MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

A Note on the Convergence of Pole Estimation by Rational Interpolation

Shinji ITO and Masaaki SUGIHARA

(Communicated by Kazuo MUROTA)

METR 2015-02 January 2015

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html

A Note on the Convergence of Pole Estimation by Rational Interpolation

Shinji ITO* and Masaaki SUGIHARA†

January 2015

Abstract

In this note, we consider the problem of estimating poles of meromorphic functions from function values at sample points on the complex plane. It is known that we can use rational interpolation for this problem since poles of rational interpolants converge to those of interpolated functions as the sample points increase. The aim of this note is to analyze the convergence rates of the poles for arbitrary distributions of sample points, in order to provide criteria about how to design sample points. The main result is that the convergence rates are bounded by a value given by logarithmic potentials which depend on the distributions of sample points. In particular, it is shown that, in some situations, equidistant sample points give better estimation than Chebyshev points.

1 Introduction

We consider the problem of estimating poles of meromorphic function f from function values at sample points. For this problem, it is known that a method using rational interpolation is applicable. In this method, we compute polynomials $p \in \mathcal{P}_M$ and $q \in \mathcal{P}_N$ satisfying

$$f(s_j) = \frac{p(s_j)}{q(s_j)}, \quad j = 1, \ldots, M+N+1 \tag{1}$$

for $M+N+1$ distinct sample points $S := \{s_j\}_{j=1}^{M+N+1}$, where $\mathcal{P}_n$ is the set of polynomials of a single indeterminate of degree at most n. We output the roots of q as the estimated poles of f. If $p \in \mathcal{P}_M$ and $q \in \mathcal{P}_N$ satisfy (1), $r(z) = \frac{p(z)}{q(z)}$ is called a rational interpolant of type (M, N) on S. Rational interpolant must satisfy

$$f(s_j)q(s_j) = p(s_j), \quad j = 1, \ldots, M+N+1, \tag{2}$$

which follows from (1). Although rational interpolant does not always exist, if it exists, we can compute it by solving (2) as a linear equation in coefficients of p and q. For more detail, see, for

*Department of Mathematical Informatics, University of Tokyo, Tokyo 113-8656, Japan. `shinji_ito@mist.i.u-tokyo.ac.jp`

†Department of Physics and Mathematics, Aoyama Gakuin University, 5-10-1 Nobe, Chuo-ku, Kanagawa 252-5258, Japan. `sugihara@gem.aoyama.ac.jp`

example, [1]. Recently, stable and fast algorithms of computing a rational interpolant have been proposed based on the equation (2) [5, 8, 11].

In the context of the approximation of functions, the polynomial interpolation is used more often than rational interpolation due to the stability and the efficiency. However, rational interpolation has a potential advantage that it may behave better in the presence of singularities. In fact, the poles of rational interpolants of a meromorphic function f converge to those of f even if the poles of f are on and outside of the interpolating region [3, 9, 14]. Specifically, Saff [9] and Warner [14] showed that rational functions interpolating meromorphic functions f with N poles behave as follows when the degree M of the denominator of the rational functions is fixed to N and the degree of the numerator is increased:

- There exists rational interpolants for sufficiently large M.
- The values of rational interpolants converge to those of f uniformly on some regions.
- The poles of rational interpolants converge to those of f.

More details are written in preliminaries. These results guarantee that a rational interpolant gives estimation of poles of f. In this note, we consider convergence rates of the poles estimated by rational interpolation.

Our approach is to use the logarithmic potential

$$U(z) = -\int \log|z - t| \mathrm{d}\mu(t), \tag{3}$$

where μ is the measure associated with the distribution of sample points. Such approach using logarithmic potentials has been exploited by Gončarov [4] and Krylov [7] in order to analyze polynomial interpolation on the real line. Davis [2] and Walsh [13] extended this approach to Hermite interpolation problem. The above results about the convergence of rational interpolation by Saff [9] and Warner [14] also depend on the logarithmic potentials. For more details about logarithmic potentials, see, for example, [6, 10, 12].

This note is organized as follows. Section 2 introduces the definition and basic properties of the logarithmic potential, and existing results about the convergence of rational interpolation. Section 3 presents the main result of this note, which states that the convergence rates of poles are bounded by a value given by the logarithmic potential. Section 4 shows results of numerical experiments, in which we compare Chebyshev sample points and equidistant sample points. In particular, it is demonstrated that equidistant points may give better estimation than Chebyshev points.

2 Preliminaries

For natural numbers n, let $S^{(n)} = \{s_1^{(n)}, \ldots, s_n^{(n)}\} \subseteq \mathbb{C}$ denote n sample points. We assume that a compact set E in the complex plane contains $S^{(n)}$ for all n. Define the elementary measures $\{\mu_n\}$ on E associated with $\{S^{(n)}\}$ by

$$\mu_n(B) = \frac{1}{n}\sum_{j=1}^{n} \chi(s_j^{(n)} \in B), \tag{4}$$

where $\chi(P) = 1$ if P is true and 0 if P is false. We assume that there exists a measure μ on E such that $\mu_n \to \mu$ in the weak$*$ topology, i.e.,

$$\lim_{n\to\infty} \int_E g(z)\mathrm{d}\mu_n(z) = \int_E g(z)\mathrm{d}\mu(z) \tag{5}$$

for arbitrary continuous functions g on E. We call this measure μ the interpolation measure. Define $U_n, U : \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ by

$$U_n(z) = -\int_E \log|z-t|\mathrm{d}\mu_n(t), \quad U(z) = -\int_E \log|z-t|\mathrm{d}\mu(t). \tag{6}$$

This U_n and U have the following property.

Lemma 1.
(a) $\liminf_{n\to\infty} U_n(z) \geq U(z)$, *uniformly on compact subsets of the complex plane.*
(b) $\liminf_{n\to\infty} U_n(z) = U(z)$, *uniformly on compact subsets of* $\mathbb{C} \setminus E$.

The proof of this lemma can be found in, for example, [14].

Define D_ρ to be the set of all complex numbers z satisfying $U(z) > \rho$. Saff [9] and Warner [14] showed that rational interpolants for meromorphic functions have the following property.

Theorem 1 ([9, 14])**.** *If $f(z)$ is holomorphic on E and meromorphic on D_ρ with precisely N poles $\{\zeta_1, \dots, \zeta_N\}$ inside D_ρ, then for all M sufficient large, there exist rational interpolants of f, of type (M, N) on $S^{(M+N+1)}$, $r_{MN}(z) = p_{MN}(z)/q_{MN}(z)$. Moreover, the denominators, $q_{MN}(z)$ converge to $\prod_{k=1}^{N}(z - \zeta_k)$, uniformly on compact subsets of the complex plane, and the rational functions, $r_{MN}(z)$, converge to $f(z)$ uniformly on compact subsets of $D_\rho \setminus \{\zeta_1, \dots, \zeta_N\}$.*

This theorem guarantees also that the roots of q_{MN} approaches the poles $\zeta_1, \dots, \zeta_N$ of f. For each k, we denote by $\zeta_k^{(M)}$ the root of q_{MN} which converges to ζ_k.

3 Convergence rates of poles of rational interpolants

The main result of this note is as follows.

Theorem 2. *If $f(z)$ is holomorphic on E and meromorphic on D_ρ with precisely N poles $\{\zeta_1, \dots, \zeta_N\}$, inside D_ρ, then the poles $\{\zeta_1^{(M)}, \dots, \zeta_N^{(M)}\}$ of rational interpolants r_{MN} satisfy*

$$\limsup_{M\to\infty} |\zeta_k^{(M)} - \zeta_k|^{\frac{1}{M}} \leq \exp(\rho - U(\zeta_k)) \tag{7}$$

for $k = 1, \dots, N$.

In preparation for the proof of this theorem, we start with the following lemma.

Lemma 2. *Suppose that p_1 is a polynomial of degree at most n and p_2 is a polynomial of degree exactly $n+2$, where n is a natural number. If Γ is a simple closed curve enclosing all roots of p_2, it holds that*

$$\int_\Gamma \frac{p_1(z)}{p_2(z)}\mathrm{d}z = 0. \tag{8}$$

Proof. This follows from the fact that $\frac{p_1(z)}{p_2(z)}$ is holomorphic on and outside Γ and its residue at the infinity is 0.

$\square$

In the following, we suppose that f satisfies the assumptions in Theorem 2. From this assumptions, there exists an open region Ω such that $E \cup D_\rho \subseteq \Omega$ and f is holomorphic on $\Omega \setminus \{\zeta_k\}$. Let η_k denote the residue of f at pole ζ_k.

Proposition 1. *Under the assumptions in Theorem 2, we have*

$$\limsup_{M\to\infty} |q_{MN}(\zeta_k)|^{\frac{1}{M}} \leq \exp(\rho - U(\zeta_k)), \quad k = 1, \ldots, N. \tag{9}$$

Proof. Fix an arbitrary number $R > \rho$. Since the closure $\overline{D_R}$ of D_R is included in D_ρ, it holds that $\overline{D_R} \cup E \subseteq \Omega$, and hence, there exists a simple closed curve $\Gamma_R \subseteq \Omega$ enclosing $D_R \cup E$. Define $F_n(z) = \prod_{j=1}^{n}(z - s_j^{(n)})$. First, we will show that

$$\left| \int_{\Gamma_R} \frac{q_{MN}(z) f(z) \prod_{k=2}^{N}(z-\zeta_k)}{F_{M+N+1}(z)} \mathrm{d}z \right| = \left| \frac{1}{2\pi \mathrm{i}} \frac{q_{MN}(\zeta_1)\eta_1 \prod_{k=2}^{N}(\zeta_1 - \zeta_k)}{F_{M+N+1}(\zeta_1)} \right|. \tag{10}$$

Since all singularities of the integrand of the left-hand side are on E except for ζ_1, we have

$$\begin{aligned}
&\int_{\Gamma_R} \frac{q_{MN}(z) f(z) \prod_{k=2}^{N}(z-\zeta_k)}{F_{M+N+1}(z)} \mathrm{d}z \\
&= \int_{\Gamma} \frac{q_{MN}(z) f(z) \prod_{k=2}^{N}(z-\zeta_k)}{F_{M+N+1}(z)} \mathrm{d}z + \int_{\Gamma_1} \frac{q_{MN}(z) f(z) \prod_{k=2}^{N}(z-\zeta_k)}{F_{M+N+1}(z)} \mathrm{d}z \\
&= \int_{\Gamma} \frac{q_{MN}(z) f(z) \prod_{k=2}^{N}(z-\zeta_k)}{F_{M+N+1}(z)} \mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \frac{q_{MN}(\zeta_1)\eta_1 \prod_{k=2}^{N}(\zeta_1 - \zeta_k)}{F_{M+N+1}(\zeta_1)},
\end{aligned} \tag{11}$$

where we take contours Γ and Γ_1 as in Figure 1. Since $p_{MN}(z) = q_{MN}(z) f(z)$ on all poles of the integrand which are enclosed by Γ, we have

$$\int_{\Gamma} \frac{p_{MN}(z) \prod_{k=2}^{N}(z-\zeta_k)}{F_{M+N+1}(z)} \mathrm{d}z = \int_{\Gamma} \frac{q_{MN}(z) f(z) \prod_{k=2}^{N}(z-\zeta_k)}{F_{M+N+1}(z)} \mathrm{d}z = 0, \tag{12}$$

where the second equality comes from Lemma 2. From (11) and (12), we have (10).

The equality (10) implies that there exists a constant $C > 0$ independent of M such that

$$C \frac{\sup_{z\in\Gamma_R} |q_{MN}(z)|}{\inf_{z\in\Gamma_R} |F_{M+N+1}(z)|} \geq \frac{|q_{MN}(\zeta_1)|}{|F_{M+N+1}(\zeta_1)|}. \tag{13}$$

Hence, we have

$$|q_{MN}(\zeta_1)|^{\frac{1}{M}} \leq \left| C \sup_{z\in\Gamma_R} |q_{MN}(z)| \right|^{\frac{1}{M}} \frac{|F_M(\zeta_1)|^{\frac{1}{M}}}{(\inf_{z\in\Gamma_R} |F_M(z)|)^{\frac{1}{M}}}. \tag{14}$$

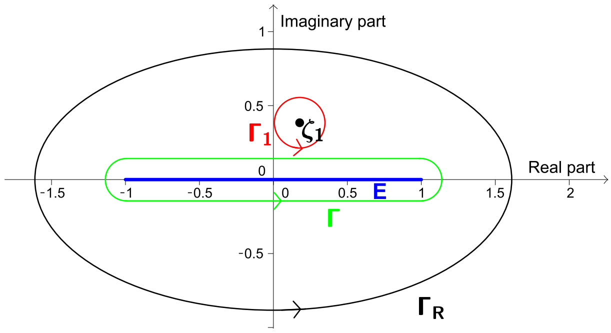


Figure 1: Contours Γ, Γ_R, Γ_1 (Γ encloses $S^{(M)}$ and does not enclose any pole of f).

From Theorem 1 we have

$$\lim_{M\to\infty}\left|C\sup_{z\in\Gamma_R}|q_{MN}(z)|\right|^{\frac{1}{M}}=1.$$

Furthermore, since Lemma 1 gives

$$\liminf_{M\to\infty}\frac{|F_M(\zeta_1)|^{\frac{1}{M}}}{(\inf_{z\in\Gamma_R}|F_M(z)|)^{\frac{1}{M}}}\leq\frac{\exp(-U(\zeta_1))}{\inf_{z\in\Gamma_R}\exp(-U(z))}\leq\exp(R-U(\zeta_1)),\tag{15}$$

we have $\liminf_{M\to}|q_{MN}(\zeta_1)|^{\frac{1}{M}}\leq\exp(R-U(\zeta_1))$. Since this holds for arbitrary $R>\rho$, we have (9). □

(Proof of Theorem 2) From Proposition 1, we have

$$\begin{aligned}
\limsup_{M\to\infty}|\zeta_k^{(M)}-\zeta_k|^{\frac{1}{M}}&=\limsup_{M\to\infty}|q_{MN}(\zeta_k)|^{\frac{1}{M}}\left|\prod_{1\leq l\leq N,l\neq k}(\zeta_k-\zeta_l^{(M)})\right|^{-\frac{1}{M}}\\
&\leq\limsup_{M\to\infty}|q_{MN}(\zeta_k)|^{\frac{1}{M}}\limsup_{M\to\infty}\left|\prod_{1\leq l\leq N,l\neq k}(\zeta_k-\zeta_l^{(M)})\right|^{-\frac{1}{M}}\\
&=\limsup_{M\to\infty}|q_{MN}(\zeta_k)|^{\frac{1}{M}}\leq\exp(\rho-U(\zeta_k)).
\end{aligned}$$

for each $k\in\{1,\dots,N\}$. □

4 Numerical examples

In this section, we compute rational interpolation of meromorphic functions f on the real segment $[-1,1]$ to see how the poles of rational interpolants behave. In particular, we compare the equidistant sample points on $[-1,1]$ and Chebyshev points. In experiments, we used matlab(R2010b). We solved the equation (2) in coefficients of p_{MN} and q_{MN} by using the singular value decomposition to compute rational interpolation. We applied `roots` command in Matlab to compute the roots of q_{MN}.

We used ζ_k, μ_k shown in Table 1 to define a rational function

$$f(z) = \sum_{k=1}^{20} \frac{\mu_k}{z - \zeta_k}. \tag{16}$$

The real parts and imaginary parts of μ_k, ζ_k were samples drawn from the normal distribution with mean 0 and standard deviation 1. For the potential U_1 given by Chebyshev sample points, we ordered ζ_k so that $\{U_1(\zeta_k)\}$ made a monotonically decreasing sequence:

$$U_1(\zeta_1) > U_1(\zeta_2) > \cdots > U_1(\zeta_{20}). \tag{17}$$

Theorem 2 implies that, for $N < 20$, the poles $(\zeta_1^{(M)}, \ldots, \zeta_N^{(M)})$ of rational interpolants of type (M, N) converge to $\zeta_1, \ldots, \zeta_N$ with the convergence rates at most $\exp(U(\zeta_{N+1}) - U(\zeta_k))$ for $k = 1, \ldots, N$, respectively, when we use Chebyshev sample points. For the potential U_2 given by equidistant sample points, $\zeta_1, \ldots, \zeta_5$ satisfy $U_2(\zeta_1) > U_2(\zeta_2) > \cdots > U_2(\zeta_5)$ and $U_2(\zeta_5)$ is greater than the potential values at any other poles.

We chose $N = 5$ and used Chebyshev sample points and equidistant sample points. Figure 2 shows the results of pole estimation. Curves in figures are level curves of the associated logarithmic potential. We can see that this method finds poles with largest potential. In this example, Chebyshev sample points give poorer estimation. Theorem 2 explains the reason for this phenomenon: the 5th largest potential at poles is close to the 6th. Figure 3 shows convergence rates of estimated poles for ζ_1 and ζ_5. The straight lines in the figure shows the theoretical curve associated with the upper bounds of the convergence rate which are derived from Theorem 2. In this situation, equidistant sample points give better convergence rates. For M sufficiently large, both sample points provide much the same accuracy in pole estimation.

Figures 4 and 5 show the results of pole estimation for the same function f for $N = 6$. Convergence rates differ from that for $N = 5$ since parameter r in Theorem 2 can be smaller for $N = 6$.

We also applied the same method to hyperbolic tangent function $f(z) = \tanh(z)$, which is a meromorphic function with poles

$$\zeta_1 = \frac{\pi \mathrm{i}}{2}, \quad \zeta_2 = -\frac{\pi \mathrm{i}}{2}, \quad \zeta_3 = \frac{3\pi \mathrm{i}}{2}, \quad \zeta_4 = -\frac{3\pi \mathrm{i}}{2}, \quad \zeta_5 = \frac{5\pi \mathrm{i}}{2}, \ldots.$$

For Chebyshev sample points and equidistant sample points, $\{\zeta_k\}$ satisfies

$$U(\zeta_1) = U(\zeta_2) > U(\zeta_3) = U(\zeta_4) > U(\zeta_5) = U(\zeta_6) > \cdots.$$

Hence, if N is an even number, the poles $(\zeta_1^{(M)}, \ldots, \zeta_N^{(M)})$ of rational interpolants of type (M, N) converge to $\zeta_1, \ldots, \zeta_N$ with the convergence rates at most $\exp(U(\zeta_{N+1}) - U(\zeta_k))$ for $k = 1, \ldots, N$, respectively. Figures 6 and 7 show the results of pole estimation for tanh function, where we set $N = 2$. In this situation, Chebyshev sample points and equidistant sample points provide much the same accuracy. The reason is that the two logarithmic potentials behave similarly on regions far from sample points. It is observed that the estimation errors increase for too large M, presumably due to numerical error.

Table 1: The poles and residues randomly generated for this experiment. U_1 and U_2 are the logarithmic potentials associated with Chebyshev sample points and equidistant sample points, respectively.

k	ζ_k	μ_k	$U_1(\zeta_k)$	$U_2(\zeta_k)$
1	$0.1873\cdots + 0.1240\cdots\mathrm{i}$	$0.0335\cdots - 0.1332\cdots\mathrm{i}$	$0.5672\cdots$	$0.7954\cdots$
2	$-0.4390\cdots - 0.1977\cdots\mathrm{i}$	$0.3502\cdots - 0.2248\cdots\mathrm{i}$	$0.4760\cdots$	$0.6136\cdots$
3	$-0.6003\cdots - 0.2725\cdots\mathrm{i}$	$-0.9792\cdots + 0.3075\cdots\mathrm{i}$	$0.3674\cdots$	$0.4340\cdots$
4	$0.7394\cdots - 0.2779\cdots\mathrm{i}$	$-0.5336\cdots - 0.8655\cdots\mathrm{i}$	$0.3180\cdots$	$0.3325\cdots$
5	$0.3035\cdots - 0.4686\cdots\mathrm{i}$	$-0.8314\cdots + 1.6555\cdots\mathrm{i}$	$0.2235\cdots$	$0.3321\cdots$
6	$-1.1658\cdots - 0.1941\cdots\mathrm{i}$	$-2.0518\cdots + 0.9642\cdots\mathrm{i}$	$0.0488\cdots$	$-0.0296\cdots$
7	$0.1049\cdots - 0.8396\cdots\mathrm{i}$	$-0.8236\cdots - 0.0200\cdots\mathrm{i}$	$-0.0722\cdots$	$-0.0024\cdots$
8	$-0.8880\cdots + 0.8252\cdots\mathrm{i}$	$-0.2620\cdots - 0.8479\cdots\mathrm{i}$	$-0.2199\cdots$	$-0.2060\cdots$
9	$-0.6669\cdots + 0.9609\cdots\mathrm{i}$	$0.2820\cdots + 1.0187\cdots\mathrm{i}$	$-0.2425\cdots$	$-0.2120\cdots$
10	$-0.5445\cdots - 1.0582\cdots\mathrm{i}$	$-0.2857\cdots + 2.5260\cdots\mathrm{i}$	$-0.2801\cdots$	$-0.2449\cdots$
11	$0.4900\cdots + 1.0984\cdots\mathrm{i}$	$-1.1564\cdots - 1.2571\cdots\mathrm{i}$	$-0.2966\cdots$	$-0.2601\cdots$
12	$0.1001\cdots + 1.3790\cdots\mathrm{i}$	$-1.7502\cdots - 1.1201\cdots\mathrm{i}$	$-0.4340\cdots$	$-0.3996\cdots$
13	$-0.0825\cdots + 1.4367\cdots\mathrm{i}$	$-1.3337\cdots - 0.7145\cdots\mathrm{i}$	$-0.4669\cdots$	$-0.4346\cdots$
14	$0.7222\cdots + 1.3546\cdots\mathrm{i}$	$-1.5771\cdots - 0.0348\cdots\mathrm{i}$	$-0.4910\cdots$	$-0.4700\cdots$
15	$0.2157\cdots + 1.7119\cdots\mathrm{i}$	$0.7015\cdots - 2.0026\cdots\mathrm{i}$	$-0.6188\cdots$	$-0.5953\cdots$
16	$-1.7947\cdots - 1.2078\cdots\mathrm{i}$	$-0.2991\cdots - 0.5890\cdots\mathrm{i}$	$-0.7550\cdots$	$-0.7601\cdots$
17	$-1.1480\cdots - 2.1384\cdots\mathrm{i}$	$-0.3539\cdots + 0.5201\cdots\mathrm{i}$	$-0.9110\cdots$	$-0.9028\cdots$
18	$2.5855\cdots - 1.0722\cdots\mathrm{i}$	$0.5080\cdots - 0.7982\cdots\mathrm{i}$	$-1.0068\cdots$	$-1.0143\cdots$
19	$-1.9330\cdots - 1.9609\cdots\mathrm{i}$	$1.1275\cdots + 1.3514\cdots\mathrm{i}$	$-1.0150\cdots$	$-1.0140\cdots$
20	$0.8404\cdots + 2.9080\cdots\mathrm{i}$	$0.0229\cdots - 0.2938\cdots\mathrm{i}$	$-1.1302\cdots$	$-1.1227\cdots$

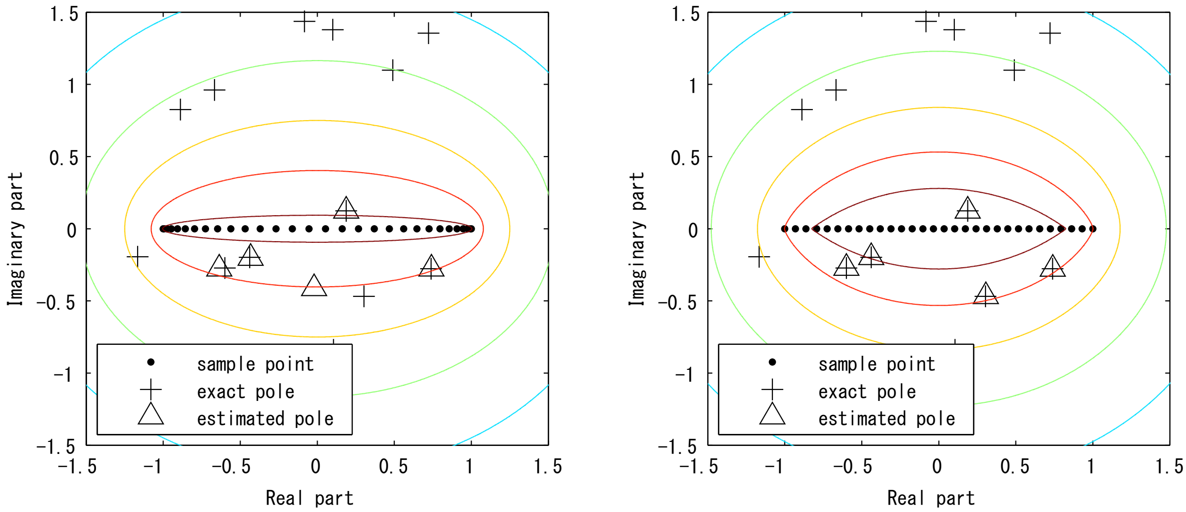
5 Conclusion

In this note, we considered the convergence of poles of rational interpolants for meromorphic functions. We established an upper bound of the convergence rates of the poles by using logarithmic potentials. This result will be useful in designing sample points for pole estimation.

A possible future work is to develop a method of designing sample points for pole estimation. In order to do this, it will be necessary to investigate the relation between sample points and numerical errors in rational interpolation.

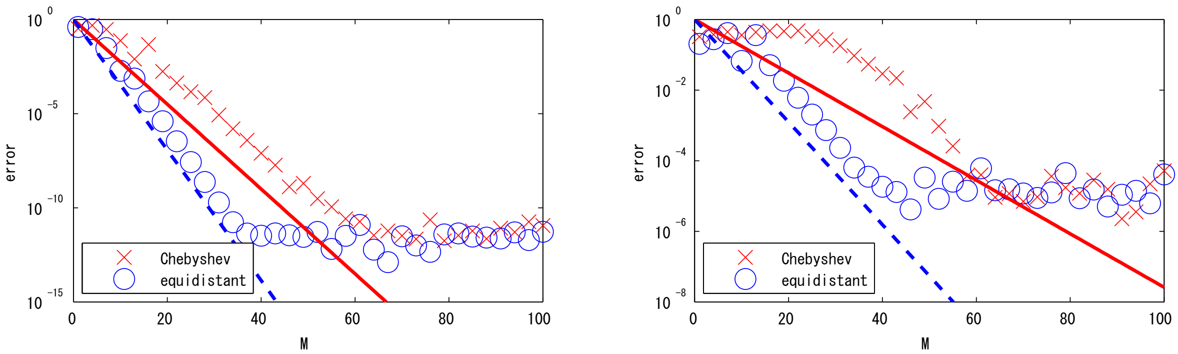
Acknowledgement

The authors would like to thank Prof. Kazuo Murota and Dr. Yuji Nakatsukasa for helpful comments and discussion. This research is supported by Grants-in-Aid for Scientific Research and by the JST, ERATO, Kawarabayashi Large Graph Project.



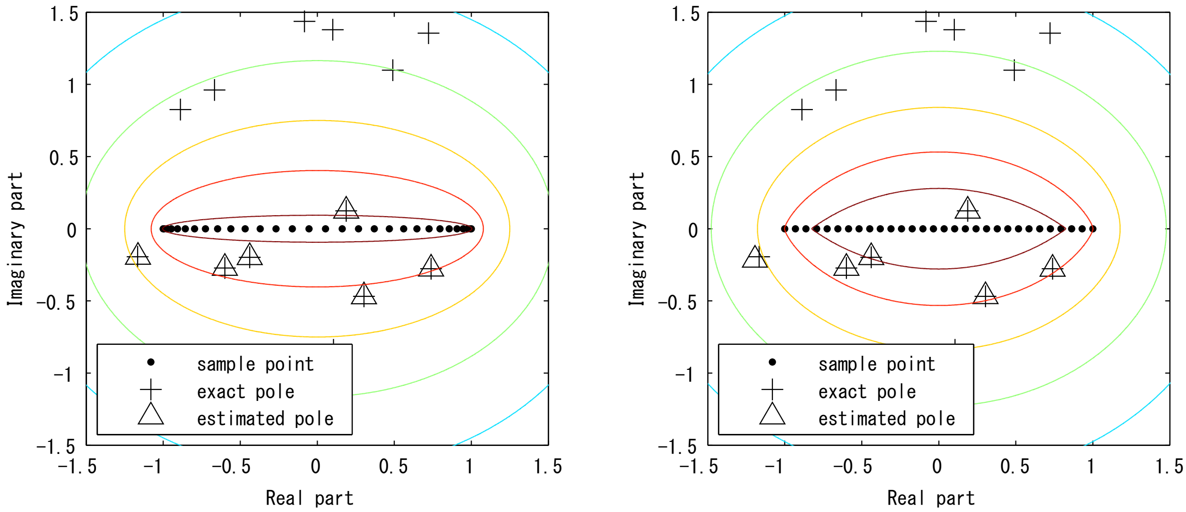
Chebyshev sample points   equidistant sample points

Figure 2: Pole estimation with $M = 24$, $N = 5$.



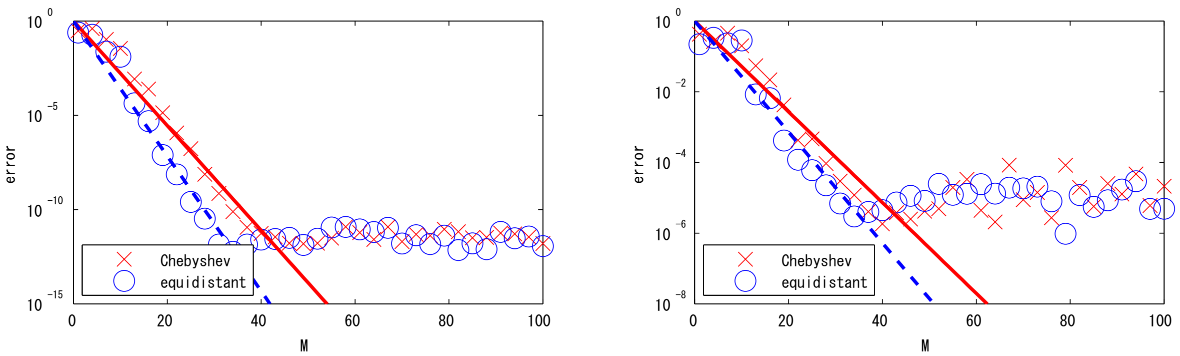
pole $\zeta_1 = 0.19 + 0.12\mathrm{i}$ pole $\zeta_5 = 0.30 - 0.47\mathrm{i}$

Figure 3: Convergence rates of pole estimation with $N = 5$. Straight lines are theoretical curves derived from Theorem 2.



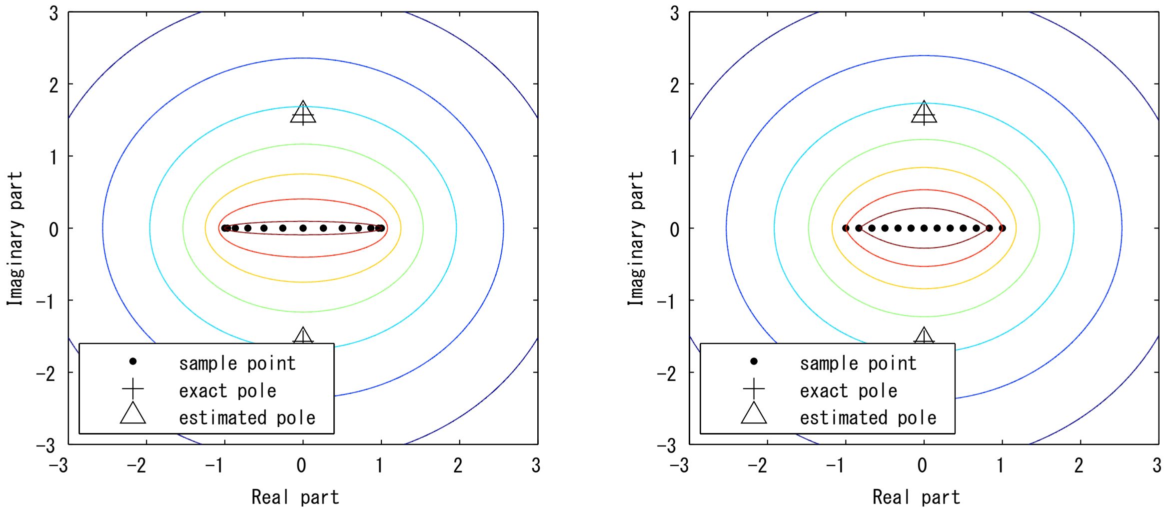
Chebyshev sample points equidistant sample points

Figure 4: Pole estimation with $M = 23$, $N = 6$.



pole $\zeta_1 = 0.19 + 0.12\mathrm{i}$ pole $\zeta_5 = 0.30 - 0.47\mathrm{i}$

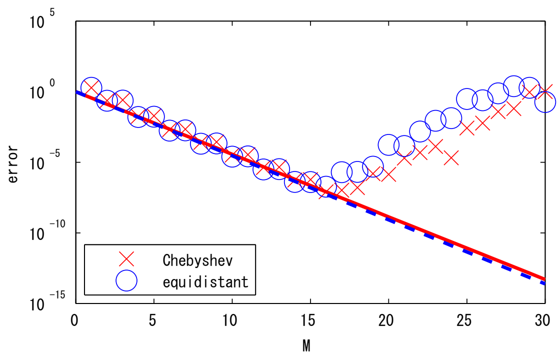
Figure 5: Convergence rates of pole estimation with $N = 6$. Straight lines are theoretical curves derived from Theorem 2.



Chebyshev sample points

equidistant sample points

Figure 6: Pole estimation with $M = 10$, $N = 2$ for tanh function.



pole $\zeta_1 = \frac{\pi i}{2}$

Figure 7: Convergence rates of pole estimation with $N = 5$. Straight lines are theoretical curves derived from Theorem 2.

References

[1] D. Braess. *Nonlinear Approximation Theory*, volume 7 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, 1986.

[2] P. J. Davis. *Interpolation and Approximation.* Blaisdell Publishing Co. Ginn and Co. New York-Toronto-London, 1963.

[3] R. de Montessus. Sur les fractions continues algébriques. *Bull. Soc. Math. France*, 30:28–36, 1902.

[4] V. L. Gončarov. *Teoriya interpolirovaniya i približeniya funkciĭ.* Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954.

[5] P. Gonnet, R. Pachón, and L. N. Trefethen. Robust rational interpolation and least-squares. *Electron. Trans. Numer. Anal.*, 38:146–167, 2011.

[6] E. Hille. *Analytic Function Theory. Vol. II.* Introductions to Higher Mathematics. Ginn and Co., Boston, Mass.-New York-Toronto, Ont., 1962.

[7] V. I. Krylov. *Approximate Calculation of Integrals.* Translated by Arthur H. Stroud. The Macmillan Co., New York-London, 1962.

[8] R. Pachón, P. Gonnet, and J. van Deun. Fast and stable rational interpolation in roots of unity and Chebyshev points. *SIAM J. Numer. Anal.*, 50(3):1713–1734, 2012.

[9] E. B. Saff. An extension of Montessus de Ballore's theorem on the convergence of interpolating rational functions. *J. Approximation Theory*, 6:63–67, 1972.

[10] E. B. Saff and V. Totik. *Logarithmic Potentials with External Fields*, volume 316 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 1997.

[11] L. N. Trefethen. *Approximation Theory and Approximation Practice.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013.

[12] M. Tsuji. *Potential Theory in Modern Function Theory.* Maruzen Co., Ltd., Tokyo, 1959.

[13] J. L. Walsh. *Interpolation and Approximation by Rational Functions in the Complex Domain.* Fourth edition. American Mathematical Society Colloquium Publications, Vol. XX. American Mathematical Society, Providence, R.I., 1965.

[14] D. D. Warner. An extension of Saff's theorem on the convergence of interpolating rational functions. *J. Approximation Theory*, 18(2):108–118, 1976.