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Abstract

Imaging the electric conductivity and permittivity inside the human body with an magnetic resonance image (MRI) scanner has attracted considerable attention. However, conventional methods assume that these electrical properties are homogeneous inside the human body, which leads to a severe reconstruction error. In this paper, we present a linear, first-order partial differential equation (PDE) for the inverse of the admittivity, while the conventional PDE for the admittivity is nonlinear. This allows well-established methods for solving linear first-order PDEs to be used in MR-based imaging of electrical properties.

1 Introduction

Recently, reconstruction of the electrical properties inside the human body with the magnetic resonance image (MRI) scanner has attracted considerable attention which can provide significant parameters for diagnosis of e.g. cancer. In this novel imaging modality based on magnetic resonance, the admittivity $\gamma = \sigma + i\omega_0\epsilon$, where σ and ϵ are the electric conductivity and permittivity, respectively, is estimated from the transverse component of the applied radio-frequency (RF) magnetic field at the Larmor frequency ω_0 measured by an MRI scanner. However, conventional methods assume that the permittivity γ is locally homogeneous and neglect the term of $\nabla\gamma$ [1], [2]. It has been noted that the reconstruction error arising from neglecting the spatial distribution of γ may cause technical difficulties in interpreting images of inhomogeneous objects [3]. In this paper, we propose a reconstruction method in which the spatially varied permittivity is fully taken into account. We show that by defining the inverse of the permittivity as $\rho = \frac{1}{\gamma}$ we can obtain a linear, first-order partial differential equation (PDE) for ρ, in contrast to the nonlinear PDE for γ. Hence, by using established methods for solving linear-first order PDEs, spatially distributed electrical properties can be reconstructed much easier.

2 Conventional methods for MREPT

Let μ be the permeability of a body, which is assumed to be constant inside and outside the body. When the transverse RF magnetic field is applied with an MRI scanner, the electric and magnetic fields are generated according to the time-harmonic Maxwell equation

$$\nabla \times \boldsymbol{H} = \gamma \boldsymbol{E}, \tag{1}$$

$$\nabla \times \boldsymbol{E} = -i\omega_0 \mu \boldsymbol{H}. \tag{2}$$

Taking rotation of Eq. (1), we have

$$\nabla(\nabla \cdot \boldsymbol{H}) - \Delta \boldsymbol{H} = \nabla\gamma \times \boldsymbol{E} + \gamma \nabla \times \boldsymbol{E}. \tag{3}$$

The first term on the left-hand side of Eq. (3) vanishes since $\nabla \cdot \boldsymbol{B} = \mu \nabla \cdot \boldsymbol{H} = 0$. Also, using Eqs. (1) and (2), the right-hand side of Eq. (3) can be rewritten in terms of $\boldsymbol{H}$ only, giving

$$-\Delta \boldsymbol{H} = \frac{\nabla\gamma}{\gamma} \times (\nabla \times \boldsymbol{H}) - i\omega_0 \mu \gamma \boldsymbol{H}. \tag{4}$$

For $\boldsymbol{H} = (H_x, H_y, H_z)^T$, let $H^+ = H_x + iH_y$. We can measure H^+ by the B_1 mapping technique with an MRI scanner [1]. The MREPT problem considered in this paper is as follows: given H^+, determine γ.

Nachman [4] proposed to take the inner product of Eq. (4) and $\nabla \times \boldsymbol{H}$ in order to eliminate the first term of the right-hand side, and derived

$$\gamma = \frac{\Delta \boldsymbol{H} \cdot (\nabla \times \boldsymbol{H})}{i\omega_0 \mu \boldsymbol{H} \cdot (\nabla \times \boldsymbol{H})}.$$

However, it has been noted that $\boldsymbol{H}$ and $\nabla \times \boldsymbol{H} = \gamma \boldsymbol{E}$ can be orthogonal and hence the formula becomes unstable [3]. Katscher *et al.* [1] assumed that $\nabla\gamma \simeq \boldsymbol{0}$, i.e., that the spatial change of γ is sufficiently small, and neglected the first term on the right-hand side of Eq. (4), and then derived the reconstruction formula:

$$\gamma = \frac{\Delta \boldsymbol{H}}{i\omega_0 \mu \boldsymbol{H}}. \tag{5}$$

They proposed to integrate this in a small domain D to obtain

$$\gamma \simeq \frac{\int_D \Delta \boldsymbol{H} \cdot \boldsymbol{n} dS}{i\omega_0 \mu \int \boldsymbol{H} \cdot \boldsymbol{n} dS} = -\frac{\int_{\partial D} (\nabla \times \boldsymbol{H}) \cdot d\boldsymbol{s}}{i\omega_0 \mu \int \boldsymbol{H} \cdot \boldsymbol{n} dS}, \tag{6}$$

where $\boldsymbol{n}$ is the unit normal to D. However, Seo *et al.* [3] evaluated the effect of neglecting $\nabla\gamma$ and showed that a significant reconstruction error can occur for an inhomogeneous object. In this paper, we propose a novel method to estimate γ in which $\nabla\gamma$ is fully considered.

3 Proposed method

We define the inverse of the permittivity by

$$\rho = \frac{1}{\gamma}. \tag{7}$$

Then, Ampere's law (1) can be rewritten with ρ by

$$\rho(\nabla \times \boldsymbol{H}) \quad = \quad \boldsymbol{E}. \tag{8}$$

Substituting Eq. (8) into Eq. (2), we have

$$\nabla \times (\rho(\nabla \times \boldsymbol{H})) \quad = \quad -i\omega_0\mu\boldsymbol{H},$$

which can be rewritten as

$$(\nabla \times \boldsymbol{H}) \times \nabla\rho + \Delta\boldsymbol{H}\rho - i\omega_0\mu\boldsymbol{H} = \boldsymbol{0}. \tag{9}$$

We emphasize here that Eq. (9) is a *linear* PDE for ρ, whereas Eq. (4) is a nonlinear PDE for γ.

Here, we require a PDE whose coefficients consist of only the measurable quantity $H^+ = H_x + iH_y$. By taking (x-component of Eq. (9)) + i (y-component of Eq. (9)), we have

$$((\nabla \times \boldsymbol{H}) \times \nabla)_{x+iy}\,\rho + \Delta H^+\rho - i\omega_0\mu H^+ = 0, \tag{10}$$

where $(\boldsymbol{a})_x$ denotes the x-component of $\boldsymbol{a}$ and $(\boldsymbol{a})_{x+iy}$ denotes $(\boldsymbol{a})_x + i(\boldsymbol{a})_y$. Since $H_z \simeq 0$ when using a birdcage coil in the MRI scanner [3], we have

$$\begin{aligned} ((\nabla \times \boldsymbol{H}) \times \nabla)_{x+iy} &= i(\nabla \times \boldsymbol{H})_z(\partial_x + i\partial_y) - i((\nabla \times \boldsymbol{H})_x + i(\nabla \times \boldsymbol{H})_y)\partial_z \\ &= i(\partial_x H_y - \partial_y H_x)(\partial_x + i\partial_y) + \partial_z H^+\partial_z. \end{aligned} \tag{11}$$

Furthermore, since it holds that

$$(\partial_x - i\partial_y)H^+ \quad = \quad \partial_x H_x + \partial_y H_y + i(\partial_x H_y - \partial_y H_x),$$

we can write

$$(\partial_x - i\partial_y)H^+ \quad = \quad i(\partial_x H_y - \partial_y H_x) \tag{12}$$

when $\nabla \cdot \boldsymbol{H} = 0$ and $H_z \simeq 0$. Substituing Eq. (12) into Eq. (11) gives

$$((\nabla \times \boldsymbol{H}) \times \nabla)_{x+iy} \quad = \quad (\partial_x - i\partial_y)H^+(\partial_x + i\partial_y) + \partial_z H^+\partial_z.$$

Hence, Eq. (10) can be rewritten as

$$(\partial_x - i\partial_y)H^+(\partial_x + i\partial_y)\rho + \partial_z H^+\partial_z\rho + \Delta H^+\rho - i\omega_0\mu H^+ = 0. \tag{13}$$

This is a linear PDE for ρ whose coefficients consist of H^+ and its spatial derivatives which can be measured.

Here, we let

$$\begin{aligned} \rho &\equiv f + ig, \\ \omega_0\mu H^+ &\equiv F_0 + iG_0, \\ (\partial_x - i\partial_y)H^+ &\equiv F_1 + iG_1, \\ \partial_z H^+ &\equiv F_2 + iG_2, \\ \Delta H^+ &\equiv F_3 + iG_3, \end{aligned}$$

where $f, g, F_0, G_0, F_1, G_1, F_2, G_2, F_3, G_3$ are real-valued functions. Substituting these into Eq. (13), we obtain from the real and imaginary parts

$$\begin{pmatrix} F_1\partial_x - G_1\partial_y + F_2\partial_z & -G_1\partial_x - F_1\partial_y - G_2\partial_z \\ G_1\partial_x + F_1\partial_y + G_2\partial_z & F_1\partial_x - G_1\partial_y + F_2\partial_z \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} F_3 & -G_3 \\ G_3 & F_3 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} G_0 \\ -F_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{14}$$

This is a simultaneous, first-order PDE for the real and imaginary parts of ρ.

To solve Eq. (14), existing methods for first-order PDEs which have been exhaustively studied can be used. When ρ is estimated, the permittivity γ can be determined by $\gamma = \frac{1}{\rho}$.

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