MATHEMATICAL ENGINEERING
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METR 2015–04 February 2015

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An Analysis on the Asymptotic Behavior of Multistep Linearly Implicit Schemes for the Duffing Equation

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February, 2015

Abstract

We consider the discrete gradient method for dissipative linear-gradient systems, which strictly replicates the dissipation property, yielding a remarkable stability. However, it also replicates the nonlinearity of an original equation. To overcome this, we can employ multistep linearly implicit schemes as a relaxation; however, it can in turn destroy the originally aimed stability. Matsuo–Furihata (2014) introduced a dynamical systems viewpoint to understand the behavior for a toy scalar problem. In this letter, we show that their method can work also for the two-dimensional Duffing equation. There a new concept of semi-strong Lyapunov functionals is required.

1 Introduction

We consider the numerical integration of the linear-gradient system

$$\frac{\mathrm{d}}{\mathrm{d}t} z = A \nabla G(z), \qquad z(0) = z_0, \tag{1}$$

where $z : \mathbb{R}^+ \to \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$, $G : \mathbb{R}^d \to \mathbb{R}$ and $z_0 \in \mathbb{R}^d$. If A is a negative semidefinite matrix, the equation (1) has the following dissipation property:

$$\frac{\mathrm{d}}{\mathrm{d}t} G(z(t)) = \nabla G(z) \cdot \frac{\mathrm{d}}{\mathrm{d}t} z(t) = (\nabla G)^\top A\, (\nabla G) \le 0, \tag{2}$$

where the symbol '$\cdot$' denotes the standard inner product in $\mathbb{R}^d$ and $M^\top$ denotes the transpose of a matrix M.

In particular, we deal with the discrete gradient method (Gonzalez [1]) for linear-gradient systems (1). The discrete gradient $\nabla_{\mathrm{d}} G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ of an energy function G is a map satisfying the following properties:

- $\nabla_{\mathrm{d}} G(z,z) = \nabla G(z)$ for any $z \in \mathbb{R}^d$;
- $\nabla_{\mathrm{d}} G(z_1,z_2)\cdot(z_1 - z_2) = G(z_1) - G(z_2)$ for any $z_1, z_2 \in \mathbb{R}^d$.

The former demands that the discrete gradient is actually an approximation of the original gradient ∇G. The latter, which is called "discrete chain rule," is the essential property of discrete gradients. Note that the discrete gradient for a specified function is not necessarily unique. If we once have a $\nabla_{\mathrm{d}} G$, we can construct a numerical integrator of (1) as follows:

$$\frac{z^{(n+1)} - z^{(n)}}{\Delta t} = A \nabla_{\mathrm{d}} G \left(z^{(n+1)}, z^{(n)} \right). \tag{3}$$

Thanks to the discrete chain rule, the discrete gradient integrator (3) strictly replicates the dissipation property of an original equation, i.e.,

$$\begin{aligned} \frac{G\left(z^{(n+1)}\right) - G\left(z^{(n)}\right)}{\Delta t} &= \nabla_{\mathrm{d}} G \left(z^{(n+1)}, z^{(n)} \right) \cdot \frac{z^{(n+1)} - z^{(n)}}{\Delta t} \\ &= \nabla_{\mathrm{d}} G\big(z^{(n+1)}, z^{(n)}\big) \cdot A \nabla_{\mathrm{d}} G\big(z^{(n+1)}, z^{(n)}\big) \leq 0, \end{aligned}$$

which usually results in a remarkable stability. However, it also replicates the nonlinearity of an original equation. In other words, the computational cost of the discrete gradient method is generally expensive. In view of this, a multistep linearly implicit scheme is useful as an approximation. However, it can be extremely unstable depending on how we approximate the energy function. Matsuo–Furihata [2] dealt with this problem using dynamical systems theory. Lyapunov-type theorem in discrete dynamical systems (Humphries–Stuart [3]) reveals that if the energy function can serve as a (strong) Lyapunov functional and the level set of the energy function is compact, then the ω-limit set of an arbitrarily chosen initial value is a subset of the fixed points. But their analysis was limited to a specific scalar problem

$$\frac{\mathrm{d}}{\mathrm{d}t} z = \nabla G(z), \qquad G(z) = \frac{(1 - z^2)^2}{2}$$

and it has remained open if their technique applies to other problem, in particular, high-dimensional problem.

In this report, we show that the method devised by Matsuo–Furihata [2] can work also for the two-dimensional Duffing equation. In the process of the analysis we will also point out that the standard (strong) definition of Lyapunov functionals is not adequate to the Duffing equation, and to overcome this difficulty, we will introduce a new concept of "semi-strong Lyapunov functionals."

2 Lyapunov-Type Theorem

We briefly review the Lyapunov theory based on Hale [4], and also introduce the concept of semi-strong Lyapunov functionals. First of all, we define the *discrete dynamical systems.*

Definition 1 (Discrete dynamical systems) *Let X be a Banach space and $T : X \to X$ be a continuous map. Then, the pair (X, T) is called a discrete dynamical system.*

We define the ω-limit set of a point $x_0 \in X$ as

$$\begin{aligned}\omega(x_0) &:= \bigcap_{n\geq 0} \overline{\bigcup_{k\geq n} T^k x_0} \\ &= \left\{ y \in X \;\middle|\; \exists n_k \to \infty \text{ s.t. } \lim_{k\to\infty} T^{n_k} x_0 = y \right\},\end{aligned}$$

where a set $\overline{S}$ denotes the closure of a set S. In the discrete dynamical systems, ω-limit set is invariant under certain conditions as Lemma 1 (see, e.g., Hale [4]) shows. The subset B of X is called *invariant* if $TB = B$, and *positively invariant* if $TB \subseteq B$.

Lemma 1 *Let $B \subseteq X$ be a compact and positively invariant set of T. Then, for all $x_0 \in B$, $\omega(x_0)$ is nonempty and invariant.*

A point $x \in X$ is called a *fixed point* if $Tx = x$, and $\mathcal{E}(T)$ denotes the set of such fixed points. Proposition 1 below is essentially the same as [3, Theorem 4.3].

Definition 2 (Strong Lyapunov functional) *Let $B \subseteq X$ be a positively invariant set with respect to T. A continuous map $\Phi : B \to \mathbb{R}$ is called a strong Lyapunov functional on B, if the following conditions are satisfied.*

(i) $\Phi(Tx) \leq \Phi(x)$ *holds for any* $x \in X$;

(ii) *If there exists $x \in B$ such that $\Phi(Tx) = \Phi(x)$ holds, then $x \in \mathcal{E}(T)$.*

A continuous map $\Phi : B \to \mathbb{R}$ satisfying only (i) is called a *weak Lyapunov functional.*

Proposition 1 (Humphries–Stuart [3]) *Let $B \subseteq X$ be a compact and positively invariant set of T, such that there exists a strong Lyapunov functional Φ on B. Then, for all $x_0 \in B$, $\omega(x_0)$ is nonempty and $\omega(x_0) \subseteq \mathcal{E}$ holds. Moreover, if $\mathcal{E}(T)$ is discrete, then $\omega(x_0) \in \mathcal{E}(T)$.*

We can relax the condition of strong Lyapunov functional as below.

Definition 3 (Semi-strong Lyapunov functional) *Let $B \subseteq X$ be a positively invariant set with respect to T. A continuous map $\Phi : B \to \mathbb{R}$ is called a semi-strong Lyapunov functional on B, if the following conditions are satisfied.*

(i) $\Phi(Tx) \leq \Phi(x)$ *holds for any* $x \in X$;

(ii) *If there exists $x \in B$ such that $\Phi(T^n x) = \Phi(x)$ holds for any $n \in \mathbb{N}$, then $x \in \mathcal{E}(T)$.*

The difference between Definitions 2 and 3 is the latter condition (ii). A semi-strong Lyapunov functional is also weak Lyapunov functional, however, not necessarily strong Lyapunov functional. The condition (ii) of Definition 3 is equivalent to the following condition:

- For all $x \in B \setminus \mathcal{E}(T)$, there exists $n \in \mathbb{N}$ such that $\Phi(T^n x) < \Phi(x)$ holds.

This relaxation makes sense as explained in section 4.3. Proposition 2 below is an immediate corollary of Proposition 1.

Proposition 2 *Let $B \subseteq X$ be a compact and positively invariant set of T, such that there exists a semi-strong Lyapunov functional Φ on B. Then, for all $x_0 \in B$, $\omega(x_0)$ is nonempty and $\omega(x_0) \subseteq \mathcal{E}$ holds. Moreover, if $\mathcal{E}(T)$ is discrete, then $\omega(x_0) \in \mathcal{E}(T)$.*

Proof *First, we show the following claim: $x, y \in \omega(x_0) \Rightarrow \Phi(x) = \Phi(y)$, i.e., the value of the Lyapunov functional is strictly the same in $\omega(x_0)$. For $x \in \omega(x_0)$,*

$$\Phi(x) = \Phi\big(\lim_{k\to\infty} T^{n_k} x_0\big) = \lim_{k\to\infty} \Phi(T^{n_k} x_0) = \inf_{n\geq 0} \Phi(T^n x_0).$$

Here, the last equality derives from the decreasing property of Φ. It implies the claim above is true. For all $y \in \omega(x_0)$ and for all $n \in \mathbb{N}$, $T^n y \in \omega(x_0)$ holds from Lemma 1. Therefore, $\Phi(T^n y) = \Phi(y)$ holds, which means that $y \in \mathcal{E}(T)$. The proof of the latter part, the case that $\mathcal{E}(T)$ is discrete, is essentially same as the proof of [5, Theorem 1]. □

3 Duffing Equation

3.1 Discrete Gradient Methods

We consider the numerical integration of the Duffing equation in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} x = J \nabla G(x), \quad x = \begin{pmatrix} q & p \end{pmatrix}^\top, \tag{4}$$

where $J \in \mathbb{R}^{2\times 2}$ and $G : \mathbb{R}^2 \to \mathbb{R}$ are defined as follows:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & -d \end{pmatrix}, \quad G(x) = \frac{p^2 - q^2}{2} + \frac{q^4}{4}, \tag{5}$$

and $q(0) = q_0, p(0) = p_0$ ($d > 0, q_0, p_0$ are constants). Since J is a negative semidefinite matrix, the equation (4) has the dissipation property (2).

In this case, we can adopt "discrete gradient method," which strictly keep the dissipation property. One of the discrete gradients $\nabla_{\mathrm{d}} G : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is as follows:

$$\nabla_{\mathrm{d}} G(x, y) = \begin{pmatrix} \frac{(x_q^2 + y_q^2 - 2)(x_q + y_q)}{4} & \frac{x_p + y_p}{2} \end{pmatrix}^\top,$$

where $x = (x_q, x_p)^\top, y = (y_q, y_p)^\top$. Then, we obtain the following one step method:

$$\frac{x^{(n+1)} - x^{(n)}}{\Delta t} = J \nabla_{\mathrm{d}} G\left(x^{(n+1)}, x^{(n)}\right), \tag{6}$$

where $x^{(n)} = (q^{(n)}, p^{(n)})^\top \simeq (q(n\Delta t), p(n\Delta t))^\top$. Note that the scheme (6) is implicitly nonlinear, so that we need to solve a nonlinear system at each step.

3.2 Multistep Linearly Implicit Schemes

First, we consider to approximate the energy function G for constructing a multistep linearly implicit scheme. In order to follow the discussion in [2], we define the approximation $\hat{G} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ of G as follows:

$$\begin{aligned} \hat{G}(x, y) = &-\frac{1}{2}\left(a x_q y_q + (1-a)\frac{x_q^2 + y_q^2}{2}\right) + \frac{1}{4} x_q^2 y_q^2 \\ &+ \frac{1}{2}\left(b x_p y_p + (1-b)\frac{x_p^2 + y_p^2}{2}\right), \end{aligned}$$

where $a, b \in \mathbb{R}$ are scheme parameters. Then, $\nabla_{\mathrm{d}} \hat{G} : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\nabla_{\mathrm{d}} \hat{G}(x, y, z) = \begin{pmatrix} -a y_q - (1-a)\frac{x_q + z_q}{2} + y_q^2 \frac{x_q + z_q}{2} \\ b y_p + (1-b)\frac{x_p + z_p}{2} \end{pmatrix}.$$

Here, we can construct a multistep linearly implicit scheme with scheme parameters a, b:

$$\frac{x^{(n+1)} - x^{(n-1)}}{2\Delta t} = J \nabla_{\mathrm{d}} \hat{G}\left(x^{(n+1)}, x^{(n)}, x^{(n-1)}\right). \tag{7}$$

This scheme is dissipative with respect to $\hat{G}$:

$$\begin{aligned}
&\frac{1}{\Delta t}\left\{\hat{G}\left(x^{(n+1)}, x^{(n)}\right) - \hat{G}\left(x^{(n)}, x^{(n-1)}\right)\right\} \\
=&2\nabla_{\mathrm{d}}\hat{G}\left(x^{(n+1)}, x^{(n)}, x^{(n-1)}\right) \cdot \frac{x^{(n+1)} - x^{(n-1)}}{2\Delta t} \\
=&2\nabla_{\mathrm{d}}\hat{G}^{\top} J \nabla_{\mathrm{d}}\hat{G} \leq 0. \qquad (8)
\end{aligned}$$

But it does not generally mean the dissipation property with respect to G. In the left panel of Fig. 1, both G and $\hat{G}$ show the proper dissipation. In the right, however, $\hat{G}$ suffers from improper dissipation with G blowing up.

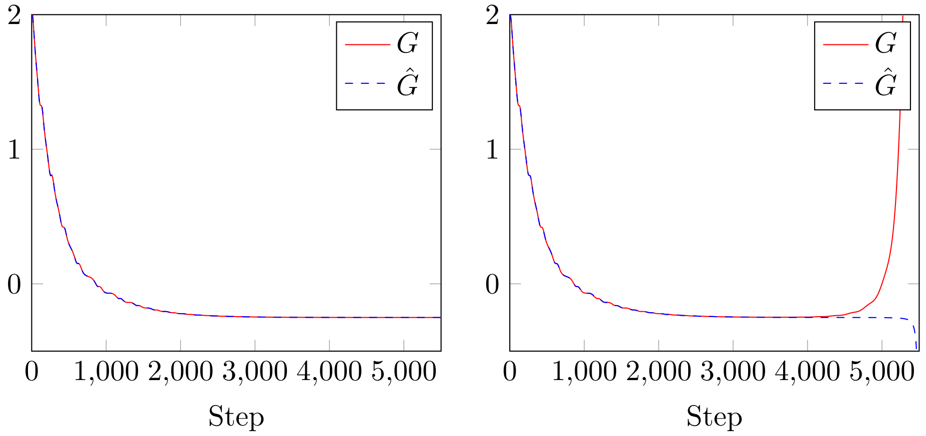

Figure 1: The time evolution of energy functions $G, \hat{G}$: the left panel corresponds to the sceme (7) under $a = 1.5, b = 0$, and the right panel corresponds to $a = 1, b = 1$. The initial value and the time-step are $x^{(0)} = (2, 0)$ and $\Delta t = 0.01$.

We regard the multistep linearly implicit scheme (7) as the discrete dynamical system (X, T), where $X = \mathbb{R}^2 \times \mathbb{R}^2$ and continuous mapping T defined as

$$T : \begin{pmatrix} x^{(2n)} \\ x^{(2n+1)} \end{pmatrix} \mapsto \begin{pmatrix} x^{(2n+2)} \\ x^{(2n+3)} \end{pmatrix}.$$

4 Analysis of the Asymptotic Behavior

We show that Proposition 2 can be applied to the scheme (7) for certain range of scheme parameters, which reveals that (7) is asymptotically stable and the solution converges to one of the true fixed points.

4.1 Fixed Points

In this section, we deal with the set of the fixed points $\mathcal{E}(a, b) \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ with respect to (7). If $\mathcal{E}(a, b)$ is essentially different from the set of the fixed points of original equation (4) for some scheme parameters $a, b \in \mathbb{R}$, the asymptotic behavior of the scheme with this a, b must be incorrect.

Since J is nonsingular, it holds that

$$\mathcal{E}(a,b) = \{(x,y) \mid \nabla_{\mathrm{d}}\hat{G}(x,y,x) = \nabla_{\mathrm{d}}\hat{G}(y,x,y) = 0\}.$$

Then, $\mathcal{E}(a,b) = \mathcal{E}_q(a) \times \mathcal{E}_p(b)$ holds, where we define

$$\begin{aligned}
&\mathcal{E}_q(a) := \{(x_q, y_q) \mid f_q(x_q, y_q; a) = f_q(y_q, x_q; a) = 0\},\\
&\mathcal{E}_p(b) := \{(x_p, y_p) \mid f_p(x_p, y_p; b) = f_p(y_p, x_p; b) = 0\},\\
&f_q(x, y; a) := -ay - (1-a)x + y^2 x,\\
&f_p(x, y; b) := by + (1-b)x.
\end{aligned}$$

When $a > 1$ or $\frac{1}{2} \leq a < 1$, it holds that $\mathcal{E}_q(a) = \{(0,0), (\pm 1, \pm 1)\}$, which corresponds to the only true fixed points $0, \pm 1$. When $b \neq \frac{1}{2}$, it holds that $\mathcal{E}_p(b) = \{(0,0)\}$, which corresponds to the true fixed point 0. Therefore, we can summarize this fact as follows.

Lemma 2 *When $a \in [\frac{1}{2}, +\infty) \setminus \{1\}$ and $b \neq \frac{1}{2}$, $\mathcal{E}(a,b)$ is same as the set of fixed points of (4).*

4.2 Level Sets

In order to consider the level sets of $\hat{G}$, we rewrite the energy function $\hat{G}$ as

$$\begin{aligned}
\hat{G}(x,y) = (a-1)\frac{(x_q - y_q)^2}{4} + \frac{(x_q y_q - 1)^2}{4} - \frac{1}{4}\\
+ \left(\frac{1}{2} - b\right)\frac{(x_p - y_p)^2}{4} + \frac{(x_p + y_p)^2}{8}.
\end{aligned}$$

From the expression of $\hat{G}$ above, it is easy to show that the following lemma holds.

Lemma 3 *The set $B(c)$ defined as*

$$B(c) := \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \hat{G}(x,y) \leq c\}$$

is bounded if and only if $a > 1$ and $b < \frac{1}{2}$.

4.3 Semi-strong Lyapunov Functional

In this section, we show that $\hat{G}$ cannot serves as a strong Lyapunov functional; instead, it can serve as a semi-strong Lyapunov functional.

As stated in Section 3.2, the scheme (7) has a dissipation property with respect to $\hat{G}$. Therefore, the property (i) of strong and semi-strong Lyapunov functional is satisfied. Unfortunately, however, $\hat{G}$ cannot serve as a strong Lyapunov functional when $b \neq \frac{1}{2}$ (the case $b = \frac{1}{2}$ can be ignored as the result of the previous sections).

Lemma 4 *When $b \neq \frac{1}{2}$, $\hat{G}$ cannot serve as a strong Lyapunov functional.*

Proof *To show this, we seek for $w, x, y, z \in \mathbb{R}^2$ such that $\hat{G}(w,x) = \hat{G}(y,z)$, $(w,x) = T(y,z)$ and $(w,x) \neq (y,z)$ hold. We see from (8) that*

$$\begin{aligned}\frac{\hat{G}(w,x) - \hat{G}(y,z)}{2\Delta t} &= \nabla_{\mathrm{d}}\hat{G}(w,x,y)^\top J \nabla_{\mathrm{d}}\hat{G}(w,x,y) \\ &\quad + \nabla_{\mathrm{d}}\hat{G}(x,y,z)^\top J \nabla_{\mathrm{d}}\hat{G}(x,y,z).\end{aligned}$$

Thus, $\nabla_{\mathrm{d}}\hat{G}(w,x,y) = (\alpha, 0)^\top$ and $\nabla_{\mathrm{d}}\hat{G}(x,y,z) = (\beta, 0)^\top$ hold for some $\alpha, \beta \in \mathbb{R}$ if and only if $\hat{G}(w,x) = \hat{G}(y,z)$. Then, $(w,x) = T(y,z)$ means that

$$w = y + \begin{pmatrix} 0 \\ -2\alpha\Delta t \end{pmatrix}, \qquad x = z + \begin{pmatrix} 0 \\ -2\beta\Delta t \end{pmatrix}, \tag{9}$$

which by eliminating x and y implies

$$\begin{cases} by_p + (1-b)z_p = (2b\beta + (1-b)\alpha)\,\Delta t; \\ -ay_q - (1-a)z_q + y_q^2 z_q = \alpha; \\ bz_p + (1-b)y_p = (1-b)\beta\Delta t; \\ -az_q - (1-a)y_q + z_q^2 y_q = \beta. \end{cases} \tag{10}$$

Suppose the case $y_q = 0$ and $z_q = -1$. Then, since $\alpha = (1-a)$ and $\beta = a$ hold, we obtain

$$\begin{aligned}\begin{pmatrix} z_p \\ y_p \end{pmatrix} &= \begin{pmatrix} 1-b & b \\ b & 1-b \end{pmatrix}^{-1} \cdot \Delta t \begin{pmatrix} 2ab + (1-a)(1-b) \\ a(1-b) \end{pmatrix} \\ &= \frac{(1-b)\Delta t}{2b-1} \begin{pmatrix} ab + (1-a)(1-b) \\ a-b \end{pmatrix},\end{aligned}$$

where we suppose that $b \neq \frac{1}{2}$. Therefore, there exists w, x, y, z such that $\hat{G}(w,x) = \hat{G}(y,z)$, $(w,x) = T(y,z)$ and $(w,x) \neq (y,z)$. Hence $\hat{G}$ cannot serve as strong Lyapunov functional. □

On the other hand, $\hat{G}$ can serve as a semi-strong Lyapunov functional as follows.

Lemma 5 *When $b \neq \frac{1}{2}$, $\hat{G}$ can serve as a semi-strong Lyapunov functional.*

Proof *First, we show the following claim: If we fix y_q and z_q, then, $x, y, z, w \in \mathbb{R}^2$ such that $\hat{G}(w,x) = \hat{G}(y,z)$ and $(w,x) = T(y,z)$ is uniquely defined. Since we fix y_q and z_q, α and β are uniquely defined by the second and forth equalities of (10). Then, y_p and z_p is also determined by the first and third equalities of (10) under the constraint $b \neq \frac{1}{2}$. Therefore, the equalities (9) shows that w, x is unique. Thus, the claim above holds.*

Second, we see that $\hat{G}$ can serve as a semi-strong Lyapunov functional. It is sufficient to show the following claim: if there exists $(y,z) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $\hat{G}(T^2(y,z)) = \hat{G}(y,z)$ holds, then (y,z) is a fixed point. To show this, we suppose that y,z satisfy $\hat{G}(T^2(y,z)) = \hat{G}(T(y,z)) = \hat{G}(y,z)$. Here, let $u,v,w,x \in \mathbb{R}^2$ be defined as $(w,x) = T(y,z)$ and $(u,v) = T^2(y,z)$. By the assumption, $\nabla_{\mathrm{d}}\hat{G}(w,x,y) = (\alpha,0)$, $\nabla_{\mathrm{d}}\hat{G}(x,y,z) = (\beta,0)$, hold for some $\alpha,\beta \in \mathbb{R}$. Hence, $w_q = y_q$ and $x_q = z_q$ holds. From the claim showed in the first harf of this proof, if we fix y_q and z_q, w,x,y,z satisfying $\hat{G}(w,x) = \hat{G}(y,z)$ is unique. Therefore, we obtain $(u,v,w,x) = (w,x,y,z)$ from $\hat{G}(u,v) = \hat{G}(w,x)$ and $w_q = y_q$ and $x_q = z_q$. Thus, we obtain $(w,x) = (y,z)$ and that means (y,z) is a fixed point. □

4.4 Numerical Examples

First, we summarize the discussion in sections 4.1–4.3 in the following proposition.

Proposition 3 *Suppose that $a > 1$ and $b < \frac{1}{2}$ holds. Then, for any $x^{(0)}, x^{(1)} \in \mathbb{R}^2$, the sequence $\{x^{(n)}\}_{n=0}^{\infty}$ defined by the multistep linearly implicit scheme (7) converges to one of the fixed points of the Duffing equation (4).*

Proof *When $a > 1$ and $b < \frac{1}{2}$ hold, the assumptions of Proposition 2 are all satisfied. Moreover, clearly $\mathcal{E}(T)$ is discrete. Therefore, we obtain*

$$\lim_{n\to\infty} \begin{pmatrix} x^{(2n)} \\ x^{(2n+1)} \end{pmatrix} \to x \in \left\{ \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \end{pmatrix} \right\}.$$

That means the proposition hold. □

We show some numerical examples. Since multistep scheme demands two initial values $x^{(0)}$ and $x^{(1)}$, we set $x^{(0)} = (2,0)$, and define $x^{(1)}$ by solving one step method (6). In Fig. 2, we show the time evolution with scheme parameters $(a,b) = (1.5,0)$, i.e., stable case ($\Delta t = 1/10$). In Fig. 3, we show the time evolution with scheme parameters $(a,b) = (1,1)$, i.e., unstable case ($\Delta t = 1/100$).

Next, we see the effect of the perturbation of initial values. We set $\tilde{x}^{(1)} = x^{(1)} \times 1.05$ and execute the scheme (7) with $x^{(0)}, x^{(1)}$ and $(a,b) = (1.5,0)$ (the left panel of Fig. 4). The orbit is eventually go to fixed point $(1,0)$, however, it oscillates. Since the dynamical system defined by (7) is on $\mathbb{R}^2 \times \mathbb{R}^2$, the orbit smoothly moves in $\mathbb{R}^2 \times \mathbb{R}^2$, however, it oscillates in $\mathbb{R}^2$ (the right panel of Fig. 4).

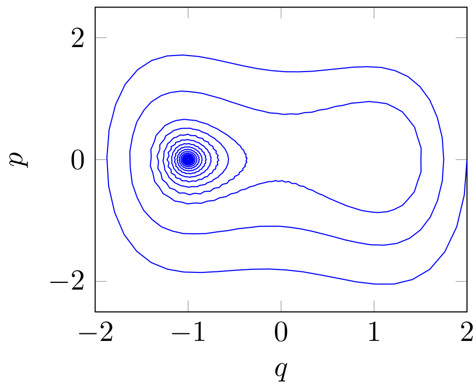

Figure 2: Stable case: scheme parameters $(a, b) = (1.5, 0)$, time-stepping $\Delta t = 1/10$.

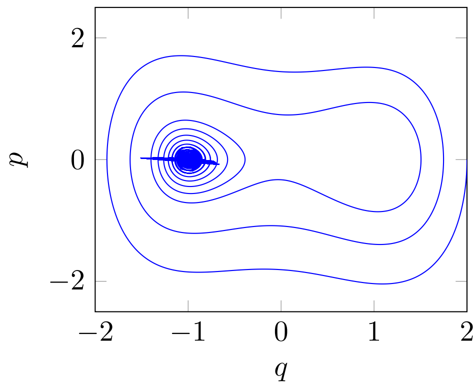

Figure 3: unstable case: scheme parameters $(a, b) = (1, 1)$, time-stepping $\Delta t = 1/100$.

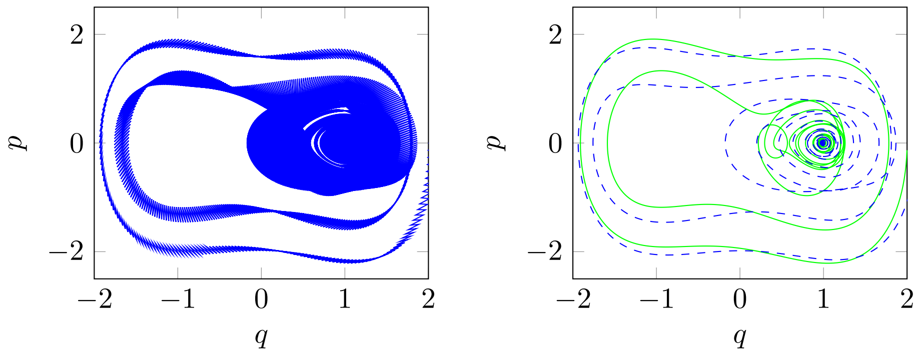

Figure 4: Stable case: $(a, b) = (1.5, 0)$, $\Delta t = 1/100$, perturbated initial value. The left panel: time evolution. The right panel: solid and dashed line correspond to $\{x^{(2n)}\}$ and $\{x^{(2n+1)}\}$, respectively.

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