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Abstract

In the context of discrete DC programming, Maehara and Murota (Mathematical Programming, Series A, 2014) posed the problem of determining the complexity of minimizing the difference of two M$^\natural$-convex set functions. In this paper, we show the NP-hardness of this minimization problem by proving a stronger result: maximizing the difference of two matroid rank functions is NP-hard.

Maehara and Murota [3] established a theoretical framework of difference of discrete convex functions and studied the problem of minimizing the difference of two discrete convex functions (discrete DC programs). The computational complexities of several types of discrete DC programs were revealed in their paper, but determining the complexity of minimizing the difference of two M$^\natural$-convex set functions was posed as an open question (see Section 4.3 and Table 1 in [3]).[1] In this paper, we show the NP-hardness of this minimization problem.

As we will describe later, a special case of this problem is to maximize the difference of two matroid rank functions. We first show the NP-hardness of this special case.

Theorem 1. *The following problem is NP-hard: for two matroids M_1 and M_2 on a common ground set E with rank functions f_1 and f_2,*

$$\underset{X \subseteq E}{\text{maximize}} \quad f_1(X) - f_2(X). \tag{1}$$

Proof. Let $\mathcal{B}_1$ be the base family of M_1. We show that Problem (1) is equivalent to the following problem:

$$\begin{aligned} &\text{maximize} && |X| - f_2(X) \\ &\text{subject to} && X \in \mathcal{B}_1. \end{aligned} \tag{2}$$

Claim 2. *Problems (1) and (2) are equivalent.*

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[1] Although the term "M$^\natural$-convex functions on $\{0,1\}^n$" is used in [3], they are equivalent to "M$^\natural$-convex set functions" by identifying a subset of the ground set and its characteristic vector.

Proof of Claim 2. Since any optimal solution of (2) is feasible in (1), the optimal value of (1) is at least that of (2). Note that $|X| - f_2(X) = f_1(X) - f_2(X)$ if $X \in \mathcal{B}_1$.

Suppose that $X \subseteq E$ is an optimal solution of (1). Let X' be a maximal subset of X such that X' is an independent set of M_1. Since $f_1(X') = f_1(X)$ and $f_2(X') \leq f_2(X)$, X' is also an optimal solution of (1). We now extend X' to a base of M_1, that is, we take $X'' \supseteq X'$ such that $X'' \in \mathcal{B}_1$. Then, since $f_1(X'') = |X''|$ and $f_2(X'') \leq f_2(X') + (|X''| - |X'|)$, we obtain $f_1(X'') - f_2(X'') \geq f_1(X') - f_2(X')$, which shows that X'' is also an optimal solution of (1). Since X'' is a feasible solution of (2), the optimal value of (2) is at least that of (1). Note that we can construct X'' from X in polynomial time, since we can remove or add elements one by one. (End of the proof of Claim 2)

Next, we reduce the problem of finding a clique of size k in a graph (Max-Clique), which is a famous NP-hard problem, to Problem (2). Suppose that we are given an instance of Max-Clique, i.e., a simple graph $G = (V, E)$ and an integer k. We define two matroids M_1 and M_2 on E as follows.

- M_1 is the uniform matroid whose bases contain exactly $\binom{k}{2}$ elements.
- M_2 is the graphic matroid of G.

By observing that $f_2(F) = |V| - c(F)$ for $F \subseteq E$, where $c(F)$ is the number of connected components of $G_F = (V, F)$, we can see that the following are equivalent.

1. $G = (V, E)$ contains a clique of size k.
2. The optimal value of Problem (2) is $\binom{k}{2} - (k - 1)$.

This shows that Max-Clique is reduced to Problem (2), and hence Problem (1) is NP-hard by Claim 2. □

Let E be a finite set and 2^E denote the set of all subsets of E. A set function $f : 2^E \to \mathbf{Z} \cup \{+\infty\}$ is said to be M$^\natural$-convex if it satisfies the exchange axiom:

For all $X, Y \subseteq E$ and $i \in X \setminus Y$,

$$f(X) + f(Y) \geq \min \Big[f(X \setminus \{i\}) + f(Y \cup \{i\}),$$
$$\min_{j \in Y \setminus X} \{ f((X \setminus \{i\}) \cup \{j\}) + f((Y \cup \{i\}) \setminus \{j\}) \} \Big].$$

We say that $f : 2^E \to \mathbf{Z} \cup \{-\infty\}$ is M$^\natural$-concave if $-f$ is M$^\natural$-convex. Note that M$^\natural$-concave set functions are essentially equivalent to valuated matroids of Dress and Wenzel [1], and M$^\natural$-convex functions on $\mathbf{Z}^n$ are introduced by Murota and Shioura [5]. It is known that matroid rank functions are M$^\natural$-concave set functions.

Lemma 3 (See [2, p. 51] and [4, Theorem 5.1])**.** *Matroid rank functions are M$^\natural$-concave.*

This lemma shows that Problem (1) is a special case of minimizing the difference of two M$^\natural$-convex set functions. Hence, by Theorem 1, we obtain the following theorem, which answers the open question posed by Maehara and Murota [3].

Theorem 4. *The following problem is NP-hard: for two M$^\natural$-convex set functions f_1 and f_2,*

$$\underset{X \subseteq E}{\text{minimize}} \quad f_1(X) - f_2(X).$$

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