MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

A Note on “Order-Based Cost Optimization in Assemble-to-Order Systems”

Mohammadreza BOLANDNAZAR, Woonghee Tim HUH,
S. Thomas McCORMICK, and Kazuo MUROTA

METR 2015–06 February 2015

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html

A Note on "Order-Based Cost Optimization in Assemble-to-Order Systems"

Mohammadreza BOLANDNAZAR[*], Woonghee Tim HUH[†], S. Thomas McCORMICK[‡], and Kazuo MUROTA[§]

February 2015

Abstract

One of the main results of "Order-Based Cost Optimization in Assemble-to-Order Systems" by Y. Lu and J-S. Song, *Operations Research*, **53**, 151–169 (2005) is Proposition 1 (c), which states that the cost function of an assemble-to-order (ATO) inventory system satisfies a discrete convexity property called L$^\natural$-convexity. We construct a counterexample showing that this result is incorrect. We then show how to use two existing algorithms to solve the underlying problem in pseudopolynomial time, and that such problems cannot in general be solved in polynomial time. Lastly we note that our counterexample can be adapted to show that some ideas for trying to show discrete convexity in this ATO model do not work.

Key words: inventory/production: stochastic, multi-item, assemble-to-order; optimization: submodularity, discrete convexity.

[*]Sauder School of Business, University of British Columbia, bolandnazar@sauder.ubc.ca
[†]Sauder School of Business, University of British Columbia, tim.huh@sauder.ubc.ca
[‡]Sauder School of Business, University of British Columbia, tom.mccormick@sauder.ubc.ca
[§]Graduate School of Information Science and Technology, University of Tokyo, murota@mist.i.u-tokyo.ac.jp

1 Introduction

Lu and Song (2005) modeled an assemble-to-order (ATO) inventory system with random demands and lead time. We follow their notation and terminology. Their system consists of a set $\mathcal{I} = \{1, 2, \ldots, m\}$ of *items* (components) and a family of *orders* (final products ordered by customers). Order type K is assembled just-in-time from a subset of items, and so $K \subseteq \mathcal{I}$. Type-K orders arrive as a Poisson process at rate λ^K. Their general model allows random, item-dependent lead times for replenishment of items, but here we assume that lead time is the item-independent deterministic number L of time units.

Incoming demand is serviced via a first-come, first-served (FCFS) policy, not only within each order type, but also across multiple order types: If a type-K order arrives and every $i \in K$ is in stock, then the order is assembled and fulfilled. If one or more $i \in K$ are not in stock, the order is backlogged: any $i \in K$ that are in stock are *earmarked* and set aside for this order until the arrival of the out-of-stock $i \in K$. Any earmarked unit of an item cannot be used to satisfy another unit of demand that it is not earmarked for.

Lu and Song analyze base-stock inventory policies, based on the inventory position (inventory that is on-hand and not earmarked plus inventory on order minus backorders) of each item. The decision variables are s_i, the base-stock inventory level for item $i \in \mathcal{I}$. Notice that each s_i can take only non-negative integer values. There is a unit inventory holding cost rate h_i for $i \in \mathcal{I}$, and a backorder cost rate b^K for type-K orders. Defining $\tilde{b}^K = b^K + \sum_{i \in K} h_i$ and $B^K(\boldsymbol{s})$ as the number of backorders for each order-type K, then equation (7) of Lu and Song (2005) develops the expression

$$C(\boldsymbol{s}) = \sum_i h_i s_i + \sum_K \tilde{b}^K \mathsf{E}(B^K(\boldsymbol{s})), \tag{1}$$

whose minimization is equivalent to minimizing the expected total cost of the system under base-stock levels $\boldsymbol{s}$. The objective is to minimize $C(\boldsymbol{s})$ over non-negative $s \in \mathbb{Z}^m$.

A main result in Lu and Song (2005) is Proposition 1, which states three properties of $C(\boldsymbol{s})$. Property (a) is that $C(\boldsymbol{s})$ is *coordinate-wise discretely convex*, i.e., $C(\boldsymbol{s} + e_i) - C(\boldsymbol{s})$ (where e_i is the i-th unit vector) is non-decreasing in i. To describe property (b), we note that points $\boldsymbol{s} \in \mathbb{Z}^m$ form a *lattice* under the operations $\boldsymbol{s}' \wedge \boldsymbol{s}''$ ("meet") as the coordinate-wise minimum, and $\boldsymbol{s}' \vee \boldsymbol{s}''$ ("join") as the component-wise maximum. Then Proposition 1 shows that $C(\boldsymbol{s})$ is *lattice submodular*, i.e.,

$$C(\boldsymbol{s}') + C(\boldsymbol{s}'') \geq C(\boldsymbol{s}' \wedge \boldsymbol{s}'') + C(\boldsymbol{s}' \vee \boldsymbol{s}'').$$

Property (c) in Proposition 1 is L$^\natural$-*convexity*, which is one of several concepts of convexity over integer vectors that make up the field of *discrete convexity*, see Murota (2003). These concepts are important because such notions have been used to show structural properties of various inventory systems as long ago as Miller (1971), then more recently by, e.g., Zipkin (2008), Huh and Janakiraman (2010), Pang et al. (2012), and Li and Yu (2014). Also, functions having one of these properties typically have polynomial minimization algorithms which are useful in, e.g., scheduling applications such as Begen and Queyranne (2011) or Koole and van der Sluis (2010).

A defining property of L$^\natural$-convexity called *Discrete Midpoint Convexity* was first stated by Favati and Tardella (1990) to characterize "submodular integrally convex" functions. A different definition in terms of submodularity in $m + 1$ variables was stated by Murota (1998) to characterize "L$^\natural$-convexity". The equivalence between Murota's definition of L$^\natural$-convexity and Favati and Tardella's definition of submodular integrally convex was shown in Fujishige and Murota (2000). We say that $C(\boldsymbol{s})$ is L$^\natural$-convex if for all $\boldsymbol{s}', \boldsymbol{s}'' \in \mathbb{Z}^m$ with $\max_i |s_i' - s_i''| \leq 2$ it satisfies the Discrete Midpoint Convexity property

$$C(\boldsymbol{s}') + C(\boldsymbol{s}'') \geq C\left(\left\lceil \frac{\boldsymbol{s}' + \boldsymbol{s}''}{2} \right\rceil\right) + C\left(\left\lfloor \frac{\boldsymbol{s}' + \boldsymbol{s}''}{2} \right\rfloor\right), \tag{2}$$

where $\lceil x \rceil$ is the least integer $k \geq x$, and $\lfloor x \rfloor$ is the greatest integer $k \leq x$, see Figure 1. It is known (see Murota (2003)) that L$^\natural$-convexity implies lattice submodularity, but that the reverse is not true. (In

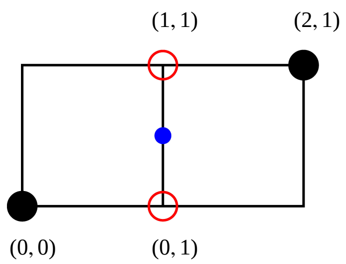

Figure 1: The blue point is the actual midpoint of the two black points, whereas the red points are the discrete midpoints.

fact, knowing that (2) is true when $\max_i |s'_i - s''_i| \leq 1$ suffices to imply lattice submodularity, see Topkis (1978).) Also, there are polynomial algorithms for minimizing L$^\natural$-convex functions.

A goal of this note is to construct and explain a counterexample showing that Proposition 1 (c) of Lu and Song (2005) is wrong in its claim that $C(\boldsymbol{s})$ is L$^\natural$-convex, and to show what algorithms are possible in the absence of L$^\natural$-convexity. Section 2 constructs an instance of the ATO model that violates (2). Section 3 shows how to adapt two known algorithms to solve the problem in pseudopolynomial time, and that no algorithm can minimize every submodular and coordinate-wise convex function in polynomial time. Finally, Section 4 briefly discusses that our counterexample can be adapted to show that some ideas for trying to show discrete convexity in this ATO model do not work.

2 A Counterexample

Our counterexample instance is a special case of the ATO model given in Lu and Song (2005). It has $\mathcal{I} = \{1, 2\}$ (so $m = 2$), and order types $P = \{1, 2\}$ and $Q = \{1\}$. Let $\boldsymbol{s} = (s_1, s_2)$ denote the vector of base-stock levels, and define $B^P(s_1, s_2)$ and $B^Q(s_1, s_2)$ as the steady-state distributions of backorders for demands P and Q, respectively. Then (1) for this instance specializes to

$$C(\boldsymbol{s}) = h_1 s_1 + h_2 s_2 + \tilde{b}^P \mathsf{E}[B^P(s_1, s_2)] + \tilde{b}^Q \mathsf{E}[B^Q(s_1, s_2)]$$

(where h_1 and h_2 denote the per-unit holding cost rates of items 1 and 2, and $\tilde{b}^P$ and $\tilde{b}^Q$ denote the modified per-unit backorder cost rates for order types P and Q). Further assume that $\tilde{b}^Q = h_1 = h_2 = 0$ and $\tilde{b}^P = 1$, and then (1) for this instance further specializes to

$$C(\boldsymbol{s}) = \mathsf{E}[B^P(s_1, s_2)]. \tag{3}$$

To show that $C(\boldsymbol{s})$ is not always L$^\natural$-convex, we consider particular values for $\boldsymbol{s}'$ and $\boldsymbol{s}''$ to construct a violation of (2). Define $\boldsymbol{s}' = (0, 0)$ and $\boldsymbol{s}'' = (2, 1)$. Since $\lceil \frac{1}{2}(\boldsymbol{s}' + \boldsymbol{s}'') \rceil = (1, 1)$ and $\lfloor \frac{1}{2}(\boldsymbol{s}' + \boldsymbol{s}'') \rfloor = (1, 0)$ (see Figure 1), from (3) it suffices to show that

$$\mathsf{E}[B^P(0, 0)] + \mathsf{E}[B^P(2, 1)] < \mathsf{E}[B^P(1, 1)] + \mathsf{E}[B^P(1, 0)] \tag{4}$$

in order to show that $C(\boldsymbol{s})$ is not always L$^\natural$-convex.

If $(s_1, s_2) = (0, 0)$, then every unit of demand for either order type triggers the ordering of required items, and thus is backordered during the lead time. If $(s_1, s_2) = (1, 0)$, there is no unit of item 2 in stock, and every unit of demand for order type P triggers the ordering of item 2; it follows that each demand for P is backordered for the duration of the replenishment lead time L. Therefore, $\mathsf{E}[B^P(0, 0)] = \mathsf{E}[B^P(1, 0)]$.

Now we compare $\mathsf{E}[B^P(2,1)]$ and $\mathsf{E}[B^P(1,1)]$. Clearly, $\mathsf{E}[B^P(2,1)] \le \mathsf{E}[B^P(1,1)]$. We now show that this inequality is strict. Fix time t, and suppose that there is no demand in the interval $(t-2L, t-L]$ (and so there are no backorders of either order type at $t-L$), and the demand realization during the interval $(t-L, t]$ is one unit of Q followed by one unit of P. Clearly, this event occurs with a strictly positive probability. Furthermore, conditioned on this event, the system managed with the base-stock vector $(s_1, s_2) = (2,1)$ has no backorder at t, whereas the system managed with $(s_1, s_2) = (1,1)$ has one unit of P backordered at t (this happens because a unit of item 1 is used to satisfy demand for Q before another unit can be used for P by the FCFS allocation policy). It follows that $\mathsf{E}[B^P(2,1)] < \mathsf{E}[B^P(1,1)]$. Putting these results together, we obtain (4), which implies that $C(\boldsymbol{s})$ is not always $\mathrm{L}^\natural$-convex.

3 Two Algorithms

Proposition 6 in Lu and Song (2005) shows how to compute an upper bound $\boldsymbol{u}$ on the optimal $\boldsymbol{s}$ so that the optimal $\boldsymbol{s}^*$ is in the *box* $B(\boldsymbol{u}) = [\boldsymbol{0}, \boldsymbol{u}] := \{\boldsymbol{s} \in \mathbb{Z}^m \mid \boldsymbol{0} \le \boldsymbol{s} \le \boldsymbol{u}\}$. Define $U = \max_i u_i$. Thus we are interested in minimizing a submodular (but not necessarily $\mathrm{L}^\natural$-convex) function f on a box. Notice that this box is a lattice, as it is closed under meet and join. Every lattice has a unique minimum element (the meet of all elements); for this box lattice the minimum element is the $\boldsymbol{0}$ vector.

3.1 From a Distributive Lattice to Submodular Function Minimization

In fact this box lattice is a *distributive lattice*, as this meet and join distribute over each other. Due to Birkhoff's Representation Theorem (Birkhoff (1937)), every finite distributive lattice $\mathcal{D}$ has a representation as a family $\mathcal{R}$ of subsets of a finite set E. The family $\mathcal{R}$ is a *ring family*, i.e., it is closed under intersection and union. A lattice submodular function f on $\mathcal{D}$ naturally induces an ordinary submodular function $\tilde{f}$ on $\mathcal{R}$, where meet and join in $\mathcal{D}$ become intersection and union in $\mathcal{R}$. Then minimizing f over $\mathcal{D}$ is equivalent to minimizing $\tilde{f}$ over $\mathcal{R}$. This is useful because minimizing a submodular function like $\tilde{f}$ on a ring family like $\mathcal{R}$ is a well-understood problem. There are standard methods for reducing this problem to *submodular function minimization* (SFM, see (Orlin 2009, Section 8), (Schrijver 2000, Section 6), or (McCormick 2006, Section 5.2)). There are various strongly and weakly polynomial SFM algorithms available.

Here is how this reduction from a distributive lattice $\mathcal{D}$ to a ring family $\mathcal{R}$ works, see (Murota 2003, Note 10.15) or (Queyranne and Tardella 2004, Section 4). We call $d \in \mathcal{D}$ *join-irreducible* if $d = d' \vee d''$ implies that $d = d'$ or $d = d''$. Define $J(\mathcal{D})$ as the set of join-irreducible elements of $\mathcal{D}$ other than its minimum element; this will be the ground set E for the ring family. For the box $B(\boldsymbol{u}) = [\boldsymbol{0}, \boldsymbol{u}]$ it is known that $J(B(\boldsymbol{u})) = \{\boldsymbol{s} \in B(\boldsymbol{u}) \mid$ all but one component of $\boldsymbol{s}$ are zero$\}$. We denote the partial order associated with $\mathcal{D}$ by "$\preceq$". Now for $\boldsymbol{s} \in \mathcal{D}$ define $\phi(\boldsymbol{s}) = \{j \in J(\mathcal{D}) \mid j \preceq \boldsymbol{s}\}$, a subset of $J(\mathcal{D})$.

Then the family of subsets $\phi(\mathcal{D}) = \{\phi(\boldsymbol{s}) \mid \boldsymbol{s} \in \mathcal{D}\}$ is a ring family, with $\phi(\boldsymbol{s}') \cap \phi(\boldsymbol{s}'') = \phi(\boldsymbol{s}' \wedge \boldsymbol{s}'')$ and $\phi(\boldsymbol{s}') \cup \phi(\boldsymbol{s}'') = \phi(\boldsymbol{s}' \vee \boldsymbol{s}'')$. For $S \in \phi(\mathcal{D})$ corresponding to $\phi(\boldsymbol{s})$ (i.e., $\phi(\boldsymbol{s}) = S \subseteq J(\mathcal{D})$), it is natural to define $\tilde{f}(S) = f(\boldsymbol{s})$. Then the lattice submodularity of f on $\mathcal{D}$ translates into ordinary submodularity of $\tilde{f}$ on the ring family $\phi(\mathcal{D})$. Therefore the problem of minimizing f on $\mathcal{D}$ translates into the problem of minimizing $\tilde{f}$ on the ring family $\mathcal{R} = \phi(\mathcal{D})$.

Thus we can solve $\min_{\boldsymbol{s} \in B(\boldsymbol{u})} C(\boldsymbol{s})$ by solving SFM for $\tilde{C}$ over $\phi(B(\boldsymbol{u}))$ in time polynomial in $|J(B(\boldsymbol{u}))|$. It is straightforward to see that all operations needed by an SFM algorithm are easy to compute via evaluations of $C(\boldsymbol{s})$ on $B(\boldsymbol{u})$. We call this algorithm the *Ring Family SFM Algorithm*. Unfortunately, the size of is $J(B(\boldsymbol{u}))$ is $O(mU)$. Since U is actually exponential in the data of the ATO instance, this means that Ring Family SFM is only pseudopolynomial, and not polynomial.

It is natural to wonder whether our ATO objective function $C(\boldsymbol{s})$ has any extra structure that can help. Recall that (Lu and Song 2005, Proposition 1 (a)–(b)) shows that $C(\boldsymbol{s})$ is not only submodular, but also coordinate-wise convex. The next proposition shows that no polynomial algorithm can minimizes every submodular and coordinate-wise convex function.

Proposition 3.1. *There exists a function f defined on $B(\boldsymbol{u})$ that is submodular and coordinate-wise convex that cannot be minimized in polynomial time.*

Proof. Proof: Choose $m = 2$, some positive integer p, and set $\boldsymbol{u} = (p, p)$. For $0 \leq i \leq p$ choose some integers $v_i \in [0, p]$ and define $f(i, j) = p(i - j)^2$ if $i \neq j$, and $f(i, j) = v_i$ if $i = j$. It is easy to check that this f is submodular and coordinate-wise convex for any set of v_i.

Since f has the required properties no matter what values we assign to the v_i, any optimization algorithm must evaluate $f(i,i)$ for all $0 \leq i \leq p$ in order to be sure that it has truly found the minimum value. Thus any algorithm must spend at least p evaluations on the $f(i,i)$. But here $U = p$ is not polynomial in the size of the input, and so no algorithm can be polynomial. □

Notice that Proposition 3.1 is stronger than saying that the problem is NP-hard, in that it says that no polynomial algorithm can exist independent of whether P ≠ NP or not. We can easily show that the function f in Proposition 3.1 is not L$^\natural$-convex: For $\boldsymbol{s}' = (0,0)$ and $\boldsymbol{s}'' = (2,2)$ we get that $\lceil \frac{1}{2}(\boldsymbol{s}' + \boldsymbol{s}'') \rceil) = \lfloor \frac{1}{2}(\boldsymbol{s}' + \boldsymbol{s}'') \rfloor = (1,1)$. Thus we get that $f(0,0) + f(2,2) = v_0 + v_2$, whereas $f(1,1) + f(1,1) = 2v_1$, and we can easily choose values such that $v_0 + v_2 \ngeq 2v_1$. This demonstrates that L$^\natural$-convexity is indeed a significantly stronger concept than submodularity plus coordinate-wise convexity.

3.2 The Favati-Tardella Heuristic

Despite the negative result in Proposition 3.1, in practice the bound $\boldsymbol{u}$ will probably not be too large, and so the Ring Family SFM algorithm could be useful despite being only pseudopolynomial. Within the realm of pseudopolynomial algorithms, another possibly simpler algorithm can be adapted from an algorithm in Favati and Tardella (1990). Their algorithm was developed for L$^\natural$-convex functions ((Lu and Song 2005, p. 156) suggest using it), but even without L$^\natural$-convexity we can use it as a heuristic to try to shrink the box $[\boldsymbol{0}, \boldsymbol{u}]$ before applying the Ring Family SFM Algorithm. We call this the *Favati-Tardella Heuristic*.

The properties needed for the Favati-Tardella Heuristic are lattice submodularity and coordinate-wise convexity, which (Lu and Song 2005, Proposition 1 (a)–(b)) shows are true for $C(\boldsymbol{s})$. The idea is to start with $\bar{\boldsymbol{l}} = \boldsymbol{0}$ and $\bar{\boldsymbol{u}} = \boldsymbol{u}$ and to sequentially shrink them towards each other. The Favati-Tardella Heuristic works as follows:

Step 0: Initially put $\bar{\boldsymbol{l}} := \boldsymbol{0}$, $\bar{\boldsymbol{u}} := \boldsymbol{u}$.

Step 1: For each $i = 1, \ldots, m$, find integer $\theta \in [0, \bar{u}_i - \bar{l}_i]$ that minimizes $f(\bar{\boldsymbol{l}} + \theta \boldsymbol{e}_i)$, and update $\bar{\boldsymbol{l}} := \bar{\boldsymbol{l}} + \theta \boldsymbol{e}_i$.

Step 2: For each $i = 1, \ldots, m$, find integer $\theta \in [0, \bar{u}_i - \bar{l}_i]$ that minimizes $f(\bar{\boldsymbol{u}} - \theta \boldsymbol{e}_i)$, and update $\bar{\boldsymbol{u}} := \bar{\boldsymbol{u}} - \theta \boldsymbol{e}_i$.

Step 3: Find $\boldsymbol{z}^+ \in \{0,1\}^m$ that minimizes $f(\bar{\boldsymbol{l}} + \boldsymbol{z}^+)$, and update $\bar{\boldsymbol{l}} := \bar{\boldsymbol{l}} + \boldsymbol{z}^+$.

Step 4: Find $\boldsymbol{z}^- \in \{0,1\}^m$ that minimizes $f(\bar{\boldsymbol{u}} - \boldsymbol{z}^-)$, and update $\bar{\boldsymbol{u}} := \bar{\boldsymbol{u}} - \boldsymbol{z}^-$.

Step 5: If $\boldsymbol{z}^+ = \boldsymbol{z}^- = \boldsymbol{0}$, then output $[\bar{\boldsymbol{l}}, \bar{\boldsymbol{u}}]$ and stop. Otherwise, go to Step 1.

The minimizations in Steps 1 and 2 can be done by binary search (a fast and simple algorithm) since f is coordinate-wise convex. The minimizations in Steps 3 and 4 can be done by SFM (a slower and more complicated algorithm) since the lattice submodularity of f implies that $f(\bar{\boldsymbol{l}} + \boldsymbol{z}^+)$ and $f(\bar{\boldsymbol{u}} - \boldsymbol{z}^-)$ are submodular in $\boldsymbol{z}^+$. Thus each iteration takes polynomial time, though the number of iterations can again be $O(mU)$, i.e., pseudopolynomial. Section 7 of Favati and Tardella (1990) reports on some computational experience with this algorithm on quadratic functions (where SFM can be solved via min cut), where it was found that re-doing the binary search steps after each round of SFM steps often sped up the optimization. We are guaranteed that the output interval $[\bar{\boldsymbol{l}}, \bar{\boldsymbol{u}}]$ contains a global minimum by (Favati and Tardella 1990, Proposition 6.1). Thus if we are lucky and $\bar{\boldsymbol{l}} = \bar{\boldsymbol{u}}$ then the common value is the minimum. If not, then we can apply the Ring Family SFM Algorithm to the smaller interval $[\bar{\boldsymbol{l}}, \bar{\boldsymbol{u}}]$.

4 Discussion

In view of the lack of L$^\natural$-convexity, it is tempting to look for another type of discrete convexity relevant to the ATO objective function $C(\boldsymbol{s})$ in (1). A natural candidate is *integral convexity* as defined in (Favati and Tardella 1990, Definition 3.1), as the class of integral convex functions includes L-convex functions, L$^\natural$-convex functions, M-convex functions, and M$^\natural$-convex functions (see Murota (2003)). An integral convex function is not necessarily lattice submodular, but within the class of lattice submodular functions, integral convexity is equivalent to L$^\natural$-convexity (Favati and Tardella 1990, Corollary 5.2.2). Since $C(\boldsymbol{s})$ is submodular, our counterexample showing that it is not L$^\natural$-convex also shows that it is not integrally convex.

It is also tempting to think that the violation of Discrete Midpoint Convexity (2) happens only for sufficiently small vectors $\boldsymbol{s}$. But this is not the case: Section 3.5 of Bolandnazar (2013) shows that for $m = 2$ and any choice of $s_1 > 0$ and $s_2 \geq s_1 + 1$, our counterexample can be adapted to show a violation of Discrete Midpoint Convexity (2) for $\boldsymbol{s}' = (s_1, s_2)$ and $\boldsymbol{s}'' = (s_1 + 1, s_2 + 2)$. Thus counterexamples to (2) exist for arbitrarily large values of $\boldsymbol{s}$.

Finally we mention that our counterexample has been constructed from an insight about equation (4) of Lu and Song (2005). That equation expresses $B^K(\boldsymbol{s})$ as the maximum among sums of Bernoulli random variables Y_k^i, i.e., $B^K(\boldsymbol{s}) = \max_{i \in K} \sum_k Y_k^i$, where the range of k in the summation is determined by $\boldsymbol{s}$. These Y_k^i are independent within each item i, but not necessarily independent across items (Bolandnazar 2013, Section 3.4).

Acknowledgment

The research of the first three authors was partially supported by NSERC Grants, and that of the fourth author by KAKENHI. We thank Fabio Tardella for insights leading to Proposition 3.1 and its proof, and for other helpful suggestions.

References

Begen, M. and Queyranne, M. (2011). Appointment Scheduling with Discrete Random Durations. *Math. of Oper. Res.*, 36:240–257.

Birkhoff, G. (1937). Rings of Sets. *Duke Math. J.*, 3:443–454.

Bolandnazar, M. (2013). On Properties of the Order-Based Cost Function in Assemble-to-Order Systems. M. Sc. thesis in the Faculty of Graduate Studies, University of British Columbia, Vancouver, BC.

Favati, P. and Tardella, F. (1990). Convexity in Nonlinear Integer Programming. *Ricerca Operativa*, 53:3–44.

Fujishige, S. and Murota, M. (2000). Notes on L-/M-convex Functions and the Separation Theorems. *Math. Prog. A*, 88:129–146.

Huh, W. T. and Janakiraman, G. (2010). On the Optimal Policy Structure in Serial Inventory Systems with Lost Sales. *Operations Research*, 58:486–491.

Koole, G. and van der Sluis, E. (2010). Optimal Shift Scheduling with a Global Service Level Constraint. *IIE Transactions*, 35:1049–1055.

Li, W. and Yu, P. (2014). Multimodularity and Its Applications in Three Stochastic Dynamic Inventory Problems *Manufacturing and Service Operations Management*, 16:455–463.

Lu, Y. and Song, J-S. (2005). Order-Based Cost Optimization in Assemble-to-Order Systems. *Operations Research*, 53:151–169.

McCormick, S. T. (2006). Submodular Function Minimization. In Aardal, K., Nemhauser, G., Weismantel, R. (eds.) Discrete Optimization, *Handbooks in Operations Research and Management Science*, Chap. 7, 12:32–391. Elsevier Science Publishers, Berlin.

Miller, B. (1971). On Minimizing Nonseparable Functions Defined on the Integers with an Inventory Application. *SIAM J. Appl. Math.* 21:166–185.

Murota, K. (1998). Discrete Convex Analysis. *Math. Prog.*, 83:313–371.

Murota, K. (2003). *Discrete Convex Analysis*. SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics, Philadelphia.

Orlin, J. B. (2009). A Faster Strongly Polynomial Algorithm for Submodular Function Minimization. *Math. Prog. A*, 118:237–251.

Pang, Z., Chen, F. Y. and Feng, Y. (2012). A Note on the Structure of Joint Inventory-Pricing Control with Leadtimes. *Operations Research*, 60:581–587.

Queyranne, M., and Tardella, F. (2004). Submodular Function Minimization in $\mathbb{Z}^n$ and Searching in Monge Arrays. Unpublished manuscript, presented by M. Queyranne at CTW04 (Cologne Twente Workshop 2004), *Electronic Notes in Discrete Mathematics*, 17:5.

Schrijver, A. (2000). A Combinatorial Algorithm Minimizing Submodular Functions in Strongly Polynomial Time. *J. Combin. Theory Ser. B* 80:346–355.

Topkis, D. M. (1978) Minimizing a Submodular Function on a Lattice. *Oper. Res.*, 26:305–321.

Zipkin, P. (2008) On the Structure of Lost-Sales Inventory Models. *Operations Research*, 56:937–944.