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Kei KIMURA and Kazuhisa MAKINO

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

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Satisfiability Preserving Assignments and Their Local and Linear Forms

Kei KIMURA
Department of Mathematical Informatics, University of Tokyo
kei_kimura@mist.i.u-tokyo.ac.jp

Kazuhisa MAKINO
Research Institute of Mathematical Sciences, Kyoto University
makino@kurims.kyoto-u.ac.jp

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Abstract

In this paper, we study several variable-based satisfiability preserving assignments to the constraint satisfaction problem. In particular, we consider fixable, autark and satisfiable partial assignments, as well as their local and linear forms. We show the inclusion relationships among the original and local forms of the satisfiability preserving assignments, and discuss maximality for linear satisfiability preserving assignments, which are defined as linear cones of the associated real space. As an application, we present a pseudo-polynomial time algorithm that computes a linear fixable assignment for integer linear systems, which also implies the well known pseudo-polynomially for integer linear systems such as two variables par inequality (TVPI), Horn and q-Horn systems.

1 Introduction

The constraint satisfaction problem (CSP, in short) is given as a triple (V, D, F), where V is a set of variables, D is variable domain, and F is a set of constraints on the variables that specify the permitted or forbidden combinations of value assignment to variables. A solution to the CSP instance is an assignment to all variables such that all constraints are satisfied. The CSP can express a number of problems in diverse fields, and is recognized as one of the most fundamental problems in computer science, see e.g., [7, 8, 29, 31]. The CSP has been studied extensively from both theoretical and practical point of view. For example, it is known that the CSP has dichotomy property if the size of the domain is bounded by 3 [4, 30]. Namely, it is polynomially solvable if all the constraints satisfy a certain property, and NP-hard, otherwise. Moreover, due to practical importance, many heuristic algorithms have been proposed [8, 29, 31].

The Boolean satisfiability problem (SAT) can be regarded as Boolean CSP, in which each constraint is represented by a clause (i.e., disjunction of some literals). The SAT itself is an important problem in computer science. It is well-known that the SAT is NP-hard [5], but several restricted classes such as 2-SAT [9], Horn SAT [14], renamable Horn SAT [25], q-Horn SAT [3], and LinAut SAT [20, 33] are solvable in polynomial time.

Integer linear systems are also formulated as the CSP. Given a matrix $G = (g_{ij}) \in \mathbb{Z}^{m \times n}$, a vector $h \in \mathbb{Z}^m$, and $D = \{0, 1, \ldots, d\}$, where $\mathbb{Z}$ denote the set of all integers, compute an integer vector $x \in D^n$ such that $Gx \geq h$. It is a well-studied topic, especially in the field of mathematical programming. The problem is strongly NP-hard, but several (semi-)tractable subclasses are known to exist. For example, the problem can be solved in polynomial time, if n is bounded by some constant [24], or G is totally unimodular [16]. Moreover, it can be solved in pseudo-polynomial time if (1) m is bounded by some constant [27], (2) G is quadratic [1, 15] (also called TVPI, i.e., each row of G contains at most two non-zero elements), (3) G is Horn [12, 34] (i.e., each row of G contains at most one positive element), or (4) G is q-Horn [19]. It is also known that the problem is weakly NP-hard, even if m is bounded by some constant or G is quadratic Horn [23] (also called monotone quadratic).

As mentioned above, many practical algorithms have been proposed to solve the CSP (including the Boolean satisfiability problem and integer linear systems). One of the important methods is based on how to reduce the problem size by partial variable assignments. For example, satisfiability preserving assignments are used to reduce the problem size, where a partial assignment is called *satisfiability preserving* if the satisfiability of the problem does not change after substituting it to the corresponding variables. Once a satisfiability preserving assignment is available, instead of solving the original problem instance, we may solve the problem instance obtained from the original one by the assignment. Here we note that the size of the resulting problem instance is smaller than the size of the original one. Another example is found under the name of backdoor set. A backdoor set is a subset of variables whose instantiation leads a given formula to an easy problem [6, 11, 35]. In this paper, we study several kinds of satisfiability preserving assignments, investigate their relations and also consider linear forms which can be obtained efficiently.

Satisfiability preserving assignments has been considered in many fields such as artificial intelligence, optimization, and so on. For example, in artificial intelligence, interchangeability is one of the first form of satisfiability preserving assignment [10]. Another example is an autark assignment, which plays a central role in the paper and thus will be stated in depth in the next paragraph. Ones of the most general notions of satisfiability preserving assignments are tuple substitutability [17] and full dynamic substitutability [28]. Although it is preferable to obtain these assignments, it is known that finding these assignments is NP-hard in general. Thus, local forms and the restriction to the problem are also considered. See also [2, 18].

For the SAT problem, autark assignments were introduced by Monien and Speckenmeyer [26] to provide a fast exponential time algorithm. An autark assignment is a partial assignment A that satisfies any clause c if c contains a variable assigned by A. Later, Kullmann investigated autark assignments in terms of resolution refutation, and introduced a linear form of autark assignments, which can be found in polynomial time via solving linear programming problem [20]. It is known that to find a non-trivial autark assignment is NP-hard and well-known tractable classes such as 2-, Horn and q-Horn SAT are solved by finding linear autark assignments.

The results in this paper

In this paper, we investigate several satisfiability preserving assignments such as fixable, autark and satisfiable partial assignments. We first reveal the inclusion relations among satisfiability preserving assignments and their local forms. In particular, we show that for any prime CNF expression, a partial assignment is fixable if and only if it is locally fixable. We also study linear forms of satisfiability preserving assignments for Boolean CSP.

For Boolean CSP of n variables, we consider n-dimensional real space $\mathbb{R}^n$, and we identify a vector $y \in \mathbb{R}^n$ with a partial assignment x such that $x_j = 1$ if $y_j > 0$, $x_j = 0$ if $y_j < 0$, and $x_j = *$ (i.e., unassigned) if $y_j = 0$. Then, to compute satisfiability preserving assignments efficiently, we consider inner polyhedral conic approximations of them, i.e., polyhedral cones that are contained in the region corresponding to satisfiability preserving assignments. We note that linear autark assignments defined by Kullmann [20] are indeed inner polyhedral conic approximations of a special type. We investigate which polyhedral cone well-approximates satisfiability preserving assignments. For the SAT problem, i.e., given a CNF $\varphi = \wedge c_i$, we show the following statements.

(1) For any inner open polyhedral conic approximation K of partially satisfiable assignments of φ, some matrix M which satisfies $M_{ij} > 0$ (resp., < 0) only if c_i contains x_j (resp., $\overline{x}_j$) produces an inner open polyhedral conic approximation that contains K.

(2) For any inner closed polyhedral conic approximation K of autark assignments of φ, some matrix M which satisfies $M_{ij} > 0$ (resp., < 0) if and only if c_i contains x_j (resp., $\overline{x}_j$) produces an inner closed polyhedral conic approximation that contains K.

(3) Any matrix M with each row containing a nonzero entry which satisfies $M_{ij} > 0$ (resp., < 0) only if c_i contains x_j (resp., $\overline{x}_j$) produces an inner open polyhedral conic approximation that contains all partially satisfiable assignments of φ, if φ is monotone.

(4) Any matrix M which satisfies $M_{ij} > 0$ (resp., < 0) if and only if c_i contains x_j (resp., $\overline{x}_j$) produces an inner closed polyhedral conic approximation that contains all autark assignments of φ, if φ is monotone.

(5) Any matrix M which satisfies $M_{ij} > 0$ (resp., < 0) if and only if c_i contains x_j (resp., $\overline{x}_j$) produces a maximal inner polyhedral conic approximation of autark assignments, if φ is prime and monotone.

From these results, matrices of a certain form are only used to find good inner approximations of satisfiable and autark assignments. In particular, (5) is used to find a good inner approximation of fixable assignments of integer linear systems, since each inequality is monotone (unate, more precisely). We also note that (2) is an extension of the result by Kullmann [21] which states that autark assignments are covered with inner polyhedral conic approximations of a certain form.

As for algorithmic results, we consider integer linear systems. By applying (5) to each constraint, we show that a non-trivial linear fixable assignment can be found in pseudo-polynomial time. Here our algorithm makes use of the ellipsoid method and dynamic programming similarly to Knapsack problem. For unit 0-1 integer linear systems, we show that a non-trivial linear fixable assignment can be found in polynomial time. We also show that well-known efficiently solvable classes such as quadratic, Horn and q-Horn systems can be solved by repeatedly finding linear fixable assignments.

2 Constraint satisfaction problem and satisfiability preserving assignments

In this section, we fix the notations to represent the constraint satisfaction problem (CSP) and the Boolean satisfiability problem (SAT), and define several satisfiability preserving assignments. We then discuss their relationships.

Let V denote a set of variables with size $n = |V|$, and let D denote a finite set, called the *domain* of variables in V. For a subset $X \subseteq V$, an element $A \in D^X$ is called a *partial assignment* to X, and an (*full*) *assignment*, if $X = V$ holds in addition. We also denote it by $X = A$. For simplicity, we write $x = p$ if $X = \{x\}$ and $p \in D$. A subset C of partial assignments to some $X \subseteq V$ (i.e., $C \subseteq D^X$) is sometimes called a *constraint*. Constraint satisfaction problem (CSP) is defined by a triplet (V, D, F) such that F denotes a set of constraints C_i with $m = |F|$, where we assume that $C_i \subseteq D^{V(C_i)}$. An (full) assignment $A \in D^V$ is called a *solution* (or a *satisfiable full assignment*) to the CSP instance (V, D, F) if for every constraint $C \in F$, the restriction of A to $V(C)$, denoted by $\pi_{V(C)}(A)$, is a member of C. CSP is the problem of finding a solution to a given instance (V, D, F). It is clear that the set of solutions to a CSP instance specifies a subset S of D^V. Thus, for an $S \subseteq D^V$, (V, D, F) (or F if V and D are clear from the context) is called an *expression* of S if the set of its solutions equals to S.

The Boolean satisfiability problem (SAT) is regarded as a special case of CSP, in which $V = \{x_1, \ldots, x_n\}$ denotes the set of propositional variables, i.e., $D = \{0, 1\}$, and F is given by a conjunctive normal form (CNF) $\varphi = \bigwedge_{i=1}^{m}(\bigvee_{j \in J_+^{(i)}} x_j \vee \bigvee_{j \in J_-^{(i)}} \overline{x}_j)$, where $J_+^{(i)}, J_-^{(i)} \subseteq V$ and $J_+^{(i)} \cap J_-^{(i)} = \emptyset$ for all $i = 1, \ldots, m$. Namely, the constraints in F correspond to clauses in φ. In this paper, we sometimes do not distinguish between CNFs φ and SAT instances I. Moreover, the problem of solving an integer linear system is also contained in CSP, where each constraint is given as a linear inequality.

Let S be a subset of D^V. For a partial assignment $X = A$, let $S[X = A] = \{\pi_{V \setminus X}(B) \mid B \in S, \pi_X(B) = A\}$. For a CSP instance (D, V, F) with the solution set $S \subseteq D^V$, a partial assignment $X = A$ is called *satisfiability preserving* if the satisfiability of the problem does not change after substituting it to the corresponding variables, i.e., $S[X = A]$ is empty if and only if so is S.

We now define several satisfiability preserving assignments.

Definition 1. *Let (D, V, F) be a CSP instance with solution set S. A partial assignment $X = A$ is called* satisfiable (*resp.,* fixable) *for (D, V, F)*

if $S[X = A] = D^{V \setminus X}$ *(resp.,* $S[X = A] \supseteq S[X = B]$ *for any partial assignment* B *to* X*).*

We note that the fixability was introduced by Bordeaux et al. [2] for single variable assignments. We also note that both satisfiable and fixable assignments are satisfiability preserving.

Example 1. *Let* $(D = \{0, 1, 2\}, V = \{x_1, x_2, x_3\}, F)$ *be a CSP instance whose solution set* S *is given as*

$$S = \{(0, 1, 1), (0, 2, 1), (1, 1, 0), (1, 1, 1), (1, 1, 2)\}.$$

Consider two partial assignments $(x_1, x_2) = (1, 1)$ *and* $(x_1, x_3) = (0, 1)$*. Then* $(x_1, x_2) = (1, 1)$ *is satisfiable for* (D, V, F)*, since* $S[(x_1, x_2) = (1, 1)] = D^{\{x_3\}}$*, while* $(x_1, x_3) = (0, 1)$ *is fixable, since* $S[(x_1, x_3) = A] = \{1, 2\}$ *if* $A = (0, 1)$*,* $\{1\}$ *if* $A = (1, 0), (1, 1), (1, 2)$*, and* $\emptyset$ *otherwise.*

Let us then consider several *local* satisfiability preserving assignments, i.e., satisfiability preserving assignments that are defined in terms of constraints in CSP.

Definition 2. *For a CSP instance* (D, V, F)*, a partial assignment* $X = A$ *is respectively called* (i) *locally satisfiable,* (ii) *autark, and* (iii) *locally fixable if* (i) $C[X = A] = D^{V(C) \setminus X}$ *for any constraint* C *in* F*,* (ii) $C[X = A] = D^{V(C) \setminus X}$ *for any constraint* C *in* F *with* $V(C) \cap X \neq \emptyset$*, and* (iii) $C[X = A] \supseteq C[X = B]$ *holds for any partial assignment* $X = B$ *and any constraint* C *in* F*.*

The concept of autark assignments was introduced by Monien and Speckenmeyer [26] for the Boolean case $D = \{0, 1\}$ to construct fast exact algorithms for SAT, and was investigated by Kullmann [20]. Kullmann [22] also introduce autark assignments for generalized clause sets for the non-Boolean case.

Example 2. *Let* $(D = \{0, 1, 2, 3\}, V = \{x_1, x_2, x_3\}, F)$ *be a CSP instance where* F *is given by a linear system*

$$\begin{array}{rcr} 2x_1 + 3x_2 - x_3 & \geq & 3 \\ 3x_1 - 4x_3 & \geq & -2 \\ - x_2 - x_3 & \geq & -5. \end{array} \tag{1}$$

Then three partial assignments $(x_2, x_3) = (1, 0)$*,* $x_2 = 2$ *and* $x_1 = 3$ *are respectively locally satisfiable, autark and locally fixable for* (D, V, F)*.*

2.1 Relationships among satisfiability preserving assignments

In this section, we consider the relationship among satisfiability preserving assignments defined in Section 2.

Definition 3. *For a CSP instance $I = (D, V, F)$, we define $P_{\text{l-sat}}(I)$, $P_{\text{sat}}(I)$, $P_{\text{autark}}(I)$, $P_{\text{l-fix}}(I)$, $P_{\text{fix}}(I)$ and $P_{\text{presv}}(I)$ as the set of all locally satisfiable, satisfiable, autark, locally fixable, fixable and satisfiability preserving assignments of I, respectively.*

We first show the following result for general CSP.

Theorem 1. *Let $I = (D, V, F)$ be a CSP instance. Then the following relations hold.*

$$P_{\text{l-sat}}(I) = P_{\text{sat}}(I) \subseteq P_{\text{autark}}(I) \subseteq P_{\text{l-fix}}(I) \subseteq P_{\text{fix}}(I) \subseteq P_{\text{presv}}(I). \qquad (2)$$

Before proving the theorem, we show by the following examples that no other relationship holds.

Example 3. *Consider four Boolean CSP instances $I = (D, V, F)$ with two variables, i.e., $D = \{0, 1\}$ and $V = \{x_1, x_2\}$.*

(i) *Let F be given as $\varphi = x_1x_2$. Then $x_1 = 1$ is contained in $P_{\text{autark}}(I)$, but not in $P_{\text{sat}}(I)$.*
(ii) *Let F be given as a single inequality $x_1 + x_2 \geq 2$. Then $x_1 = 1$ is contained in $P_{\text{l-fix}}(I)$, but not in $P_{\text{autark}}(I)$.*
(iii) *Let F be given as $\varphi = x_1(x_1 \vee x_2)$. Then $x_2 = 0$ is contained in $P_{\text{fix}}(I)$, but not in $P_{\text{l-fix}}(I)$.*
(iv) *Let F be given as $\varphi = (x_1 \vee x_2)$. Then $x_1 = 0$ is contained in $P_{\text{presv}}(I)$, but not in $P_{\text{fix}}(I)$.*

Proof of Theorem 1. It is not difficult from the definitions that

$$P_{\text{l-sat}}(I) = P_{\text{sat}}(I) \subseteq P_{\text{autark}}(I) \text{ and } P_{\text{fix}}(I) \subseteq P_{\text{presv}}(I). \qquad (3)$$

For $P_{\text{autark}}(I) \subseteq P_{\text{l-fix}}(I)$, let $X = A$ be an autark assignment. If $X \cap V(C) \neq \emptyset$ for a constraint C, then $D^{V \setminus X}$ includes $C[X = B]$ for any $B \in D^X$. On the other hand, if $X \cap V(C) = \emptyset$ for a constraint C, then $C[X = A] = C[X = B]$ for any $B \in D^X$. Thus, we have $P_{\text{autark}}(I) \subseteq P_{\text{l-fix}}(I)$.

For $P_{\text{l-fix}}(I) \subseteq P_{\text{fix}}(I)$, let $X = A$ be a local fixable assignment, and S be the solution set of I. Since each $C[X = A]$ specifies a subset $S_{C[X=A]} \subseteq D^{V \setminus X}$, i.e., $S_{C[X=A]} = \{\pi_{V \setminus X}(B) \mid B \in D^V, \pi_{V(C)}(B) \in C, \pi_X(B) = A\}$, we have $S[X = A] = \bigcap_C S_{C[X=A]}$. Note that for any $B \in D^X$, $C[X = A] \supseteq C[X = B]$ is equivalent to $S_{C[X=A]} \supseteq S_{C[X=B]}$. Thus $S[X = A] \supseteq S[X = B]$ holds for any $B \in D^X$, which implies that $X = A$ is fixable. $\square$

We note that the locality depends on properties on the constraints. We show that one more relation

$$P_{\text{autark}}(I) \supseteq P_{\text{l-fix}}(I) \tag{4}$$

is satisfied for any SAT instance I.

Theorem 2. *Let I denote a SAT instance (i.e., a Boolean CSP instance where the constraints are given by a CNF). Then we have*

$$P_{\text{l-sat}}(I) = P_{\text{sat}}(I) \subseteq P_{\text{autark}}(I) = P_{\text{l-fix}}(I) \subseteq P_{\text{fix}}(I) \subseteq P_{\text{presv}}(I). \tag{5}$$

Proof. By Theorem 1, it suffices to show that a local fixable assignment $X = A$ for a CNF φ is autark. For a clause $c \in \varphi$ with $V(c) \cap X \neq \emptyset$, we have a partial assignment $X = B$ such that $c[X = B] = D^{V(c) \setminus X}$. This, together with local fixability of $X = A$ implies $c[X = A] = D^{V(c) \setminus X}$. Hence $X = A$ is autark. □

A CNF φ is called *prime* if the solution set is different from the one of any CNF obtained from φ by replacing a clause $c \in \varphi$ by its proper subclause. If Boolean constraints are in addition represented by a prime CNF, it holds that

$$P_{\text{l-fix}}(I) \supseteq P_{\text{fix}}(I). \tag{6}$$

Theorem 3. *Let I denote a SAT instance such that the constraints are given by a prime CNF. Then we have*

$$P_{\text{l-sat}}(I) = P_{\text{sat}}(I) \subseteq P_{\text{autark}}(I) = P_{\text{l-fix}}(I) = P_{\text{fix}}(I) \subseteq P_{\text{presv}}(I). \tag{7}$$

Proof. Let φ be a prime CNF that represents a Boolean CSP instance, and let $X = A$ be a partial assignment. Assuming that $X = A$ is not locally fixable, we show that it is not fixable.

Let S be the set of solutions of φ. Since $X = A$ is not locally fixable, there exists a clause c in φ such that $c[X = A] \not\supseteq c[X = B]$ for some partial assignment $X = B$. This implies that $X = A$ is not a satisfiable assignment for c, i.e., $c[X = A] \neq \{0, 1\}^{V(c) \setminus X}$. On the other hand, by primality of φ, we have a satisfiable (full) assignment $V = A^*$ for φ that does not satisfy $c[X = A]$. This implies that $\pi_{V \setminus X}(A^*) \notin S[X = A]$ and $\pi_{V \setminus X}(A^*) \in S[X = \pi_X(A^*)]$, which proves that $X = A$ is not fixable for S. □

Let us finally mention integer linear systems. Note that any CNF can be represented by a 0-1 linear system, since a 0-1 assignment satisfies a clause $c = (\bigvee_{j \in J_+} x_j \vee \bigvee_{j \in J_-} \overline{x}_j)$ if and only if it satisfies $\sum_{j \in J_+} x_j + \sum_{j \in J_-} (1 - x_j) \geq 1$. Thus it follows from Example 3 that (4) or (6) is not always satisfied by integer linear systems.

3 Linear satisfiability preserving assignments

In this section, we restrict our attention to Boolean CSP and deal with linear forms of satisfiability preserving assignments.

For a vector $y \in \mathbb{R}^n$, let $V(y) = \{x_j \mid y_j \neq 0\}$ and define a partial assignment $V(y) = A(y)$ by $A(y)_j = 1$ if $y_j > 0$ and $A(y)_j = 0$ if $y_j < 0$. For $R \subseteq \mathbb{R}^n$, let $P(R) = \{V(y) = A(y) \mid y \in R\}$. Conversely, for a set of partial assignments P, we define the region of P by $R(P) = \{y \in \mathbb{R}^n \mid V(y) = A(y) \in P\}$. Note that $R(P)$ is a cone, where $R \subseteq \mathbb{R}^n$ is a cone if $\lambda y \in R$ holds for any $\lambda > 0$ and $y \in R$.

The following properties show that satisfiable partial assignments and satisfiability preserving assignments of CSP are characterized by the region of its solutions. Let $B \subseteq \mathbb{R}$ be the unit ball with center 0, i.e., $B = \{y \in \mathbb{R}^n \mid |y| \leq 1\}$. For a subset Y of $\mathbb{R}^n$, its closure is defined as $\mathrm{cl}(Y) = \{y \in \mathbb{R}^n \mid \forall\varepsilon > 0,\ (y + \varepsilon B) \cap Y \neq \emptyset\}$, and its interior is defined as $\mathrm{int}(Y) = \{y \in \mathbb{R}^n \mid \exists\varepsilon > 0,\ (y + \varepsilon B) \subseteq Y\}$. We note that these definitions are equivalent to the usual definitions of closure and interior defined in terms of convergence of vectors.

Proposition 1. *Let $I = (D = \{0,1\}, V, F)$ be a CSP instance with solution set $S(\neq \emptyset)$. Let y be a vector in $\mathbb{R}^n$. Then we have the following equivalences.*

(i) *$V(y) = A(y)$ is a satisfiable assignment of I if and only if y is contained in the interior of the closure of $R(S)$.*

(ii) *$V(y) = A(y)$ is a satisfiability preserving assignment of I if and only if y is contained in the closure of $R(S)$.*

Proof. (i): For the only-if part, assume that $V(y) = A(y)$ is satisfiable. Let ε be a positive real such that $\varepsilon < |y_j|$ for any $y_j \neq 0$, and let y' be a vector in $(y + \varepsilon B) \cap R(\{V = A \mid A \in \{0,1\}^n\})$. Since y' has no zero element, $V(y') = A(y')$ is a satisfiable full assignment. Hence for any $y' \in (y + (\varepsilon/2)B)$ and any $0 < \varepsilon' \leq \varepsilon/2$, we have $(y' + \varepsilon' B) \cap R(S) \neq \emptyset$, since $(y' + \varepsilon' B) \subseteq (y + \varepsilon B)$ and clearly there exists a point in $(y' + \varepsilon' B)$ which corresponds to a full assignment. Hence $(y + (\varepsilon/2)B) \subseteq \mathrm{cl}(R(S))$ holds, which implies that y is contained in $\mathrm{int}(\mathrm{cl}(R(S)))$.

For the if part, suppose that y is contained in the interior of the closure of $R(S)$. Then for sufficiently small $\varepsilon > 0$, $y + \varepsilon w$ is also contained in the interior of the closure of $R(S)$ for all $w \in \{-1,1\}^n$. Note that $y + \varepsilon w$ has no zero element, i.e., $V(y + \varepsilon w) = V$. This means that $V(y + \varepsilon w) = A(y + \varepsilon w)$ is a satisfiable full assignment for any $w \in \{-1,1\}^n$. Thus, a partial assignment $V(y) = A(y)$ is arbitrarily extendable to a satisfiable full assignment and hence it is satisfiable.

(ii): For the only-if part, suppose that y is not contained in the closure of $R(S)$. We show that $V(y) = A(y)$ is not satisfiability preserving.

Since $y \notin \text{cl}(R(S))$, there exists an $\varepsilon > 0$ such that $(y+\varepsilon B)\cap\text{cl}(R(S)) = \emptyset$. Thus, for sufficiently small $\varepsilon' > 0$, $y+\varepsilon' w \notin R(S)$ and $\pi_{V(y)}(A(y+\varepsilon' w)) = A(y)$ hold for all $w \in \{-1,1\}^n$. This implies that $V(y) = A(y)$ is not extendable to any satisfiable full assignment. Hence, $V(y) = A(y)$ is not satisfiability preserving.

For the if part, suppose y is contained in the closure of $R(S)$. Then for sufficiently small $\varepsilon > 0$, $y + \varepsilon w \in R(S)$ and $\pi_{V(y)}(A(y + \varepsilon w)) = A(y)$ hold for some $w \in \{-1,1\}^n$. Thus, $V(y) = A(y)$ is extendable to a satisfiable assignment, which implies that $V(y) = A(y)$ is satisfiability preserving. □

We remark that $R(S)$ is not equal to $\text{int}(\text{cl}(R(S)))$. For example, for a SAT instance $\varphi = x_1 \vee x_2$, we have $R(S) = \{y \in \mathbb{R}^2 \mid y_1 > 0, y_2 \neq 0\} \cup \{y \in \mathbb{R}^2 \mid y_2 > 0, y_1 \neq 0\}$ and $\text{int}(\text{cl}(R(S))) = \{y \in \mathbb{R}^2 \mid y_1 > 0 \text{ or } y_2 > 0\}$.

We now introduce linear satisfiability preserving assignments. Note that Proposition 1 implies that computing a point in the closure of $R(S)$ is equivalent to computing a satisfiability preserving assignment. However, $R(S)$ and/or its closure are not given explicitly (i.e., they are given as a CSP instance (D, V, F)). Moreover, they are not convex, although they are cones. It is in general difficult to compute a point from the closure of $R(S)$. We thus consider convex polyhedral cones K contained in the closure of $R(S)$. Here convex polyhedrality helps efficient computation of a point in K. Since such polyhedral cones K are inner approximations of satisfiability preserving assignments, we study how to construct *large* such cones.

Definition 4. *Let $(D = \{0,1\}, V, F)$ be a Boolean CSP instance. A subset P of satisfiable (resp., autark, local fixable, fixable and satisfiability preserving) assignments of (D, V, F) is called linear if it is described by some convex polyhedral cone K and such an assignment in P is called K-linear.*

Linear autark assignment is a generalization of simple linear autark assignments introduced by Kullmann [20].

We identify a partial assignment $X = A$ with a vector a in $\{0,1,*\}^n$ such that $a_j = 0$ if $j \in X$ and $A_j = 0$, $a_j = 1$ if $j \in X$ and $A_j = 1$, and $a_j = *$ otherwise (i.e., $j \notin X$).

Example 4. *Let $I = (D = \{0,1\}, V = \{x_1, x_2\}, F)$ be a SAT instance, where F is given by a CNF $\varphi = x_1 \vee x_2$. Then we have*

$$\begin{aligned} &P_{\text{sat}}(I) = \{(0,1),(1,0),(1,1),(1,*),(*,1)\},\\ &P_{\text{autark}}(I) = P_{\text{l-fix}}(I) = P_{\text{fix}}(I) = P_{\text{sat}}(I) \cup \{(*,*)\},\\ &P_{\text{presv}}(I) = \{0,1,*\}^2 \setminus \{(0,0)\}. \end{aligned} \tag{8}$$

Consider three convex cones defined by linear systems:

$$\begin{aligned} K_1 &= \{y \in \mathbb{R}^2 \mid y_1 + y_2 > 0\}, \\ K_2 &= \{y \in \mathbb{R}^2 \mid y_1 + y_2 \geq 0\}, \\ K_3 &= \{y \in \mathbb{R}^2 \mid y_1 \geq 0, y_1 - y_2 \geq 0\}. \end{aligned} \tag{9}$$

Then we have

$$K_1 \subseteq R(P_{\text{sat}}(I)),\ K_2 \subseteq R(P_{\text{autark}}(I)) \text{ and } K_3 \subseteq R(P_{\text{presv}}(I)). \tag{10}$$

In fact,

$$P(K_1) = P_{\text{sat}}(I),\ P(K_2) = P_{\text{autark}}(I) \text{ and } P(K_3) \subseteq P_{\text{presv}}(I). \tag{11}$$

Lemma 1. *Let $I = (D = \{0,1\}, V, F)$ be a Boolean CSP instance.*

(i) *For an open subset $K \subseteq R(P_{\text{sat}}(I))$, there exists an open cone K^* such that $K \subseteq K^* \subseteq R(P_{\text{sat}}(I))$. If K is in addition convex (resp., linear), then K^* can be taken from convex (resp., linear) cones.*

(ii) *For a closed subset $K \subseteq R(P_{\text{presv}}(I))$, there exists a closed convex cone K^* such that $K \subseteq K^* \subseteq R(P_{\text{presv}}(I))$. If K is in addition convex (resp., linear), then K^* can be taken from convex (resp., linear) cones.*

Proof. (i): Let K^* be the conic hull of K. Then K^* is open, and we have $K^* \subseteq R(P_{\text{sat}}(I))$, since $R(P_{\text{sat}}(I))$ is a cone. Since convexity and linearity are preserved by taking the conic hull, the proof is competed.

(ii): Let K^* the closure of the conic hull of K. Then $K^* \subseteq R(P_{\text{presv}}(I))$, since $R(P_{\text{presv}}(I))$ is a closed cone by Proposition 1. It is again well known that convexity and linearity are preserved by taking the closure of the conic hull. □

We note that for a SAT instance I and a non-open convex subset K of $R(P_{\text{sat}}(I))$ there does not always exists an open convex subset of $R(P_{\text{sat}}(I))$ which contains K, as the next example shows.

Example 5. *Let $I = (D = \{0,1\}, V = \{x_1 x_2\}, F)$ be a SAT instance, where F is given by $\varphi = x_1 \vee x_2$. For a non-open convex subset $K = \{y \in \mathbb{R}^2 \mid y_1 + y_2 > 0\} \cup \{(1,-1)\}$ of $R(P_{\text{sat}}(I))$, there does not exists an open convex subset of $R(P_{\text{sat}}(I))$ which contains K, since a convex set containing both K and $\{(1,-1)\} + \varepsilon B$ also contains a point not in $R(P_{\text{sat}}(I))$ for any $\varepsilon > 0$.*

For a SAT instance, we now introduce special forms of linear satisfiability preserving assignments obtained from CNF expressions. For two vectors y and $z \in \mathbb{R}^W$, we write $y \gg z$, if $y_j > z_j$ for all $j \in W$. Note that

$y > z$ is different from $y \gg z$, where $y > z$ denotes that $y_j \geq z_j$ for all $j \in W$ and $y \neq z$. For a CNF $\varphi = \bigwedge_{i=1}^{m} c_i$, let M_φ denote an $m \times n$ matrix such that (i,j) entry is 1 if $x_j \in c_i$, -1 if $\overline{x}_j \in c_i$, and 0 otherwise (i.e., $x_j \notin V(c_i)$). For a matrix M, let $K_{\gg}(M)$ and $K_{\geq}(M)$ respectively denote open and closed polyhedral cones defined by

$$\begin{aligned} K_{\gg}(M) &= \{y \in \mathbb{R}^n \mid My \gg 0\}, \\ K_{\geq}(M) &= \{y \in \mathbb{R}^n \mid My \geq 0\}. \end{aligned} \tag{12}$$

We first show the following proposition.

Proposition 2. *Let $I = (D = \{0,1\}, V, F)$ be a SAT instance, where F is given by a CNF φ. Then we have*
(i) $K_{\gg}(M_\varphi) \subseteq R(P_{\rm sat}(I))$,
(ii) $\mathrm{cl}(K_{\gg}(M_\varphi)) \subseteq K_{\geq}(M_\varphi) \subseteq R(P_{\rm autark}(I)) \subseteq R(P_{\rm presv}(I))$.

Proof. It is known by Kullmann [20, 21] that $K_{\gg}(M_\varphi) \subseteq R(P_{\rm sat}(I))$ and $K_{\geq}(M_\varphi) \subseteq R(P_{\rm autark}(I))$. Moreover, we have $\mathrm{cl}(K_{\gg}(M_\varphi)) \subseteq K_{\geq}(M_\varphi)$, since $K_{\geq}(M_\varphi)$ is a closed set. Finally, by Theorem 1, $R(P_{\rm autark}(I)) \subseteq R(P_{\rm presv}(I))$. □

We note that $\mathrm{cl}(K_{\gg}(M_\varphi)) \subseteq R(P_{\rm sat}(I))$ and $\mathrm{cl}(K_{\gg}(M_\varphi)) = K_{\geq}(M_\varphi)$ are not always satisfied. Let $I_1 = (D = \{0,1\}, V_1 = \{x_1\}, F_1)$, where F_1 is given by $\varphi_1 = x_1$, and $I_2 = (D = \{0,1\}, V_2 = \{x_1, x_2\}, F_2)$, where F_2 is given by $\varphi_2 = (x_1 \vee \overline{x}_2)(\overline{x}_1 \vee x_2)$. Then we have $\mathrm{cl}(K_{\gg}(M_{\varphi_1})) = \{y_1 \in \mathbb{R} \mid y_1 \geq 0\}$ and $R(P_{\rm sat}(I_1)) = \{y_1 \in \mathbb{R} \mid y_1 > 0\}$, which implies $\mathrm{cl}(K_{\gg}(M_{\varphi_1})) \not\subseteq R(P_{\rm sat}(I_1))$. We also have $\mathrm{cl}(K_{\gg}(M_{\varphi_2})) = \emptyset$ and $K_{\geq}(M_{\varphi_2}) = \{y \in \mathbb{R}^2 \mid y_1 = y_2\}$, which implies $\mathrm{cl}(K_{\gg}(M_{\varphi_2})) \neq K_{\geq}(M_{\varphi_2})$.

By Proposition 2, a satisfiable partial (resp., autark) assignment is called simple linear if a corresponding vector is contained in $K_{\gg}(M_\varphi)$ (resp., $K_{\geq}(M_\varphi)$).

Example 6. *Let $(D = \{0,1\}, V = \{x_1, x_2\}, F)$ be a SAT instance, where F is given by a CNF $\varphi = (x_1 \vee x_2)(x_1 \vee x_3)(x_2 \vee x_3)$. Then satisfiable assignment $(x_1, x_2) = (1,1)$ is simple linear and autark assignment $x_3 = 1$ is simple linear, since $(2,3,0) \in K_{\gg}(M_\varphi)$ and $(0,0,2) \in K_{\geq}(M_\varphi)$.*

We note that $K_{\gg}(M_\varphi) = \emptyset$ and $K_{\geq}(M_\varphi) = \{0\}$ might hold in general.

Example 7. *Let $I = (D = \{0,1\}, V = \{x_1, x_2, x_3\}, F)$ be a SAT instance, where F is given by a CNF $\varphi = (x_1 \vee x_2 \vee x_3)(x_1 \vee \overline{x}_2 \vee \overline{x}_3)(\overline{x}_1 \vee x_2 \vee \overline{x}_3)(\overline{x}_1 \vee \overline{x}_2 \vee x_3)$. It can be verified that*

$$\begin{aligned} &P_{\rm sat}(I) = \{(0,0,1), (0,1,0), (1,0,0), (1,1,1)\}, \\ &P_{\rm autark}(I) = \{(0,0,1), (0,1,0), (1,0,0), (1,1,1), (*,*,*)\}, \\ &P_{\rm presv}(I) = \{0,1,*\}^3 \setminus \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}. \end{aligned} \tag{13}$$

On the other hand, there exists no simple linear satisfiable assignment nor non-trivial simple linear autark assignment, since

$$M_\varphi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \tag{14}$$

and hence $K_\gg(M_\varphi) = \emptyset$ *and* $K_\geq(M_\varphi) = \{0\}$.

We now define the perturbed version of simple linear satisfiable and autark assignments. For a real matrix M, let $\mathcal{Q}(M)$ denote the set of matrices N with sign pattern identical to M, i.e., $N_{ij} < 0$ (resp., $= 0$, > 0) if and only if $M_{ij} < 0$ (resp., $= 0$, > 0), and let $\mathcal{Q}_0(M)$ denote the set of matrices N such that 1) $N_{ij} < 0$ (resp., $N_{ij} > 0$), only if $M_{ij} < 0$ (resp., $M_{ij} > 0$), and 2) each row contains at least one nonzero entry.

Proposition 3. *Let* $I = (D = \{0,1\}, V, F)$ *be a SAT instance, where* F *is given by a CNF* φ. *Then we have*
(i) $K_\gg(M) \subseteq R(P_{\text{sat}}(I))$ *for all* $M \in \mathcal{Q}_0(M_\varphi)$,
(ii) $K_\geq(M) \subseteq R(P_{\text{autark}}(I))$ *for all* $M \in \mathcal{Q}(M_\varphi)$,
(iii) $\text{cl}(K_\gg(M)) \subseteq R(P_{\text{presv}}(I))$ *for all* $M \in \mathcal{Q}_0(M_\varphi)$.

Proof. Since (ii) is known by Kullmann [21], we show (i) and (iii) only.

(i): Let y be a vector in $K_\gg(M)$. Then for each $c \in \varphi$, there exists an element j such that ($x_j \in c$ and $y_j > 0$) or ($\overline{x}_j \in c$ and $y_j < 0$). In either case, $V(y) = A(y)$ should satisfy c, implying that it satisfies φ.

(iii): We have $K_\gg(M) \subseteq R(P_{\text{sat}}(I))$ for any $M \in \mathcal{Q}_0(M_\varphi)$ by (i). By taking the closure of both sides of $K_\gg(M) \subseteq R(P_{\text{sat}}(I))$, we obtain $\text{cl}(K_\gg(M)) \subseteq \text{cl}(R(P_{\text{sat}}(I)))$, and since $\text{cl}(R(P_{\text{sat}}(I))) \subseteq R(P_{\text{presv}}(I))$ by Proposition 1, we have $\text{cl}(K_\gg(M)) \subseteq R(P_{\text{presv}}(I))$. □

We note that the proof of (i) in Proposition 3 is similar to the one that $K_\gg(M) \subseteq R(P_{\text{sat}}(I))$ for $M \in \mathcal{Q}(M_\varphi)$ in [21].

Example 8. *Consider again the CSP instance* (D, V, F) *given in Example 7, i.e.,* $D = \{0,1\}$, $V = \{x_1, x_2, x_3\}$ *and* F *is given by a CNF* $\varphi = (x_1 \vee x_2 \vee x_3)(x_1 \vee \overline{x}_2 \vee \overline{x}_3)(\overline{x}_1 \vee x_2 \vee \overline{x}_3)(\overline{x}_1 \vee \overline{x}_2 \vee x_3)$. *Recall that there exists no simple linear satisfiable assignment nor non-trivial simple linear autark assignment for* φ. *On the other hand, define* $M_1 \in \mathcal{Q}_0(M_\varphi)$ *and* $M_2 \in \mathcal{Q}(M_\varphi)$ *by*

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{15}$$

Then we have

$$\begin{aligned} P(K_{\gg}(M_1)) &= P(K_{\geq}(M_2)) = \{(1,1,1),(*,*,*)\}, \\ P(\mathrm{cl}(K_{\gg}(M_1))) &= \{(1,1,1),(1,1,*),(1,*,1),(1,*,*),(*,1,1), \\ &\qquad (*,1,*),(*,*,1),(*,*,*)\}. \end{aligned} \tag{16}$$

These provide linear satisfiability preserving assignments. For example, $(x_1, x_2) = (1, 1)$ *is a* $\mathrm{cl}(K_{\gg}(M_1))$*-linear satisfiability preserving assignment.*

We note that for a CNF φ and a matrix $M \in \mathcal{Q}_0(M_\varphi)$, $K_{\geq}(M)$ is not always contained in $R(P_{\mathrm{presv}}(I))$. For example, for $\varphi = (x_1 \vee x_2)(x_1 \vee \overline{x}_2)$ and

$$M = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \in \mathcal{Q}_0(M_\varphi), \tag{17}$$

we have $K_{\geq}(M) \not\subseteq P_{\mathrm{presv}}(I)$, since $(-1, 0)$ is contained in $K_{\geq}(M)$ while not in $P_{\mathrm{presv}}(I)$ (since $x_1 = 0$ is not satisfiability preserving).

We also note $\{K_{\geq}(M) \mid M \in \mathcal{Q}(M_\varphi)\}$ and $\{\mathrm{cl}(K_{\gg}(M)) \mid M \in \mathcal{Q}_0(M_\varphi)\}$ are incomparable in general. For example, for $\varphi = (x_1 \vee \overline{x}_2)(\overline{x}_1 \vee x_2)$ and

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in \mathcal{Q}(M_\varphi), \tag{18}$$

there does not exist a matrix $M^* \in \mathcal{Q}_0(M_\varphi)$ with $K_{\geq}(M) \subseteq \mathrm{cl}(K_{\gg}(M^*))$, and for $\varphi = (x_1 \vee x_2)$ and

$$M = \begin{pmatrix} 1 & 0 \end{pmatrix} \in \mathcal{Q}_0(M_\varphi), \tag{19}$$

there does not exist a matrix $M^* \in \mathcal{Q}(M_\varphi)$ with $\mathrm{cl}(K_{\gg}(M)) \subseteq K_{\geq}(M^*)$.

The following proposition shows that linear satisfiable and autark assignments in Proposition 3 is enough to find satisfiable and autark assignments.

Proposition 4 ([21])**.** *Let* $I = (D, V, F)$ *be a SAT instance, where* F *is given by a CNF* φ*. Then, for any* $X = A \in P_{\mathrm{sat}}(I)$ *(resp.,* $P_{\mathrm{autark}}(I)$*) there exists a matrix* $M \in \mathcal{Q}(M_\varphi)$ *such that* $X = A$ *is* $K_{\gg}(M)$*-linear (resp.,* $K_{\geq}(M)$*-linear).*

We here study maximality of these linear assignments, which is useful to find a satisfiability preserving assignment.

Theorem 4. *Let* $I = (D, V, F)$ *be a SAT instance, where* F *is given by a CNF* φ*.*

(i) *Let* N *be a matrix that represents linear satisfiable assignments, i.e.,* $K_{\gg}(N) \subseteq R(P_{\mathrm{sat}}(I))$*. Then there exists a matrix* M *in* $\mathcal{Q}_0(M_\varphi)$ *such that* $K_{\gg}(N) \subseteq K_{\gg}(M)$*.*

(ii) *Let N be a matrix that represents linear autark assignments, i.e., $K_{\geq}(N) \subseteq R(P_{\text{autark}}(I))$. Then there exists a matrix M in $\mathcal{Q}(M_\varphi)$ such that $K_{\geq}(N) \subseteq K_{\geq}(M)$.*

Proof. (i): For a clause $c = \bigvee_{j \in J_+} x_j \vee \bigvee_{j \in J_-} \overline{x}_j$ in φ with $J_+ \cap J_- = \emptyset$, let $K_c = \{y \in \mathbb{R}^n \mid y_j \leq 0 \ (j \in J_+),\ y_j \geq 0 \ (j \in J_-)\}$. Then, K_c is a closed cone which contains no point in $R(P_{\text{sat}}(I))$. This implies $K_{\gg}(N) \cap K_c = \emptyset$. Since $K_{\gg}(N)$ is open, there exists a separating hyperplane between $K_{\gg}(N)$ and K_c, i.e., there exists a vector $\alpha \in \mathbb{R}^n$ such that $K_{\gg}(N) \subseteq \{y \in \mathbb{R}^n \mid \alpha y > 0\}$ and $K_c \subseteq \{y \in \mathbb{R}^n \mid \alpha y \leq 0\}$. We claim that $\alpha_j \geq 0$ for all $j \in J_+$, $\alpha_j \leq 0$ for all $j \in J_-$, and $\alpha_j = 0$ for all $j \notin J_+ \cup J_-$.

Since $-e_j$, e_j and $\pm e_j$ are respectively contained in K_c for $j \in J_+$, $j \in J_-$ and $j \notin J_+ \cup J_-$, we have $(-e_j)\alpha \leq 0$, $e_j\alpha \leq 0$ and $(\pm e_j)\alpha \leq 0$ respectively. Here e_j denotes the jth unit vector. This proves the claim.

Since $\alpha \neq 0$, the claim implies (i) in the theorem.

(ii): For a clause $c = \bigvee_{j \in J_+} x_j \vee \bigvee_{j \in J_-} \overline{x}_j$ in φ with $J_+ \cap J_- = \emptyset$, let K_c be a non-closed cone defined by $K_c = \{y \in \mathbb{R}^n \mid y_j \leq 0 \ (j \in J_+), y_j \geq 0 \ (j \in J_-), -\sum_{j \in J_+} y_j + \sum_{j \in J_-} y_j > 0\}$. Then K_c contains no point in $R(P_{\text{autark}}(I))$. This implies $K_{\geq}(N) \cap K_c = \emptyset$. There exists a vector $\alpha \in \mathbb{R}^n$ such that $K_{\geq}(N) \subseteq \{y \in \mathbb{R}^n \mid \alpha y \geq 0\}$ and $K_c \subseteq \{y \in \mathbb{R}^n \mid \alpha y < 0\}$. We claim that $\alpha_j > 0$ for all $j \in J_+$, $\alpha_j < 0$ for all $j \in J_-$, and $\alpha_j = 0$ for all $j \notin J_+ \cup J_-$.

Since $-e_j$ and e_j are respectively contained in K_c for $j \in J_+$ and $j \in J_-$, we have $(-e_j)\alpha < 0$ and $e_j\alpha < 0$ respectively, which proves the claim for $j \in J_+$ and $j \in J_-$. For $j \notin J_+ \cup J_-$, define two vectors z and z' by $z_j = 1$, $z'_j = -1$, $z_k = z'_k = 0$ if $k \notin J_+ \cup J_- \cup \{j\}$, $z_k = z'_k = -\varepsilon$ if $k \in J_+$, and $z_k = z'_k = +\varepsilon$ if $k \in J_-$ for any $\varepsilon > 0$. Then z is contained in K_c and hence we have $\alpha z < 0$. This implies that $-\varepsilon(\sum_{k \in J_+} \alpha_k - \sum_{k \in J_-} \alpha_k) < \alpha_j < \varepsilon(\sum_{k \in J_+} \alpha_k - \sum_{k \in J_-} \alpha_k)$. Since $\varepsilon > 0$ is arbitrary, we have $\alpha_j = 0$. □

While $\{K_{\gg}(M) \mid M \in \mathcal{Q}_0(M_\varphi)\}$ and $\{K_{\geq}(M) \mid M \in \mathcal{Q}(M_\varphi)\}$ represents a family of convex sets which are *large* in $R(P_{\text{sat}}(I))$ and $R(P_{\text{autark}}(I))$ respectively by Theorem 4, they are respectively not maximal convex subset of $R(P_{\text{sat}}(I))$ and $R(P_{\text{autark}}(I))$ in terms of region, as the next examples show.

For a subset R in $\mathbb{R}^n$, we denote by $\text{conv}(R)$ its convex closure.

Example 9. *Let $I = (D = \{0,1\}, V = \{x_1, x_2\}, F)$ be a SAT instance, where F is given by a CNF $\varphi = x_1 \vee x_2$. Then no $M \in \mathcal{Q}_0(M_\varphi)$ produces a maximal convex set $K_{\gg}(M)$ of $R(P_{\text{sat}}(I))$. This is because* $\text{conv}(K_{\gg}(M) \cup \{y\})$ *is respectively contained in $R(P_{\text{sat}}(I))$ for $y = (M_{12}, -M_{11}) \notin K_{\gg}(M)$ if $M_{12} \neq 0$ and $y = (0,1) \notin K_{\gg}(M)$ if $M_{12} = 0$.*

Example 10. *Let $I = (D = \{0,1\}, V = \{x_1, x_2\}, F)$ be a SAT instance, where F is given by a CNF $\varphi = x_1(x_1 \vee x_2)$. Then for any $M \in \mathcal{Q}(M_\varphi)$, $K_{\geq}(M)$ is not a maximal convex subset of $R(P_{\text{autark}}(I))$. We note that $R(P_{\text{autark}}(I)) = \{y \in \mathbb{R}^2 \mid y_1 > 0\} \cup \{y \in \mathbb{R}^2 \mid y_1 \geq 0, y_2 \geq 0\}$. $y = (M_{22}, -M_{21} - 1)$ is contained in $R(P_{\text{autark}}(I))$, but it is not contained in $K_{\geq}(M)$, since it violates $M_{21}y_1 + M_{22}y_2 \geq 0$. Moreover, we can see that $\text{conv}(K_{\geq}(M) \cup \{y\})$ is contained in $R(P_{\text{autark}}(I))$.*

The next theorem shows that there exist maximal elements in $\{K_{\gg}(M) \mid M \in \mathcal{Q}_0(M_\varphi)\}$ and $\{K_{\geq}(M) \mid M \in \mathcal{Q}(M_\varphi)\}$ in terms of partial assignment, which contrasts Examples 9 and 10.

Theorem 5. *Let $I = (D, V, F)$ be a SAT instance, where F is given by a CNF φ.*

(i) *For any convex set C contained in $R(P_{\text{sat}}(I))$, there exists a matrix $M \in \mathcal{Q}_0(M_\varphi)$ such that $P(C) \subseteq P(K_{\gg}(M))$.*

(ii) *For any convex set C contained in $R(P_{\text{autark}}(I))$, there exists a matrix $M \in \mathcal{Q}(M_\varphi)$ such that $P(C) \subseteq P(K_{\geq}(M))$.*

Proof. Since we have 3^n partial assignments, there exists finite points $y^{(1)}$, $\ldots, y^{(\ell)}$ in C such that $P(\{y^{(1)}, \ldots, y^{(\ell)}\}) = P(C)$.

For (i), let K denote the convex hull of $y^{(i)}$ $(i = 1, \ldots, \ell)$. It follows from convexity of C that $K \subseteq R(P_{\text{sat}}(I))$. By an argument similar to that in Theorem 4, we have $K \subseteq K_{\gg}(M)$ for some $M \in \mathcal{Q}_0(M_\varphi)$.

For (ii), let K denote the convex cone closure of $y^{(i)}$ $(i = 1, \ldots, \ell)$. It follows from convexity of C that $K \subseteq R(P_{\text{autark}}(I))$. By Theorem 4, there exists a matrix $M \in \mathcal{Q}(M_\varphi)$ such that $K \subseteq K_{\geq}(M)$, which proves the theorem. □

We remark that M_φ itself might not be maximal as seen in Example 7 and 8.

A CNF φ is called monotone if it contains no negative literal. For monotone SAT instance, we have the following good properties.

Theorem 6. *Let $I = (D, V, F)$ be a monotone SAT instance, i.e., a SAT instance where F is given by a monotone CNF φ. For any matrix $M \in \mathcal{Q}(M_\varphi)$, we have $P(K_{\gg}(M)) = P_{\text{sat}}(I)$ and $P(K_{\geq}(M)) = P_{\text{autark}}(I)$.*

Proof. We first show the case of $P_{\text{sat}}(I)$. For a partial satisfiable assignment $X = A$, let y be a vector in $\mathbb{R}^n$ defined as $y_j = \gamma$ if $A_j = 1$, $y_j = -1$ if $A_j = 0$, and $y_j = 0$ otherwise, where γ is sufficiently large. Then it is clear that a partial assignment $V(y) = A(y)$ is equal to $X = A$. Moreover, we have $My \gg 0$.

We can similarly show the case of autark assignments. □

Theorem 6 does not hold if we replace $\mathcal{Q}(M_\varphi)$ to $\mathcal{Q}_0(M_\varphi)$ for the case of satisfiable assignments. Indeed, for a SAT instance I represented by a monotone CNF $\varphi = x_1 \vee x_2$, let $M = (1\ 0) \in \mathcal{Q}_0(M_\varphi)$. Then we have $P(K_{\gg}(M)) = \{(1,0),(1,1),(1,*)\} \subsetneq P_{\text{sat}}(I)$.

We also remark that by Examples 9 and 10, even if φ is a monotone CNF, there does not always exist an $M \in \mathcal{Q}(M_\varphi)$ such that $K_{\gg}(M)$ (resp., $K_{\geq}(M)$) is a maximal convex subset of $R(P_{\text{sat}}(I))$ (resp., $R(P_{\text{autark}}(I))$). This contrasts Theorem 6.

We next consider prime monotone CNFs. In this case, maximality holds in terms of not only partial assignments but also region for autark assignments.

Theorem 7. *Let $I = (D, V, F)$ be a prime monotone SAT instance, i.e., a SAT instance, where F is given by a prime monotone CNF φ. Then any $M \in \mathcal{Q}(M_\varphi)$ produces a maximal convex subset $K_{\geq}(M)$ of $P_{\text{autark}}(I)$.*

Proof. We claim that $\text{conv}(K_{\geq}(M) \cup \{y\}) \not\subseteq R(P_{\text{autark}}(I))$ holds for any $y \notin K_{\geq}(M)$. Fix a vector $y \notin K_{\geq}(M)$. Then $\sum_{x_j \in c} M_{cj} y_j + \varepsilon = 0$ holds for some $c \in \varphi$ and $\varepsilon > 0$. If $|c| = 1$, then $M_{cj} y_j < 0$ for $x_j \in c$, implying that $y_j < 0$, which implies that $V(y) = A(y)$ does not satisfy c. Therefore, y is not contained in $R(P_{\text{autark}}(I))$, which proves the claim. We therefore assume that $|c| \geq 2$.

Let z be a vector in $\mathbb{R}^n$ defined as $z_j = (1/M_{cj})(\sum_{x_k \in c, k \neq j} M_{ck} y_k + ((|c| - 1)/|c|)\varepsilon)$ if $x_j \in c$, and $z_j = \gamma$ otherwise, where γ is sufficiently large. We show that $z \in K_{\geq}(M)$ and $(y+z)/2 \notin R(P_{\text{autark}}(I))$, which prove the claim since the former condition implies that $(y+z)/2 \in \text{conv}(K_{\geq}(M) \cup \{y\})$.

In order to prove $z \in K_{\geq}(M)$, we show that $\sum_{x_j \in c'} M_{c'j} z_j \geq 0$ for each $c' \in \varphi$. If $c' = c$, we have

$$\begin{aligned} \textstyle\sum_{x_j \in c} M_{cj} z_j &= \textstyle\sum_{x_j \in c} (\sum_{x_k \in c, k \neq j} M_{ck} y_k + \frac{|c|-1}{|c|}\varepsilon) \\ &= \textstyle\sum_{x_j \in c} \sum_{x_k \in c} M_{ck} y_k - \sum_{x_j \in c} M_{cj} y_j + (|c| - 1)\varepsilon \\ &= (|c| - 1)(\textstyle\sum_{x_j \in c} M_{cj} y_j + \varepsilon) \\ &= 0. \end{aligned} \tag{20}$$

On the other hand, if $c' \neq c$, we have $\sum_{x_j \in c'} M_{c'j} z_j \geq 0$, since c' contains at least one variable not contained in $V(c)$ by primality of φ and γ is sufficiently large.

Next, we show that $(y+z)/2 \notin R(P_{\text{autark}}(I))$. For each $x_j \in c$,

$$\begin{aligned} y_j + z_j &= y_j + \tfrac{1}{M_{cj}}(\textstyle\sum_{x_k \in c, k \neq j} M_{ck} y_k + \frac{|c|-1}{|c|}\varepsilon) \\ &= \tfrac{1}{M_{cj}}(\textstyle\sum_{x_j \in c} M_{cj} y_j + \frac{|c|-1}{|c|}\varepsilon) \\ &= \tfrac{1}{M_{cj}}(-\varepsilon + \frac{|c|-1}{|c|}\varepsilon) < 0. \end{aligned} \tag{21}$$

Hence a partial assignment $V((y+z)/2) = A((y+z)/2)$ does not satisfy c, which implies that $(y+z)/2$ is not contained in $R(P_{\text{autark}}(I))$. This completes the proof. □

Different from Theorems 4, 5 and 6, $K_{\gg}(M)$ for $M \in \mathcal{Q}_0(M_\varphi)$ is not maximal convex set of $P_{\text{sat}}(I)$, which follows Example 9.

We also note that primality alone does not always yield maximality.

Example 11. *Let $I = (D = \{0,1\}, V = \{x_1, x_2\}, F)$ be a SAT instance, where F is given by a prime CNF $\varphi = (x_1 \vee \overline{x}_2)(\overline{x}_1 \vee x_2)$. Then no $M \in \mathcal{Q}(M_\varphi)$ with $M \neq M_\varphi$ produces a maximal convex subset $K_{\geq}(M)$ of $R(P_{\text{autark}}(I))$.*

Indeed, let $R_+ = \{y \in \mathbb{R}^2 \mid y_1 > 0, y_2 > 0\}$ and $R_- = \{y \in \mathbb{R}^2 \mid y_1 < 0, y_2 < 0\}$. Then we have $R(P_{\text{autark}}(I)) = R_+ \cup R_- \cup \{0\}$. For any $M \in \mathcal{Q}(M_\varphi)$, it holds that $K_{\geq}(M) \subseteq R_+ \cup \{0\}$ or $K_{\geq}(M) \subseteq R_- \cup \{0\}$. If $K_{\geq}(M) \subseteq R_+ \cup \{0\}$, then there exists a $y \in R_+$ which is not contained in $K_{\geq}(M)$, since $K_{\geq}(M)$ is closed and R_+ is open. It follows from convexity of $R_+ \cup \{0\}$ that $\text{conv}(K_{\geq}(M) \cup \{y\}) \subseteq R_+ \cup \{0\}$. Hence, $K_{\geq}(M)$ is not a maximal convex subset of $R(P_{\text{autark}}(I))$. The case of $K_{\geq}(M) \subseteq R_- \cup \{0\}$ can be proven in a similar way.

We remark that maximality in Theorem 7 is used to heuristically compute a local fixable assignment, which can be found in the next section.

4 Computing a satisfiability preserving assignment for CSPs given by 0-1 integer linear systems

In this section, we discuss the complexity of computing a satisfiability preserving assignment of Boolean CSPs given by integer linear systems. We first note that computing a non-trivial satisfiability preserving assignment defined in Section 2 is NP-hard, since we can solve CSP by iteratively computing such assignments. For the linear forms discussed in the previous section, it is still NP-hard to compute an assignment which is contained in *some* linear satisfiability preserving assignments, since for any satisfiable partial assignment there exists a linear cone that corresponds to such an assignment. Hence, we consider restricting cones that represents linear satisfiability preserving assignments. For CNFs φ, it is natural to consider linear autark assignments derived from matrices in $\mathcal{Q}(M_\varphi)$, in particular, simple linear autark assignments (i.e., cones derived from M_φ). For the 0-1 integer linear systems, we note that each constraint is unate (i.e., it can be represented by a monotone CNF by changing the polarity of variables appropriately). Simple linear autark (i.e., fixable) assignments for its prime CNF expression are maximal in terms of both region and partial assignments, which follows from results in the previous section. Here we note that a prime CNF might have exponentially many clauses, if the constraint inequality involve non-constantly many variables. This implies that we cannot explicitly have linear inequalities that represent simple linear fixable assignments. In this paper, we study computing a linear locally-fixable assignment which is contained in the intersection of such simple linear autark assignments. We show that it can be computed in pseudo-polynomial time, even if some constraints have non-constantly many variables. From this, we can show that well known pseudo-polynomially solvable classes are also solvable by repeatedly computing such linear locally-fixable assignments, which is discussed in the next section.

We consider a Boolean CSP instance $I = (D = \{0,1\}, V, F)$, where F is given by a linear system

$$Gx \geq h, \tag{22}$$

where G is a matrix in $\mathbb{Z}^{m \times n}$ and h is a vector in $\mathbb{Z}^m$. As mentioned above, each linear inequality in the system (22) is unate. Let I_i denote the CSP instance whose constraint is represented by the ith inequality of (22). By Theorem 7, $R(P_{\rm fix}(I_i))$ is well-approximated by $K_{\geq}(M_\varphi)$ for a unique prime CNF expression φ of I_i, where primality implies $P_{\rm autark}(I_i) = P_{\rm fix}(I_i)$ by Theorem 3.

Consider a single inequality:

$$\sum_{j \in J_+} g_j x_j - \sum_{j \in J_-} g_j x_j \geq \eta, \tag{23}$$

where $J_+, J_- \subseteq V$, $J_+ \cap J_- = \emptyset$, $g_j > 0$ for $j \in J_+ \cup J_-$, and $\eta \in \mathbb{Z}$.

Let U denote the set of extreme infeasible points of (23), i.e., $u \in U$ satisfies that $u + e_j$ is feasible to (23) if $j \in J_+$ and $u_j = 0$, and $u - e_j$ is feasible to (23) if $j \in J_-$ and $u_j = 1$. Then it is known that $\varphi = \bigwedge_{u \in U}(\bigvee_{j \in J_+ : u_j = 0} x_j \vee \bigvee_{j \in J_- : u_j = 1} \overline{x}_j)$ is a unique prime CNF, which implies the following lemma.

Lemma 2. *Let φ denote a unique prime CNF expression φ of* (23)*. Then we have*

$$K_{\geq}(M_\varphi) = \left\{ y \in \mathbb{R}^n \,\middle|\, \sum_{j \in J_+ : u_j = 0} y_j - \sum_{j \in J_- : u_j = 1} y_j \geq 0 \ (u \in U) \right\}. \tag{24}$$

Let φ_i denote a prime CNF for the ith inequality in (23) for each i. Then we note that $\bigcap_{i=1}^{m} K_{\geq}(M_{\varphi_i})$ represents linear locally-fixable assignments. In order to find a non-trivial one, we solve the following two LP problems for each $j \in V$:

$$\begin{array}{lll} \text{maximize} & \pm y_j & \\ \text{subject to} & M_{\varphi_i} y \geq 0 & (i = 1, \ldots, m), \\ & -1 \leq y_j \leq 1 & (j = 1, \ldots, n). \end{array} \tag{25}$$

The constraints $-1 \leq y_j \leq 1$ for $j = 1, \ldots, n$ in (25) ensure that the feasible region is bounded. We note that the number of the inequalities in the LP problem (25) might be exponential in the size of the input. For example, for a linear inequality $\sum_{j=1}^{n} x_j \geq \lfloor n/2 \rfloor$, the set U which corresponds to the extreme infeasible points is given by

$$U = \left\{ u \in \{0,1\}^n \,\middle|\, \sum_{i=1}^{n} u_j = \lfloor n/2 \rfloor - 1 \right\}. \tag{26}$$

Hence $|U| = \binom{n}{\lfloor n/2 \rfloor - 1} = \Omega(2^{n/2 - \log n})$ holds, which implies that the LP problem (25) has exponentially many inequalities. Instead of solving (25) directly, we make use of the ellipsoid method [13]. It is known [13] that optimization is polynomially equivalent to solving the separation problem. Namely, given a $y \in \mathbb{R}^n$, we compute an inequality in (25) that is violated by y, if y is infeasible. Since $-1 \leq y_j \leq 1$ can be checked efficiently, we consider how to check if $M_{\varphi_i} y \geq 0$ is satisfied for each i.

Let $J_+^{(i)} = \{j \in V \mid g_{ij} > 0\}$, $J_-^{(i)} = \{j \in V \mid g_{ij} < 0\}$ and $U^{(i)}$ denote the set of extreme infeasible points of the ith constraint in (22). Then by Lemma 2, the following LP problem computes $\min_k\{(M_{\varphi_i})_k y\}$, where $(M_{\varphi_i})_k$ denotes the kth row vector of M_{φ_i}.

$$\begin{array}{ll} \text{minimize} & \sum_{j \in J_+^{(i)}} y_j(1 - u_j) - \sum_{j \in J_-^{(i)}} y_j u_j \\ & \quad = -\sum_{j \in J_+^{(i)} \cup J_-^{(i)}} y_j u_j + \sum_{j \in J_+^{(i)}} y_j \\ \text{subject to} & u \in U^{(i)}. \end{array} \tag{27}$$

Hence the optimal value is at least zero if and only if $M_{\varphi_i} y \geq 0$. It is easy to see that $u \in U^{(i)}$ if and only if

1) $\sum_{j \in J_+^{(i)} \cup J_-^{(i)}} g_{ij} u_j \leq h_i - 1$,

2) $u \in \{0,1\}^n$,

3) u is extreme, i.e., $\sum_{j \in J_+^{(i)} \cup J_-^{(i)}} g_{ij} u_j + g_{ik} \geq h_i$ if $k \in J_+^{(i)}$ and $u_k = 0$, and $\sum_{j \in J_+^{(i)} \cup J_-^{(i)}} g_{ij} u_j - g_{ik} \geq h_i$ if $k \in J_-^{(i)}$ and $u_k = 1$.

Although condition 3) seems to be difficult to deal with, we show that the following dynamic programming approach can solve (27) in pseudo-polynomial time.

We assume that $0 < |g_{i1}| \leq \cdots \leq |g_{in}|$ holds without loss of generality. For a vector $u \in \{0,1\}^n$, we say that u *chooses an element* j if either ($u_j = 1$ and $j \in J_+^{(i)}$) or ($u_j = 0$ and $j \in J_-^{(i)}$). Let k be the first element that is not chosen by u. Then we can see that u is an extreme infeasible point of the ith constraint if and only if $h_i - g_{ik} + 1 \leq \sum_{j=1}^n g_{ij} u_j \leq h_i$ (resp., $h_i + g_{ik} + 1 \leq \sum_{j=1}^n g_{ij} u_j \leq h_i$) when $k \in J_+^{(i)}$ (resp., $k \in J_-^{(i)}$). Therefore, in order to solve (27), it is enough to solve the following LP problem for each $k = 1, \ldots, n+1$.

$$\begin{array}{ll} \text{maximize} & \sum_{j \in V} y_j u_j \\ \text{subject to} & h_i - g_{ik} + 1 \leq \sum_{j \in V} g_{ij} u_j \leq h_i \\ & \quad (\text{resp., } h_i + g_{ik} + 1 \leq \sum_{j \in V} g_{ij} u_j \leq h_i), \\ & k \text{ is the first unchosen element by } u, \\ & u \in \{0,1\}^n. \end{array} \tag{28}$$

Here $k = n+1$ means that u chooses all the elements.

Let k be an integer in $[n]$. For an integer ℓ with $k \leq \ell \leq n$ and an integer $w \in \mathbb{Z}$, let $A(\ell, w)$ denote the objective value of

$$\begin{array}{ll} \text{maximize} & \sum_{j \in [\ell]} y_j u_j \\ \text{subject to} & \sum_{j \in [\ell]} g_{ij} u_j = w, \\ & k \text{ is the first unchosen element by } u, \\ & u \in \{0,1\}^\ell. \end{array} \tag{29}$$

Let us consider the case in which $k \in J_+^{(i)}$, since the case of $k \in J_-^{(i)}$ can be treated similarly. Then we have

$$\begin{aligned} A(k, w) &= \begin{cases} \sum_{j \in J_+^{(i)} : j \leq k-1} y_j & \text{if } w = \sum_{j \in J_+^{(i)} : j \leq k-1} g_{ij} \\ -\infty & \text{otherwise,} \end{cases} \\ A(\ell, w) &= \max\{A(\ell-1, w), A(\ell-1, w - g_{i\ell}) + y_\ell\} \quad \ell = k+1, \ldots, n. \end{aligned} \tag{30}$$

Hence we can see that $\max_{h_i - g_{ik} + 1 \leq w \leq h_i} A(n, w)$ is the optimal value of (28). Since the dynamic programming algorithm requires $\mathrm{O}((n-k)n\eta_i)$

time for each k, where $\eta_i = \max\{h_i, |g_{i1}|, \ldots, |g_{in}|\}$, we can solve the separation problem in $\mathrm{O}(n^3 m\eta)$ time, where $\eta = \max_i \eta_i$.

We next consider the *unit* integer linear system, i.e., the integer linear system such that the input matrix G is given by $G \in \{0, -1, +1\}^{m\times n}$. Note that this includes global cardinality constraint [32] and is related to MAX(MIN) ONES [7] in CSP.

Let

$$\sum_{j\in J_+} x_j + \sum_{j\in J_-} (1 - x_j) \geq k \tag{31}$$

be an inequality in the system. Then, it has a unique prime CNF expression φ represented by

$$\varphi = \bigwedge_{\substack{I_+\subseteq J_+, I_-\subseteq J_-:\\ |I_+|+|I_-|=|J_+|+|J_-|-k+1}} \Big(\bigvee_{j\in I_+} x_j \vee \bigvee_{j\in I_-} \overline{x}_j\Big). \tag{32}$$

Thus we have the following lemma.

Lemma 3. *Let φ be a unique prime CNF expression of* (31)*. Then we have*

$$\begin{aligned} K_{\geq}(M_\varphi) = \{y \in \mathbb{R}^n \mid &\textstyle\sum_{j\in I_+} y_j - \sum_{j\in I_-} y_j \geq 0, \\ & I_+ \subseteq J_+, I_- \subseteq J_-, |I_+| + |I_-| = |J_+| + |J_-| - k + 1\}. \end{aligned} \tag{33}$$

Again the number of the constraints in the LP problem (25) might be exponential in the size of the input. Hence we use the ellipsoid method to solve the LP problem (25).

Lemma 4. *We can solve the separation problem for the LP problem* (25) *in polynomial time if $G \in \{0, -1, +1\}^{m\times n}$.*

Proof. Suppose that a vector $y \in \mathbb{R}^n$ is given. Then we note that $y \in K_{\geq}(M_{\varphi_1}) \cap \cdots \cap K_{\geq}(M_{\varphi_m})$ if and only if

$$\begin{aligned} \min\{&\textstyle\sum_{j\in I_+} y_j - \sum_{j\in I_-} y_j \geq 0 \mid I_+ \subseteq J_+^{(i)}, I_- \subseteq J_-^{(i)}, \\ & |I_+| + |I_-| = |J_+^{(i)}| - h_i + 1\} \geq 0 \end{aligned} \tag{34}$$

holds for all $i = 1, \ldots, m$. The left hand side of (34) can be computed simply by sorting the set $\{y_j \mid j \in J_+^{(i)}\} \cup \{-y_j \mid j \in J_-^{(i)}\}$ and summing up the smallest $|J_+^{(i)}| - h_i + 1$ elements. Hence we can solve the separation problem in $\mathrm{O}(mn\log n)$ time. □

In summary, we have shown the following theorem.

Theorem 8. *We can find a non-trivial linear locally-fixable assignment for* (22)*, if exists, in pseudo polynomial time. Moreover, it is possible in polynomial time, if the input matrix is contained in* $\{0, -1, +1\}^{m\times n}$.

5 Computing a satisfiability preserving assignment for CSPs given by general integer linear systems

In this section, we consider linear satisfiability preserving assignments for non-Boolean CSPs given by integer linear systems. We make use of the following encoding that transform CSP instances with $D = \{0, 1, \ldots, d\}$ to Boolean CSP instances. For each variable x_j in V, we take d Boolean variables $x_{j1}, \ldots, x_{jd}$, which guarantees that $x_j = p$ if and only if $x_{j1} = \cdots = x_{jp} = 1$ and $x_{j,p+1} = \cdots = x_{jd} = 0$. Let

$$Gx \geq h,\, x \in \{0, 1, \ldots, d\}^n \tag{35}$$

denote an integer linear system such that the ith inequality is represented by

$$\sum_{j\in J_+^{(i)}} \alpha_{ij} x_j - \sum_{j\in J_-^{(i)}} \alpha_{ij} x_j \geq h_i \tag{36}$$

for $\alpha_{ij} > 0\,(j \in J_+^{(i)} \cup J_-^{(i)})$. Let U_i denote the set of extreme infeasible points of (36), i.e., $u \in U_i$ satisfies that $u + e_j$ is feasible to (36) if $j \in J_+^{(i)}$ and $u_j \neq d$, and $u - e_j$ is feasible to (36) if $j \in J_-^{(i)}$ and $u_j \neq 0$.

Define a CNF φ_i by

$$\varphi_i = \bigwedge_{u\in U_i} \Big(\bigvee_{j\in J_+^{(i)}:u_j\neq d} x_{j,u_j+1} \quad\vee \bigvee_{j\in J_-^{(i)}:u_j\neq 0} \overline{x}_{j,u_j} \Big). \tag{37}$$

We note that $Gx \geq h$ and $x \in D^n$ if and only if all $\varphi_i\,(i = 1, \ldots, m)$ and $x_{jp} \geq x_{j,p+1}\,(j = 1, \ldots, n; p = 1, \ldots, d-1)$ are satisfied. Since $x_{jp} \geq x_{j,p+1}$ can be represented by a CNF ψ_{jp} discussed in Section 4, by applying the ellipsoid method proposed in Section 4, we can compute in pseudo-polynomial time a linear locally-fixable assignment contained in the intersection of simple linear autark assignments of φ_i and ψ_{jp}.

A system $Gx \geq h$ is said to be *quadratic* if each row of G contains at most two nonzero elements, *Horn* if each row of G contains at most one positive element, and *q-Horn* if the optimal value of the following LP problem is at most 1.

$$\begin{array}{lll} \text{minimize} & Z & \\ \text{subject to} & \displaystyle\sum_{j:g_{ij}>0} \alpha_j + \sum_{j:g_{ij}<0} (1-\alpha_j) \le Z & (i=1,\ldots,m), \\ & 0 \le \alpha_j \le 1 & (j=1,\ldots,n). \end{array} \tag{38}$$

Also, a CNF $\varphi = \bigwedge_{i=1}^{m}\left(\bigvee_{j\in J_+^{(i)}} x_j \vee \bigvee_{j\in J_-^{(i)}} \overline{x}_j\right)$ is called *quadratic* (or a 2-CNF) if $|J_+^{(i)} \cup J_-^{(i)}| \le 2$ holds for all $i=1,\ldots,m$, *Horn* if $|J_+^{(i)}| \le 1$ holds for all $i=1,\ldots,m$, and *q-Horn* if the optimal value of the following LP problem is at most 1.

$$\begin{array}{lll} \text{minimize} & Z & \\ \text{subject to} & \sum_{j\in J_+^{(i)}} \alpha_j + \sum_{j\in J_-^{(i)}} (1-\alpha_j) \le Z & (i=1,\ldots,m), \\ & 0 \le \alpha_j \le 1 & (j=1,\ldots,n). \end{array} \tag{39}$$

We note that $\bigwedge_{i=1}^{m} \varphi_i \wedge \bigwedge_{j=1,\ldots,n;p=1,\ldots,d-1} \psi_{jp}$ is Horn (resp., quadratic (i.e., a 2-CNF) and q-Horn) if so is the original integer linear system $Gx \ge h$. It is known [20] that any satisfiable quadratic, Horn and q-Horn CNF can be solved by repeatedly finding non-trivial simple linear autark assignment. Therefore, we have the following theorem.

Theorem 9. *Let $Gx \ge h$ and $x \in D^n$ be an integer linear system. If it is quadratic, Horn or q-Horn, then it can be solved in pseudo-polynomial time by repeatedly finding linear locally-fixable assignments.*

Finally, we note that the following naive encoding does not work, since quadratic (resp., Horn and q-Horn) systems does not always transformed to the 0-1 quadratic (resp., Horn and q-Horn) systems.

$$\left\{ \begin{array}{rll} G(\sum_{p=1}^{d} x_p) & \ge h, & \\ x_{jp} & \ge x_{j,p+1} & (j=1,\ldots,n; p=1,\cdots,d-1), \\ x_p & \in \{0,1\}^n & (p=1,\cdots,d), \end{array} \right. \tag{40}$$

where $x_p = (x_{1p},\cdots,x_{np})^\top$ for $p=1,\ldots,d$.

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