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Yutaro Yamaguchi*

Abstract

The problem of packing non-zero A-paths in group-labeled graphs was introduced by Chudnovsky, Geelen, Gerards, Goddyn, Lohman, and Seymour (2006), which commonly generalizes several disjoint paths problems, e.g., Mader's disjoint $\mathcal{S}$-paths problem. Tanigawa and Yamaguchi (2013) showed a reduction of this problem to matroid matching, which leads to alternative proofs for a min-max formula due to Chudnovsky et al. and its efficient solvability. In addition, Yamaguchi (2014) provided a reduction to linear matroid parity under some condition for groups, which leads to faster algorithms than the first one due to Chudnovsky, Cunningham, and Geelen (2008). In this paper, we extend these reductions to a weighted version, and present various cases that can be solved in polynomial time by weighted linear matroid parity algorithms thanks to Iwata (2013) and Pap (2013).

1 Introduction

1.1 Packing Non-zero A-paths

Let Γ be a group. A Γ-*labeled graph* is a directed graph $G = (V, E)$ with each edge labeled by an element of Γ, i.e., with a mapping $\psi_G \colon E \to \Gamma$ called a *label function*. The *label* of an undirected walk $W = (v_0, e_1, v_1, e_2, v_2, \ldots, e_l, v_l)$ in G (i.e., $v_i \in V$ for each $i = 0, 1, \ldots, l$ and $e_i = v_{i-1}v_i \in E$ or $e_i = v_iv_{i-1} \in E$ for each $i = 1, 2, \ldots, l$) is defined as $\psi_G(W) := \psi_G(e_l) \cdots \psi_G(e_2) \cdot \psi_G(e_1)$ if $e_i = v_{i-1}v_i$ for every i, and otherwise by replacing the corresponding label $\psi_G(e_i)$ with its inverse $\psi_G(e_i)^{-1}$ for each i with $e_i = v_iv_{i-1}$. A walk is called *balanced* (or a *zero* walk) if its label is the identity element 1_Γ of Γ, and *unbalanced* (or a *non-zero* walk) otherwise. For a prescribed terminal set $A \subseteq V$, an A-*path* is a path (a walk intersecting each vertex at most once) that starts and ends in A and does not intersect A in between.

The problem of *packing non-zero A-paths*, introduced by Chudnovsky et al. [4], is to find a maximum number of vertex-disjoint non-zero A-paths in a given Γ-labeled graph. It generalizes Mader's disjoint $\mathcal{S}$-paths problem [14] (see [17, Chapter 73]) and packing odd-length A-paths, depending on the choice of the group Γ (see [4, Section 2]). For this problem, a min-max formula and a polynomial-time algorithm were given by Chudnovsky et al. [4] and Chudnovsky et al. [3], respectively.

In this paper, we deal with a weighted version of this problem: to minimize the total length of a designated number of vertex-disjoint non-zero A-paths for a given edge length.

Problem 1 (Shortest Disjoint Non-zero A-paths).

Input A Γ-labeled graph $G = (V, E)$, a terminal set $A \subseteq V$, a nonnegative edge length $\ell \in \mathbb{R}^E_{\geq 0}$, and a positive integer k.

Find A family $\mathcal{P}$ of vertex-disjoint non-zero A-paths in G such that $|\mathcal{P}| = k$ and the total length $\ell(E(\mathcal{P}))$ of the edges used in $\mathcal{P}$ is minimized.

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Even for such a weighted version of Mader's disjoint $\mathcal{S}$-paths problem, any polynomial-time algorithm was not known, while Karzanov [9] had shown one for a similar problem in the edge-disjoint A-paths setting (which is a special case of Mader's setting), whose full proof had been left to an unpublished paper [8]. Karzanov's problem can be solved by finding shortest k vertex-disjoint $\mathcal{S}$-paths for possible k, where the number of iterations is at most $|A|/2$ and can be reduced to $\mathrm{O}(\log |A|)$ by binary search. It should be remarked that Hirai and Pap [6] discussed a generalization of Karzanov's setting, in which each pair of two terminals has weight.

1.2 Matroid Matching

The matroid matching problem introduced by Lawler [11] commonly generalizes the matroid intersection problem and the non-bipartite matching problem. This problem cannot be solved in polynomial time in general, but is known to be efficiently solvable as well as to admit a good characterization when the matroid in question is linearly represented (or in a more general situation) due to Lovász [12, 13].

We here describe the problem setting. A pair of a finite set S and an integer-valued set function $f\colon 2^S \to \mathbb{Z}$ is called a *2-polymatroid* if

(1) $f(\emptyset) = 0$,

(2) $f(X) \leq f(Y)$ for each $X \subseteq Y \subseteq S$,

(3) $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for each $X, Y \subseteq S$, and

(4) $f(\{e\}) \leq 2$ for each $e \in S$.

A subset $X \subseteq S$ is called a *matching* in a 2-polymatroid (S, f) if $f(X) = 2|X|$, and a *base* if $f(X) = 2|X| = f(S)$.

The *matroid matching problem* is to find a maximum matching in a given 2-polymatroid. In this paper, we utilize a weighted version: to minimize the total weight of a base.

Problem 2 (Weighted Matroid Matching)**.**
Input A 2-polymatroid (S, f) and a weight vector $w \in \mathbb{R}^S$.
Find A base $X \subseteq S$ in (S, f) such that the total weight $w(X)$ is minimized.

A 2-polymatroid (S, f) is said to be *linearly represented* over a field $\mathbb{F}$ if we have a matrix $Z = (Z_e)_{e \in S} \in \mathbb{F}^{d \times 2S}$ obtained by concatenating $d \times 2$ matrices $Z_e \in \mathbb{F}^{d \times 2}$ $(e \in S)$ such that $f(X) = \operatorname{rank} Z(X)$ for every $X \subseteq S$, where d is a positive integer and $Z(X) = (Z_e)_{e \in X}$ denotes the submatrix of Z obtained by selecting the corresponding columns. A subset $X \subseteq S$ is called a *parity base* for Z if $\operatorname{rank} Z(X) = 2|X| = \operatorname{rank} Z$.

In the linearly represented case, the matroid matching problem is called the *linear matroid parity problem*, for which various polynomial-time algorithms have been proposed, e.g., by Gabow and Stallmann [5], Orlin [15], and Cheung et al. [2]. As announced by Iwata [7] and Pap [16], the following weighted version is also solved in polynomial time.

Problem 3 (Weighted Linear Matroid Parity)**.**
Input A finite set S, a matrix $Z \in \mathbb{F}^{d \times 2S}$ over a field $\mathbb{F}$, and a weight vector $w \in \mathbb{R}^S$.
Find A parity base $X \subseteq S$ for Z such that the total weight $w(X)$ is minimized.

1.3 Results

With the aid of a generalized frame matroid, Tanigawa and the author [18] showed a reduction of the problem of packing non-zero A-paths to the matroid matching problem, which also implies the min-max formula [4] and the polynomial-time solvability [3]. In addition, the author [19] clarified a neccesary and sufficient condition for the groups in question to admit a natural, more direct reduction to the linear matroid parity problem, which leads to much faster algorithms.

Using a trick shown in the next section, the following theorem can be derived from a reduction of packing non-zero A-paths to matroid matching due to Tanigawa and the author [18].

Theorem 1. Shortest Disjoint Non-zero A-paths *reduces to* Weighted Matroid Matching.

In the same way, the next theorem can be obtained from a more direct reduction to linear matroid parity due to the author [19]. Define $\mathrm{PGL}(n,\mathbb{F}) := \mathrm{GL}(n,\mathbb{F})/\langle I_n \rangle$, where $\mathrm{GL}(n,\mathbb{F})$ denotes the general linear group of degree n over a field $\mathbb{F}$ (i.e., the set of all nonsingular $n \times n$ matrices over $\mathbb{F}$ with the ordinary multiplication), $I_n \in \mathrm{GL}(n,\mathbb{F})$ the $n \times n$ identity matrix, and $\langle y \rangle$ the 1-dimensional subspace of a linear space Λ spanned by a vector $y \in \Lambda \setminus \{\mathbf{0}\}$ (i.e., $\langle y \rangle = \{\, ky \mid k \in \mathbb{F} \,\}$ when Λ is a linear space over a field $\mathbb{F}$).

Theorem 2. *Let* Γ *be a group and* $\mathbb{F}$ *a field. If there exists a homomorphism* $\rho\colon \Gamma \to \mathrm{PGL}(2,\mathbb{F})$ *such that* $\rho(\alpha)\binom{1}{0} \notin \left\langle \binom{1}{0} \right\rangle$ *for every* $\alpha \in \Gamma \setminus \{1_\Gamma\}$*, then* Shortest Disjoint Non-zero A-paths *in* Γ*-labeled graphs reduces to* Weighted Linear Matroid Parity *over* $\mathbb{F}$*.*

Thanks to Iwata [7] and Pap [16], Theorem 2 leads to the first polynomial-time algorithm for the weighted version of Mader's disjoint $\mathcal{S}$-paths problem (see Section 3.1). In Section 3, we present other solvable cases, and it is still open whether Shortest Disjoint Non-zero A-paths (as well as Weighted Matroid Matching derived from it by Theorem 1) is efficiently solvable or not.

2 Proofs

2.1 Construction of Auxiliary Graph

First we construct a common auxiliary Γ-labeled graph from a given Γ-labeled graph $G = (V, E)$ with terminal set $A \subseteq V$. We may assume that $|A| \geq 2k$, since otherwise there cannot be a feasible solution.

The construction is summarized as follows. Add $|A| - 2k$ extra terminals so that each extra terminal is adjacent to every original terminal by an edge with an arbitrary non-zero label. Besides, add two other extra terminals b_1, b_2 so that there is an edge from b_1 to b_2 with a non-zero label and b_1 is adjacent to all original non-terminal vertices.

Formally, for the vertex set, let a_i ($i = 1, 2, \ldots, |A| - 2k$) and b_j ($j = 1, 2$) be distinct vertices not in V, and define $A_1 := \{\, a_i \mid i = 1, 2, \ldots, |A| - 2k \,\}$, $A_2 := \{b_1, b_2\}$, $V' := V \cup A_1 \cup A_2$, and $A' := A \cup A_1 \cup A_2$. Next, for the edge set, let $E_1 := \{\, e_{it} = a_i t \mid a_i \in A_1,\ t \in A \,\}$, $E_2 := \{\, e_v = b_1 v \mid v \in V \setminus A \,\}$, and $E' := E \cup E_1 \cup E_2 \cup \{e' = b_1 b_2\}$. Finally, for the label function of $G' := (V', E')$, extend $\psi_G\colon E \to \Gamma$ to $\psi_{G'}\colon E' \to \Gamma$ as follows: for each edge $e \in E'$,

$$\psi_{G'}(e) := \begin{cases} \psi_G(e) & (e \in E), \\ \alpha & (e \in E' \setminus E), \end{cases}$$

where $1_\Gamma \neq \alpha \in \Gamma$.

2.2 Proof of Theorem 1

For the resulting graph $G' = (V', E')$ with terminal set $A' \subseteq V'$, consider the 2-polymatroid constructed in [18], whose ground set is the edge set E'. Define a weight vector $w \in \mathbb{R}^{E'}$ as follows: for each $e \in E'$,

$$w_e := \begin{cases} \ell_e & (e \in E), \\ 0 & (e \in E' \setminus E). \end{cases} \tag{1}$$

Note that to minimize the total weight makes incentive to take edges in $E' \setminus E$ rather than in E, since $\ell_e \geq 0$ for every $e \in E$.

We omit the construction procedure of the 2-polymatroid correspinding to G' (see [18]), and only show its properties as follows. We denote the 2-polymatroid by (E', f), and by $C(F)$ the family of maximal connected subsets of an edge set $F \subseteq E'$.

Claim 3 (From Tanigawa–Yamaguchi [18, Lemma 3.1])**.**

$$f(E') = 2|V'| - |A'| = 2(|V| - k + 1).$$

Claim 4 (Tanigawa–Yamaguchi [18, Lemma 3.2])**.** *An edge set $F \subseteq E'$ satisfies $f(F) = 2|F|$ if and only if*

- *F contains no cycle, and*
- *for each $F' \in C(F)$, we have $|V(F') \cap A'| \leq 2$ and the A'-path between the two terminals is non-zero if $|V(F') \cap A'| = 2$.*

Suppose that G contains k vertex-disjoint non-zero A-paths. By extending the edge set of such paths using edges in $E_1 \cup E_2 \cup \{e'\}$, we can obtain a spanning forest $F \subseteq E'$ such that $|V(F') \cap A'| = 2$ and the A'-path between the two terminals is non-zero for each $F' \in C(F)$, since each unused terminal can be connected to an extra terminal in A_1 by an edge in E_1 (and vice versa), each unused non-terminal can be connected to the extra terminal $b_1 \in A_2$ by an edge in E_2, and $b_1, b_2 \in A_2$ are adjacent by a non-zero edge e'. Then, the number of connected components of $G'[F]$ is $k + (|A| - 2k) + 1 = |A| - k + 1$, and hence, by Claims 3 and 4, we have

$$f(F) = 2|F| = 2\left(|V'| - (|A| - k + 1)\right) = 2(|V| - k + 1) = f(E'),$$

which means that F is a base in (E', f). Therefore, for each family $\mathcal{P}$ of k vertex-disjoint non-zero A-paths in G, there exists a base $F_{\mathcal{P}}$ in (E', f) with $w(F_{\mathcal{P}}) = \ell(E(\mathcal{P}))$ (recall the definition (1) of the weight $w \in \mathbb{R}^{E'}$).

To the contrary, for each base F in (E', f), there exists a family $\mathcal{P}_F$ of k vertex-disjoint non-zero A-paths in G with $E(\mathcal{P}_F) \subseteq F$ (hence, we have $\ell(E(\mathcal{P}_F)) \leq w(F)$) as follows. Thus we have done, i.e., shortest k vertex-disjoint non-zero A-paths in G can be obtained by finding a minimum-weight base in (E', f).

Claim 5. *For a base F in (E', f), the subgraph $G'[F]$ is a spanning forest with $|A| - k + 1$ connected components, each of which contains exactly one non-zero A'-path.*

Proof. By the first condition in Claim 4, F contains no cycle. Since $|A'| = |A| + (|A| - 2k) + 2 = 2(|A| - k + 1)$ and each $F' \in C(F)$ intersects at most two terminals in A' by the second condition in Claim 4, there are at least $|A| - k + 1$ connected components in $G'[F]$. Hence, we have

$$|F| \leq |V'| - (|A| - k + 1) = |V| - k + 1. \tag{2}$$

Recall that $2|F| = f(F) = f(E') = 2(|V| - k + 1)$ by Claim 3, which implies that the equality holds in (2). This means that $G'[F]$ has exactly $|A| - k + 1$ connected components, each of which contains exactly two terminals in A' and the A'-path between the two terminals is non-zero by the second condition in Claim 4. Then, F is obviously spanning. $\square$

Claim 6. *For a base F in (E', f), the subgraph $G[F \cap E]$ contains k vertex-disjoint non-zero A-paths in G.*

Proof. By Claim 5, there are $|A| - k + 1$ connected components in $G'[F]$ each of which contains exactly one non-zero A-paths. Since there is only one edge $e' = b_1 b_2 \in E'$ incident to the extra terminal $b_2 \in A_2$, the connected component containing b_2 must contain $b_1 \in A_2$. Besides, since each edge in E' incident to each extra terminal $a_i \in A_1$ ends an original terminal in A, the connected component containing a_i must contain some original terminal in A. The number of such connected components is $|A_1| = |A| - 2k$, and hence the number of the connected components containing non-zero A-paths in G is $|A| - k + 1 - (|A| - 2k + 1) = k$. $\square$

2.3 Proof of Theorem 2

For the Γ-labeled graph $G' = (V', E')$ with terminal set $A' \subseteq V'$ constructed in Section 2.1, consider the matrix constructed in [19] with a 1-dimensional subspace $Y = \langle \binom{1}{0} \rangle \subseteq \mathbb{F}^2$, whose column set is associated with the edge set E'. Define a weight vector $w \in \mathbb{R}^{E'}$ in the same way as the previous section (see (1)).

Formally, let us construct a matrix $Z \in \mathbb{F}^{2V' \times 2E'}$ as follows, where $Z_{v,e} \in \mathbb{F}^{2\times 2}$ denotes the 2×2 submatrix of Z corresponding to a vertex $v \in V'$ and an edge $e \in E'$. Fix a homomorphism $\rho\colon \Gamma \to \mathrm{PGL}(2, \mathbb{F})$ such that $\rho(\alpha)\binom{1}{0} \notin \langle \binom{1}{0} \rangle$ for every $\alpha \in \Gamma \setminus \{1_\Gamma\}$. For each edge $e = uv \in E'$ and each vertex $w \in V'$, define

$$Z'_{w,e} := \begin{cases} I_2 & (w = u), \\ -\rho(\psi_{G'}(e)) & (w = v), \\ O_2 & (w \in V' \setminus \{u, v\}), \end{cases}$$

where $O_2 \in \mathbb{F}^{2\times 2}$ denotes the 2×2 zero matrix. Next, for each non-terminal vertex $v \in V' \setminus A'$ and each edge $e \in E'$, let $Z_{v,e} := Z'_{v,e}$. Finally, for each terminal $t \in A'$ and each edge $e \in E'$, let $Z_{t,e}$ be the matrix obtained from $Z'_{t,e}$ by eliminating the first row using the vector $\binom{1}{0} \in Y$, i.e., replacing each entry in the first row with 0.

Then, the following claim holds for the resulting matrix $Z \in \mathbb{F}^{2V' \times 2E'}$. Recall that $C(F)$ denotes the family of maximal connected subsets of an edge set $F \subseteq E'$.

Claim 7 (Yamaguchi [19, Lemmas 3.2 and 3.3])**.** *An edge set $F \subseteq E'$ satisfies* $\operatorname{rank} Z(F) = 2|F|$ *if each $F' \in C(F)$ satisfies the following condition* (b) *or* (c)*, and only if* (a), (b)*, or* (c):

(a) $|V(F') \cap A'| = 0$ *and $G'[F']$ contains exactly one non-zero cycle*;

(b) $|V(F') \cap A'| \leq 1$ *and $G'[F']$ contains no cycle*;

(c) $|V(F') \cap A'| = 2$*, $G'[F']$ contains no cycle, and the A'-path between the two terminals is non-zero.*

This claim implies the following claim.

Claim 8. *An edge set $F \subseteq E'$ is a parity base for Z if F is spanning in G' with each $F' \in C(F)$ satisfying Condition* (c) *in Claim* 7*, and only if F is spanning in G' with* (a) *or* (c)*.*

Proof. By the construction of Z, we have

$$\operatorname{rank} Z \leq 2|V'| - |A'| = 2(|V| - k + 1).$$

Suppose that G contains k vertex-disjoint non-zero A-paths. By extending the edge set of such paths using edges in $E_1 \cup E_2 \cup \{e'\}$, we can obtain a spanning forest $F \subseteq E'$ with each $F' \in C(F)$ satisfying Condition (c) (cf. the argument just after Claim 4). Since the number of connected components of $G'[F]$ is $k + (|A| - 2k) + 1 = |A| - k + 1$, by the "if" part of Claim 7, we have

$$\operatorname{rank} Z(F) = 2|F| = 2\left(|V'| - (|A| - k + 1)\right) = 2(|V| - k + 1) \geq \operatorname{rank} Z \geq \operatorname{rank} Z(F).$$

Hence, $\operatorname{rank} Z = 2(|V| - k + 1)$, and the "if" part follows from Claim 7.

The converse direction is also derived from Claim 7. Note that, for a parity base $F \subseteq E'$ for Z, there are at most $|A| - k + 1$ connected components in $G'[F]$ that contain no cycle because of the rank, and hence there cannot be a connected component of type (b). ☐

By the same discussion as that between Claims 4 and 5 in Section 2.2, shortest k vertex-disjoint non-zero A-paths in G can be obtained by finding a minimum-weight parity base for Z. Note again that, we have incentive to take extra edges in $E' \setminus E$ rather than original edges in E because of the definition (1) of the weight w, and hence the total length of any connected component of type (a) in Claim 7 in a minimum-wight parity base for Z is 0.

3 Applications

In this section, we present various cases to which Theorem 2 is applicable. Then, Shortest Disjoint Non-zero A-paths can be solved in polynomial time via Weighted Linear Matroid Parity with the aid of Iwata [7] and Pap [16]. We also refer the readers to [19, Section 5] for further explanation for the following examples.

3.1 Infinite Cyclic Group $\mathbb{Z}$ (Including Mader's $\mathcal{S}$-paths)

For the infinite cyclic group $\mathbb{Z}$, there exists a desired homomorphism $\rho\colon \mathbb{Z} \to \mathrm{PGL}(2, \mathbb{Q})$ defined as follows:

$$\rho(i) := \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \quad (i \in \mathbb{Z}).$$

It should be remarked that the weighted version of Mader's disjoint $\mathcal{S}$-paths can be formulated as Shortest Disjoint Non-zero A-paths in $\mathbb{Z}$-labeled graphs, and hence it can be solved via Weighted Linear Matroid Parity. We again emphasize that this is the first polynomial-time algorithm for this problem.

3.2 Finite Cyclic Groups $\mathbb{Z}_n$ (Including Odd-length A-paths)

For a finite cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ $(n \geq 2)$, there exists a desired homomorphism $\rho\colon \mathbb{Z}_n \to \mathrm{PGL}(2, \mathbb{R})$ defined as follows:

$$\rho(i) := \begin{pmatrix} \cos\dfrac{i\pi}{n} & -\sin\dfrac{i\pi}{n} \\ \sin\dfrac{i\pi}{n} & \cos\dfrac{i\pi}{n} \end{pmatrix} \quad (i \in \mathbb{Z}_n).$$

It should be remarked that the problem of shortest disjoint odd-length A-paths can be formulated as Shortest Disjoint Non-zero A-paths in $\mathbb{Z}_2$-labeled graphs, and hence it can be solved via Weighted Linear Matroid Parity. In the case of $\mathbb{Z}_2 = \{0, 1\}$, there exists a simpler representation ρ' over an arbitrary field such that

$$\rho'(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho'(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

3.3 Finitely Generated Abelian Groups

Suppose that $k = 1$ and a finitely generated abelian group Γ is given as decomposed into p cyclic groups (cf. the fundamental theorem of finitely generated abelian groups). In this case, because of the above two examples, one can solve Shortest Disjoint Non-zero A-paths by solving Weighted Linear Matroid Parity repeatedly p times.

While Kobayashi and Toyooka [10] gave an algebraic algorithm for the case when $k = 1$ and Γ is a fixed finite abelian group inspired by the work of Björklund and Husfeldt [1], our result provides a combinatorial solution to a more general case.

3.4 Dihedral Groups D_n

Even when Γ is non-abelian, there exists a solvable case. For the dihedral group D_n of degree $n \geq 3$, i.e., $D_n = \langle r, R \mid r^n = R^2 = \mathrm{id},\ rR = Rr^{n-1} \rangle$, there exists a desired homomorphism $\rho\colon D_n \to \mathrm{PGL}(2, \mathbb{R})$ defined as follows:

$$\rho(r^i R^j) := \begin{pmatrix} \cos\dfrac{i\pi}{n} & -\sin\dfrac{i\pi}{n} \\ \sin\dfrac{i\pi}{n} & \cos\dfrac{i\pi}{n} \end{pmatrix} \begin{pmatrix} \cos\dfrac{\pi}{n+1} & \sin\dfrac{\pi}{n+1} \\ \sin\dfrac{\pi}{n+1} & -\cos\dfrac{\pi}{n+1} \end{pmatrix}^j \quad (0 \leq i \leq n-1,\ j \in \{0, 1\}).$$

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