely, we drop the condition (iii) of Definition 1.

Definition 4 (Non-semigroup dynamical systems)**.** *Let X be a Banach space, and $F(\Delta t) : X \to X$ $(\Delta t \geq 0)$ be an operator satisfying*

1. *$F(0)$ is an identity operator;*
2. *$F(\Delta t)x$ is continuous in Δt and x.*

Then, the pair (X, F) is called a non-semigroup dynamical system.

Note that although this is quite similar to the concepts of the cocycles and nonautonomous dynamical systems, we prefer to introduce a new terminology to clarify that we are not considering pullbacks.

Next, we define the positive orbit in non-semigroup dynamical systems. This is not so straightforward since the positive orbit in non-semigroup dynamical systems depends on the time-stepping. Let ΔT denote a subset of $(0, +\infty)$ which represents a range of the time-stepping sizes, and $\mathcal{T}(\Delta T)$ denotes the set of all feasible instance of time stepping schedule $\{\Delta t_n\}$'s:

$$\mathcal{T}(\Delta T) := \left\{ \{\Delta t_n\}_{n=0}^{\infty} \mid \Delta t_n \in \Delta T \ (n = 0, 1, 2, \dots) \right\}.$$

Then, we define the *positive orbit* $\gamma^+(x; \{\Delta t_n\})$ *through* $x \in X$ *with* $\{\Delta t_n\} \in \mathcal{T}(\Delta T)$ as $\gamma^+(x; \{\Delta t_n\}) = \{x^{(m)}(\{\Delta t_n\}) \mid m \geq 0\}$, where $x^{(m)}(\{\Delta t_n\})$ is recursively defined by

$$x^{(m)}(\{\Delta t_n\}) = \begin{cases} x & (m = 0), \\ F(\Delta t_{m-1}) x^{(m-1)}(\{\Delta t_n\}) & (m > 0). \end{cases}$$

In other words, $x^{(m)}(\{\Delta t_n\})$ denotes the solution of the m-th time step, reached by the time-stepping schedule $\{\Delta t_n\}$. In this definition, we do not suppose that a family of operators F has the semigroup property, as opposed to the continuous and discrete dynamical systems. The concept of the non-semigroup dynamical systems naturally includes the discrete dynamical systems: we can simply consider $(X, F(\Delta t))$ for a constant time-stepping width $\Delta t > 0$.

We proceed to the definition of ω-limit set. As noted above, it should depend on the employed time-stepping $\{\Delta t_n\} \in \mathcal{T}(\Delta T)$.

Definition 5 (ω-limit set)**.** *The ω-limit set of $B \subseteq X$ with respect to $\{\Delta t_n\}$ is defined as follows:*

$$\omega(B; \{\Delta t_n\}) = \bigcap_{k \geq 1} \overline{\bigcup_{m \geq k} F(\Delta t_{m-1}) \cdots F(\Delta t_0) B}. \tag{10}$$

The same definition can be found in [23].

In the continuous and discrete dynamical systems context, usually there are also equivalent characterizations in terms of sequences, which are useful in practical point of view. Fortunately, the definition (10) allows a similar counterpart. As stated in Lemma 1 below, the $\omega(B; \{\Delta t_n\})$ can be characterized as

$$\omega(B; \{\Delta t_n\}) = \left\{ y \in X \;\middle|\; \exists n_k \to \infty, x_k \in B \text{ s.t. } x_k^{(n_k)}(\{\Delta t_n\}) \to y \right\}. \tag{11}$$

Lemma 1. *The ω-limit set defined by* (10) *is equivalent to the set defined by the right-hand side of* (11)*.*

Proof. We define the sets ω_1 and ω_2 as follows:

$$\omega_1 = \bigcap_{k \geq 1} \overline{\bigcup_{m \geq k} F(\Delta t_{m-1}) \cdots F(\Delta t_0) B},$$
$$\omega_2 = \left\{ y \in X \;\middle|\; \exists n_k \to \infty, x_k \in B \text{ s.t. } x_k^{(n_k)}(\{\Delta t_n\}) \to y \right\}.$$

We show that $\omega_1 = \omega_2$ holds.

1. $\omega_1 \subseteq \omega_2$:
Since $y \in \overline{\bigcup_{m\geq n} F(\Delta t_{m-1})\cdots F(\Delta t_0)B}$ holds for all $y \in \omega_1$ and $n \geq 1$, there exists a sequence $\{(m_k^n, x_k^n)\}_{k=1}^{\infty}$ such that $m_1^n \geq n$ and
$$(x_k^n)^{(m_k^n)}(\{\Delta t_n\}) \to y \text{ as } k \to \infty$$
hold. Then, for each $n \geq 1$, we define $(\tilde{m}_n, \tilde{x}_n) = (m_k^n, x_k^n)$ such that
$$\left\|(x_k^n)^{(m_k^n)}(\{\Delta t_n\}) - y\right\| \leq \frac{1}{n}.$$
For this sequence, we see $\tilde{x}_n^{(\tilde{m}_n)}(\{\Delta t_n\}) \to y$, which implies $y \in \omega_2$.

2. $\omega_1 \supseteq \omega_2$:
Let y be an element of ω_2. By the definition of ω_2, there exists m_k and $x_k \in B$ such that $x_k^{(m_k)}(\{\Delta t_n\}) \to y$ as $k \to \infty$. Therefore, $y \in \overline{\bigcup_{m\geq n} F(\Delta t_{m-1})\cdots F(\Delta t_0)B}$ holds for all $n \geq 1$, which means $y \in \omega_1$.

□

Remarks. 1. When B consists of a single point, say x_0, the above definition naturally considers the orbit $\gamma^+(x_0; \{\Delta t_n\})$. On the other hand, supposing the time schedule is defined for all $n \in \mathbb{Z}$, we can define an ω-limit set in the pullback sense: $\omega(B, n; \{\Delta t_n\}) = \bigcap_{k\leq n} \overline{\bigcup_{m\leq k} F(\Delta t_{n-1})\cdots F(\Delta t_{m-1})B}$ (see [3, §2.1] for a continuous definition; essentially the same concept in discrete setting can be found in, for example, [20]). Note that it depends on n, i.e., at which moment we consider the limit set. From the definition, we see it does not match Lyapunov-type theories, since the points $F(\Delta t_{n-1})\cdots F(\Delta t_{m-1})x_0$ ($m = n, n-1, \ldots$) does not form an orbit.

2. Recall the first choice of the time-steppings in the Lotka–Volterra example in Introduction. The above definition of ω-limit set captures the set of the three points.

We say a subset $B \subseteq X$ is *positively invariant with respect to F*, if $F(\Delta t)B \subseteq B$ holds for all $\Delta t \geq 0$. Since non-semigroup dynamical systems do not keep the semigroup property, ω-limit sets are not invariant in general. Still, the following lemma states that a weaker invariance, corresponding to Proposition 1, holds for non-semigroup dynamical systems.

Lemma 2. *Suppose that ΔT is a compact set. Suppose also that $B \subseteq X$ is a compact and positively invariant set of F. Then, for all $x \in B$ and $\{\Delta t_n\} \in \mathcal{T}(\Delta T)$, $\omega(x; \{\Delta t_n\})$ is nonempty and satisfies the following condition: for any $y \in \omega(x; \{\Delta t_n\})$, there exists $\Delta t \in \Delta T$ such that $F(\Delta t)y \in \omega(x; \{\Delta t_n\})$.*

Proof. We fix $x \in B$ and $\{\Delta t_n\} \in \mathcal{T}(\Delta T)$. In this proof, we use the notation $x^{(m)} = x^{(m)}(\{\Delta t_n\})$ for brevity. Since $\gamma^+(x; \{\Delta t_n\})$ is a sequence on the compact set B, there exists a convergent subsequence. Hence, $\omega(x; \{\Delta t_n\})$ is nonempty. Due to the characterization (11) of $\omega(x; \{\Delta t_n\})$, for arbitrarily chosen $y \in \omega(x; \{\Delta t_n\})$, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $x^{(n_k)} \to y$ ($k \to \infty$) holds.

Since F is continuous, we obtain
$$\forall \varepsilon > 0,\ \exists \delta > 0, \forall \tilde{x} \in B,\ |\Delta s_1 - \Delta s_2| < \delta \Rightarrow \|F(\Delta s_1)\tilde{x} - F(\Delta s_2)\tilde{x}\| < \frac{\varepsilon}{2}.$$
Moreover, $\Delta t_k \in \Delta T$ holds for all k by the assumption on $\{\Delta t_n\}$. Hence, there exists a subsequence of $\{n_k\}_{k=1}^{\infty}$, again denoted by $\{n_k\}_{k=1}^{\infty}$, such that $\lim_{k\to\infty} \Delta t_{n_k} \to \Delta t > 0$ for some $\Delta t \in \Delta T$, i.e.,
$$\forall \delta > 0,\ \exists k_1 \in \mathbb{N},\ \forall k \geq k_1, |\Delta t_{n_k} - \Delta t| < \delta.$$

Summing up the two claims above, we obtain

$$\forall \varepsilon > 0, \exists k_1 \in \mathbb{N}, \forall \tilde{x} \in B, \forall k \geq k_1, \|F(\Delta t_{n_k})\tilde{x} - F(\Delta t)\tilde{x}\| < \frac{\varepsilon}{2}. \tag{12}$$

On the other hand, since $\{x^{(n_k+1)}\}_{k=1}^{\infty}$ is a sequence on the compact set B, there exists a convergent subsequence $\{n'_k\}_{k=1}^{\infty}$, i.e.,

$$\forall \varepsilon > 0,\ \exists k_2 \in \mathbb{N},\ \forall k \geq k_2,\ \left\| x^{\left(n'_k+1\right)} - y' \right\| < \frac{\varepsilon}{2}. \tag{13}$$

Note that $y' \in \omega\left(x; \{\Delta t_n\}\right)$ holds.

For all $k \geq \max\{k'_1, k_2\}$, where k'_1 is a positive integer such that $n'_{k'_1} \geq n_{k_1}$ holds, the following inequality holds by (12) and (13):

$$\begin{aligned} \left\| F(\Delta t)x^{\left(n'_k\right)} - y' \right\| &\leq \left\| F(\Delta t)x^{\left(n'_k\right)} - F(\Delta t_{n'_k})x^{\left(n'_k\right)} \right\| + \left\| x^{\left(n'_k+1\right)} - y' \right\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, we obtain

$$F(\Delta t)y = F(\Delta t) \lim_{k\to\infty} x^{\left(n'_k\right)} = \lim_{k\to\infty} F(\Delta t)x^{\left(n'_k\right)} = y'.$$

It means that $F(\Delta t)y \in \omega\left(x; \{\Delta t_n\}\right)$ holds. □

The invariance that this lemma states is weak in the following sense: (i) it is only about the positive invariance (not the full invariance); (ii) the invariance is only conditional in the sense that it does hold for some Δt, and not for all $\Delta t > 0$; and furthermore, (iii) the Δt's vary depending on y.

Fortunately, however, the weak invariance is sufficient to establish a Lyapunov-type theorem for the non-semigroup dynamical systems. For the description of the main theorem, we define fixed points and Lyapunov functionals for non-semigroup dynamical systems.

Definition 6 (Fixed point)**.** *Let (X, F) be a non-semigroup dynamical system. $x \in X$ is called a fixed point of (X, F) if $F(\Delta t)x = x$ holds for all $\Delta t \geq 0$. The set $\mathcal{E}(F) \subseteq X$ denotes all such fixed points of (X, F).*

Definition 7 (Lyapunov functional)**.** *Let (X, F) be a non-semigroup dynamical system, ΔT be a subset of $(0, +\infty)$, and $B \subseteq X$ be a positively invariant set of F. A continuous map $\Phi : B \to \mathbb{R}$ is called a Lyapunov functional on B and ΔT, if the following conditions are satisfied.*

1. *$\Phi\left(F(\Delta t)x\right) \leq \Phi\left(x\right)$ holds for all $x \in B$ and $\Delta t \in \Delta T$;*
2. *If there exist $\Delta t \in \Delta T$ and $x \in B$ such that $\Phi\left(F(\Delta t)x\right) = \Phi\left(x\right)$ holds, then $x \in \mathcal{E}(F)$.*

Theorem 1 below is the main result of this paper. The proof of the second part is essentially the same as that in [16, Theorem 4.3].

Theorem 1. *Suppose that ΔT is a compact set. Let $B \subseteq X$ be a compact and positively invariant set of F, such that there exists a Lyapunov functional Φ on B and ΔT. Then, for all $x \in B$ and $\{\Delta t_n\} \in \mathcal{T}\left(\Delta T\right)$, $\omega\left(x; \{\Delta t_n\}\right)$ is nonempty and $\omega\left(x; \{\Delta t_n\}\right) \subseteq \mathcal{E}\left(F\right)$ holds. Furthermore, if the fixed points of F are isolated, then $\omega\left(x; \{\Delta t_n\}\right) = \{y\}$ for some $y \in \mathcal{E}\left(F\right)$.*

Proof. We fix $x \in B$ and $\{\Delta t_n\} \in \mathcal{T}(\Delta T)$. Then, $\omega(x; \{\Delta t_n\})$ is nonempty from Lemma 2. Since, for any $y \in \omega(x; \{\Delta t_n\})$, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $x^{(n_k)} \to y$ holds, $\Phi(y)$ is expressed as follows:

$$\Phi(y) = \Phi\left(\lim_{k\to\infty} x^{(n_k)}\right) = \lim_{k\to\infty} \Phi\left(x^{(n_k)}\right) = \inf_n \Phi\left(x^{(n)}\right).$$

Here, the final equality derives from the decreasing property of Φ. Hence, we obtain $\Phi(y_1) = \Phi(y_2)$ for all $y_1, y_2 \in \omega(x; \{\Delta t_n\})$. For all $y \in \omega(x; \{\Delta t_n\})$, there exists $\Delta t > 0$ such that $F(\Delta t)y \in \omega(x; \{\Delta t_n\})$ from Lemma 2. Therefore, $\Phi(F(\Delta t)y) = \Phi(y)$ holds, which means that $y \in \mathcal{E}(F)$.

Now we assume that the fixed points of F are isolated, i.e., there exists $\varepsilon > 0$ such that for any distinct $y_1, y_2 \in \omega(x; \{\Delta t_n\}) = \mathcal{E}(F) \cap \omega(x; \{\Delta t_n\})$, $\|y_1 - y_2\| > 2\varepsilon$. Since $\omega(x; \{\Delta t_n\})$ is compact, this assumption implies that $\omega(x; \{\Delta t_n\})$ contains a finite number of fixed points, say y_i, $i = 1, 2, \dots, I$. Then, there exists a sufficiently small δ $(0 < \delta < \varepsilon)$ such that $\|F(\Delta t)x_1 - F(\Delta t)x_2\| < \varepsilon$ holds for any $\Delta t \in \Delta T$ and for any $x_1, x_2 \in B$ satisfying $\|x_1 - x_2\| < \delta$. We define $B_i = \{\tilde{y} \in X \mid \|\tilde{y} - y_i\| < \delta\}$. Note that $B_i \cap B_j = \emptyset$ holds for any $i \neq j$. Let the set B^- denote the closure of $B \setminus \bigcup_i B_i$. Below we show the claim by contradiction: for this purpose, we assume that y_1 is not the unique member of $\omega(x; \{\Delta t_n\})$. Since y_1 is not the unique member of $\omega(x; \{\Delta t_n\})$, there exists a subsequence $\{n_k\}$ such that $x^{(n_k)} \in B_1$ and $x^{(n_k+1)} \notin B_1$ hold for all k. Then, we obtain

$$\|x^{(n_k+1)} - y_1\| = \|F(\Delta t_{n_k})x^{(n_k)} - F(\Delta t_{n_k})y_1\| < \varepsilon.$$

This means $x^{(n_k+1)} \in B^-$ for each k. However, since B^- is compact, this sequence has a convergent subsequence, which is a contradiction. □

4 Application to Dissipative Numerical Integrators

In this section, we apply the main theorem to numerical integrators. We first consider abstract dissipative integrators on X, and then the discrete gradient integrators on $\mathbb{R}^d$ as their special cases.

Let $\mathcal{E}$ and $\mathcal{E}(F)$ denote the set of the fixed points of the original equation (1) and numerical integrator (4), respectively. Lemma 3 below states that they coincide under a reasonable assumption.

Lemma 3. *Let (X, F) be a non-semigroup dynamical system defined by* (4)*. If $f(x) = \hat{f}(x, x)$ holds for all $x \in X$, then $\mathcal{E} = \mathcal{E}(F)$ holds.*

Proof. By the definitions of the fixed points, we can represent the sets $\mathcal{E}$ and $\mathcal{E}(F)$ as $\mathcal{E} = \{x \in X \mid f(x) = 0\}$ and $\mathcal{E}(F) = \{x \in X \mid \hat{f}(x, x) = 0\}$, respectively. Hence, $x \in \mathcal{E} \iff f(x) = 0 \iff \hat{f}(x, x) = 0 \iff x \in \mathcal{E}(F)$ holds by the assumption. □

The implication of Theorem 1 for the numerical integrator (4) can be summarized as follows.

Theorem 2. *Consider a set $\Delta T \subseteq (0, +\infty)$ and general dissipative integrator* (4) *satisfying the condition: $G(F(\Delta t)x) = G(x)$ holds for some $\Delta t \in \Delta T$, only when $x \in \mathcal{E}(F)$. Then, G serves as a Lyapunov functional on $B := \{x \mid G(x) \leq G(x^{(0)})\}$ and ΔT in the sense of Definition* 7*.*

If B and ΔT are compact, then, for all $\{\Delta t_n\} \in \mathcal{T}(\Delta T)$, and for all $x^{(0)}$, it holds $\omega\left(x^{(0)}; \{\Delta t_n\}\right) \subseteq \mathcal{E}$. If the fixed points of the original equation are isolated, $\omega\left(x^{(0)}; \{\Delta t_n\}\right) = \{y\}$ for some $y \in \mathcal{E}$, i.e., the numerical solution converges to a fixed point of the original equation.

Proof. Note that this F defines a non-semigroup dynamical system (X, F). We only have to prove the following claim: under the assumption of this theorem, B is a positively invariant set. For any $x \in B$ and $\Delta t \in \Delta T$, $G(F(\Delta t)x) \leq G(x) \leq G(x^{(0)})$ holds, which means $F(\Delta t)x \in B$. □

Let us more specifically consider the linear gradient case on $X = \mathbb{R}^d$ that we considered in Introduction. For such a system, the discrete gradient integrators defined by (6) automatically satisfy some of the assumptions of Theorem 2. The remaining assumptions are only related to the original differential equations, i.e., they do not depend on the choice of the discrete gradients.

Corollary 1. *Suppose L satisfies $\frac{\mathrm{d}}{\mathrm{d}t}G(z) < 0$ for $\forall z \notin \mathcal{E}$. Consider the non-semigroup dynamical system defined by (6). Then G is a Lyapunov functional on B.*

Suppose also that B and ΔT are compact. Then, for all $\{\Delta t_n\} \in \mathcal{T}(\Delta T)$, and for all $z^{(0)}$, $\omega\left(z^{(0)}; \{\Delta t_n\}\right) \subseteq \mathcal{E}$ holds. If the fixed points of the original equation are isolated, $\omega\left(z^{(0)}; \{\Delta t_n\}\right) = \{y\}$ for some $y \in \mathcal{E}$, i.e., the numerical solution converges to a fixed point of the original equation.

Let us illustrate the above results taking the following test scalar problem as an example (see, e.g., Matsuo–Furihata [26]):

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = -\nabla G(z) = z - z^3, \quad G(z) = \frac{(1-z^2)^2}{4}.$$

Here, we can construct a candidate of discrete gradient $\nabla_{\mathrm{d}} G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$\nabla_{\mathrm{d}} G\left(z^{(n+1)}, z^{(n)}\right) = -\frac{z^{(n+1)} + z^{(n)}}{2} + \left(\frac{z^{(n+1)} + z^{(n)}}{2}\right)\left(\frac{\left(z^{(n+1)}\right)^2 + \left(z^{(n)}\right)^2}{2}\right). \tag{14}$$

Then, the discrete gradient integrator reads as follows:

$$\frac{z^{(n+1)} - z^{(n)}}{\Delta t_n} = -\nabla_{\mathrm{d}} G\left(z^{(n+1)}, z^{(n)}\right).$$

The set $B = \{z \in \mathbb{R} \mid G(z) \leq C\}$ is obviously compact. Hence, if there exists a compact set $\Delta T \subseteq (0, +\infty)$ such that $\Delta t_n \in \Delta T$ for all n, the numerical solution converges to one of the fixed points $\{-1, 0, 1\}$.

Let us employ the adaptive time-stepping given by

$$\Delta t_n = \frac{\alpha}{\left|z^{(n)}\left(1 - \left(z^{(n)}\right)^2\right)\right| + \beta}$$

for $z^{(0)} = 1.5$, where $\alpha, \beta > 0$. Then, the compact set $\Delta T = \left[\frac{8\alpha}{15+8\beta}, \frac{\alpha}{\beta}\right]$ satisfies $\Delta t^{(n)} \in \Delta T$. Hence, Corollary 1 implies that the associated numerical solution should converge to one of the true fixed points, which can be confirmed numerically (see Fig. 2, 3).

5 Conclusions and Remarks

In this paper, we established a Lyapunov-type theorem for dissipative numerical integrators with adaptive time-stepping. There we employed a forward approach (instead of the pullback approach in the cocycles and nonautonomous systems studies). This destroys the invariance of the limit sets, but we proved that a weaker invariance can still hold, and that is sufficient to construct a Lyapunov-type theorem. As an application, we gave a theorem that precisely describes the asymptotic behavior of the discrete gradient integrators.

The results can be applied to various dissipative integrators keeping Lyapunov functionals. In the literature, extensive effort has been devoted to such integrators, for example, see [9, 10, 25, 8, 38] (see also [11] and the references therein).

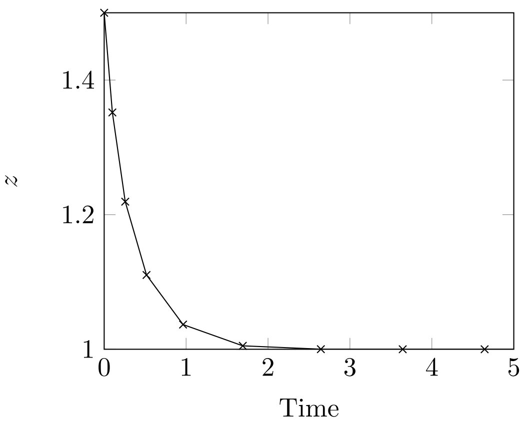

Figure 2: $\alpha = \beta = 5/24$

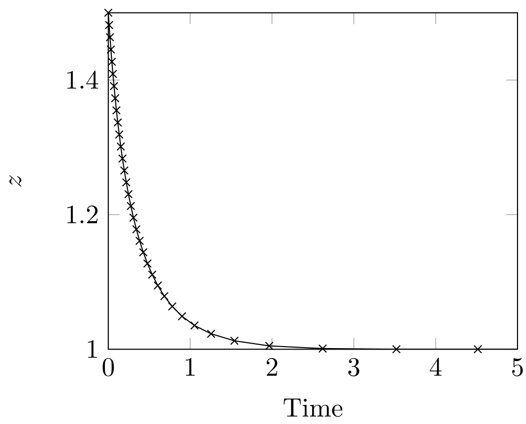

Figure 3: $\alpha = \beta = 5/264$

As mentioned earlier, the forward and pullback approaches are such that they complement each other—one gives up invariance, while the other is not suitable to consider each orbit. Compared to the rich literature on the pullback approach, it seems quite little has been known for the forward approach, at least to the best knowledge of the present authors. It should be interesting to further study to which extent the forward approach can make sense, and the relation between the results on the forward and pullback approaches. The present authors are now working on this issue, and the results will be reported elsewhere soon.

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