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Abstract

Classified stable matching, proposed by Huang, describes a matching model between academic institutes and applicants, where each institute has upper and lower quotas on classes, i.e., subsets of applicants. Huang showed that the problem to decide whether there exists a stable matching or not is NP-hard in general. On the other hand, he showed that the problem is solvable if the classes form a laminar family. For this case, Fleiner and Kamiyama gave a concise interpretation in terms of matroids and showed the lattice structure of stable matchings.

In this paper, we introduce stable matchings on generalized matroids, extending the model of Fleiner and Kamiyama. We design a polynomial-time algorithm which finds a stable matching or reports the nonexistence. We also show that, the set of stable matchings, if nonempty, forms a lattice with several significant properties. Furthermore, we extend this structural results to the polyhedral framework, which we call stable allocations on generalized polymatroids.

1 Introduction

Since the college admission model of Gale and Shapley [11], the two-sided stable matching model has been generalized in many different ways [15, 16]. One of these directions is to generalize a feasible region of each agent, which is originally defined as a family of subsets satisfying an upper quota. For example, in the ordered matroid model of Fleiner [5], a quota constraint is generalized to a matroid constraint. Also, in the stable allocation model of Baïou and Balinski [2], variables can take nonnegative reals, i.e., we determine how much time each pair spend together. In both of these models, the following well-known results of the college admission model have been extended successfully.

(a) Every instance has a stable matching.

(b) The stable matchings form a distributive lattice.

(c) The "rural hospital theorem" holds, i.e., the number of applicants assigned to each college is the same across all stable matchings [12, 20].

In these models, the feasible region of each agent contains zero point (i.e., the emptyset or the zero vector). This simple assumption is in fact essential to guarantee the existence of a stable matching. It is known that, if lower quotas are introduced to these models, some instances have no stable matching. There are various approach to the college admission model with lower quotas. Hamada, Iwama and Miyazaki [13] considered the optimization version of the problem, i.e., to minimize the number of blocking pair satisfying all the upper and lower quotas. They gave an inapproximability result and an exponential-time exact algorithm. Biró et al. [3] considered a variant of the college admission problem with lower quotas which allows some colleges to be closed. They showed the NP-completeness of deciding whether there exists a stable matching or not.

The classified stable matching model, proposed by Huang [14], is a one-to-many matching model between academic institutes and applicants. In this model, besides a preference list on applicants, each institute has a classification of applicants based on their expertise and gives an upper and lower quotas on each class. Huang showed that the problem to decide whether there is a stable matching is NP-complete. On the other hand, he proved that the problem is solvable in polynomial time if the classes form a laminar family. This special case is called the laminar classified stable matching (LCSM) problem. Providing a concise interpretation in terms of matroids, Fleiner and Kamiyama [7] gave an algorithm which solves the many-to-many version of LCSM problem and showed the lattice structure of stable matchings.

In this paper, we generalize the approach and results of Fleiner and Kamiyama. We introduce the following two models.

Stable Matchings on Generalized Matroids We consider a many-to-many matching model in which each agent has a generalized matroid, whose independent set family represents a family of acceptable subsets of opposite agents. As is shown in Section 4, this model includes LCSM model [7, 14].

A generalized matroid (g-matroid) is a generalization of a matroid defined by the "exchange axioms" [22]. In contrast to a matroid, the independent set family of g-matroid does not necessarily contain the empty set, and can express some kind of lower quotas.

For this model, we design an algorithm which finds a stable matching (or reports the nonexistence) in polynomial time, provided a membership oracle and an initial independent set for each g-matroids. Also, we show that the set of stable matchings, if nonempty, forms a lattice.

The key technique of our analysis is to construct a special matroid for each g-matroid. We call this operation the "lower extension" of g-matroids. Through this operation, we obtain a modified instance in which each g-matroid is extended to be a matroid. The modified instance can be treated in the matroid framework of Fleiner [5, 6]. Also, using the exchange axioms of g-matroids, we can obtain the following dichotomy property:

- the set of stable matchings of the original instance coincides with that of the modified instance, or
- there is no stable matching for the original instance.

This implies that we can decide whether the original instance has a stable matching or not by finding one stable matching of the modified instance.

Stable Allocations on Generalized Polymatroids We also consider the polyhedral version of the above model, in which variables take nonnegative reals and the feasible region of each agent is a generalized polymatroid. This model includes the stable allocation model of Baïou and Balinski [2].

A generalized polymatroid (g-polymatroid), introduced by Frank [8], is a polyhedron defined by a pair of submodular and supermodular functions satisfying the "cross inequalities." A g-polymatroid is identical with a polymatroid if it contains the zero vector as the minimum point.

Similarly to the binary case, we construct a modified instance in which each g-polymatroid is extended to be a polymatroid, and show the similar dichotomy property by using the cross inequalities. We analyze the modified instance by reducing it to the choice function model of Alkan and Gale [1]. More precisely, we induce a choice function from an ordered polymatroid, and show that the induced function satisfies the requirement of their model, such as persistence and size-monotonicity. In this way, we show that, the set of stable allocations, if nonempty, is a distributive lattice and a vector version of the rural hospital theorem holds.

The rest of this paper is organized as follows. The first three sections are devoted to the binary model. Section 2 provides preliminaries on matroids and g-matroids. In Section 3, we introduce the notions to represent preferences on g-matroids. Section 4 investigates the stable matching model on g-matroids, and gives algorithmic and structural results. The polyhedral framework starts with Section 5, which provides preliminaries on polymatroids and g-polymatroids. In Section 6, we define a partial order on vectors, and show how to induce a choice function from an ordered polymatroid. Section 7 provides structural properties of stable allocations on g-polymatroids.

2 Matroids and Generalized Matroids

This section gives some basics of matroids and generalized matroids. For a subset X and an element e of some finite set, we denote $X \cup \{e\}$ by $X + e$ and $X \setminus \{e\}$ by $X - e$. For any natural number k, we let $[k] := \{1, 2, \ldots, k\}$.

2.1 Matroids

A pair $(S, \mathcal{I})$ is called a *matroid* if S is a finite set and $\mathcal{I} \subseteq 2^S$ is a nonempty family satisfying the following two conditions.

(I1) If $X \subseteq Y \in \mathcal{I}$, then $X \in \mathcal{I}$.

(I2) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then $X + e \in \mathcal{I}$ for some $e \in Y \setminus X$.

Let $\mathbf{M} = (S, \mathcal{I})$ be a matroid. We call S the *ground set* and $\mathcal{I}$ the *independent set family* of $\mathbf{M}$. A subset $X \subseteq S$ is called *independent* if $X \in \mathcal{I}$. Especially, an independent set X is called a *base* of $\mathbf{M}$ if X is inclusionwise maximal in $\mathcal{I}$. We denote by $\mathcal{B}$ the family of bases. By (I2), every member of $\mathcal{B}$ has the same cardinality. The rank function $r_{\mathbf{M}} : 2^S \to \mathbf{Z}$ is defined by

$$r_{\mathbf{M}}(X) = \max\{\ |Y| : Y \in \mathcal{I},\ Y \subseteq X\ \}.$$

For a subset $X \subseteq S$, its superset $\mathrm{span}_{\mathbf{M}}(X)$ is defined by

$$\mathrm{span}_{\mathbf{M}}(X) = \{\, e \in S \mid r_{\mathbf{M}}(X) = r_{\mathbf{M}}(X + e) \,\}\,.$$

Observation 2.1. For $X \in \mathcal{I}$, $\quad \mathrm{span}_{\mathbf{M}}(X) = X \cup \{\, e \in E \mid X + e \notin \mathcal{I} \,\}$. ■

Let $S_1, S_2, \ldots, S_k$ be disjoint finite sets and $\{\mathbf{M}_i = (S_i, \mathcal{I}_i)\}_{i \in [k]}$ be the set of matroids. Define S and $\mathcal{I}$ by

$$\begin{aligned} S &= S_1 \cup S_2 \cup \cdots \cup S_k, \\ \mathcal{I} &= \{\, X \subseteq S \mid X \cap S_i \in \mathcal{I}_i \ (\forall i \in [k]) \,\}\,. \end{aligned}$$

Then, one can show that the pair $(S, \mathcal{I})$ is a matroid, and it is called *the direct sum of matroids* $\{\mathbf{M}_i\}_{i \in [k]}$.

2.2 Generalized Matroids

A pair $(S, \mathcal{J})$ is called a *generalized matroid* (*g-matroid*) if S is a finite set and $\mathcal{J} \subseteq 2^S$ is a nonempty family satisfying the following two conditions.

(J1) If $X, Y \in \mathcal{J}$ and $e \in Y \setminus X$, then $X + e \in \mathcal{J}$ or $\exists e' \in X \setminus Y : X + e - e' \in \mathcal{J}$.

(J2) If $X, Y \in \mathcal{J}$ and $e \in Y \setminus X$, then $Y - e \in \mathcal{J}$ or $\exists e' \in X \setminus Y : Y - e + e' \in \mathcal{J}$.

We call S the ground set and $\mathcal{J}$ the independent set family of a g-matroid. Conditions (J1) and (J2) are called the *exchange axioms* of g-matroids.

In our stable matching model in Section 4, each agent has a g-matroid to represent his feasible region. For example, an institute has a g-matroid whose independent set family corresponds to the family of acceptable subsets of applicants. Here, we give examples. In Appendix A (Propositions A.8 and A.9), we show that these are indeed g-matroids.

Example 2.2. Let S be a set of applicants and assume that an institute has a a classification $\mathcal{F} \subseteq 2^S$, which is a laminar family (i.e., any $A, B \in \mathcal{F}$ satisfies $A \cap B = \emptyset$ or $A \subseteq B$ or $B \subseteq A$). Each class $A \subseteq S$ of $\mathcal{F}$ has a upper quota $f(A) \in \mathbf{Z}_+$ and a lower quota $g(A) \in \mathbf{Z}_+$. Let $\mathcal{J}(f, g) \subseteq 2^S$ be a family of subsets satisfying all quotas, i.e.,

$$\mathcal{J}(f, g) = \{\ X \subseteq S : \ g(A) \leq |X \cap A| \leq f(A)\ (\forall A \in \mathcal{F})\ \}.$$

Then, the pair $(S, \mathcal{J}(f, g))$ is a g-matroid if $\mathcal{J}(f, g) \neq \emptyset$. ■

Example 2.3. Let S be a set of applicants. An institute has a set D of divisions to which applicants are assigned. Each division $d \in D$ has a set $\Gamma(d) \subseteq S$ of acceptable applicants, an upper quota $u(d)$, and a lower quota $l(d)$. Let $\mathcal{J}(\Gamma, u, l) \subseteq 2^S$ be the set of applicants assignable to divisions, i.e.,

$$\mathcal{J}(\Gamma, u, l) = \left\{ X \subseteq S \,\middle|\, \begin{array}{l} \exists \pi : X \to D \ \text{ s.t. every } d \in D \text{ satisfies} \\ \pi^{-1}(d) \subseteq \Gamma(d),\ l(d) \leq |\pi^{-1}(d)| \leq u(d) \end{array} \right\},$$

where $\pi^{-1}(d) = \{\, s \in X \mid \pi(s) = d \,\}$. Then, the pair $(S, \mathcal{J}(\Gamma, u, l))$ is a g-matroid if $\mathcal{J}(\Gamma, u, l) \neq \emptyset$. ■

2.3 Lower Extension of Generalized Matroids

In this section, we introduce the *lower extension* of g-matroids, which is an operation to construct a matroid as an extension of the given g-matroid. We first show some basic properties of g-matroids. Let $(S, \mathcal{J})$ be a g-matroid.

Lemma 2.4. If $X, Z \in \mathcal{J}$ and $X \subseteq Y \subseteq Z$, then $Y \in \mathcal{J}$.

Proof. Suppose to the contrary, there is a pair $(X, Z) \in \mathcal{J} \times \mathcal{J}$ such that $X \subsetneq Y \subsetneq Z$ for some $Y \notin \mathcal{J}$. Among such pairs, let (X, Z) minimize $|Z \setminus X|$. Apply the exchange axiom (J1) for $X, Z \in \mathcal{J}$ and $e \in Y \setminus X \subseteq Z \setminus X$. Then, as $X \setminus Z = \emptyset$, we have $X + e \in \mathcal{J}$. As $X + e \subseteq Y \subseteq Z$ and $|Z \setminus (X + e)| < |Z \setminus X|$, this contradicts the minimality of $|Z \setminus X|$. □

It is known that a g-matroid satisfies axiom (I2) of matroids.

Lemma 2.5 ([22, Lemma 2.4])**.** If $X, Y \in \mathcal{J}$ and $|X| < |Y|$, then there is an element $e \in Y \setminus X$ s.t. $X + e \in \mathcal{J}$. ■

By Lemmas 2.4 and 2.5, we obtain the following observation.

Observation 2.6. A g-matroid $(S, \mathcal{J})$ is a matroid if $\emptyset \in \mathcal{J}$. ■

Define a superfamily $\mathrm{L}(\mathcal{J})$ of $\mathcal{J} \subseteq 2^S$ by

$$\mathrm{L}(\mathcal{J}) = \{\, X \subseteq S \mid \exists Y : X \subseteq Y \in \mathcal{J} \,\}.$$

We call the pair $(S, \mathrm{L}(\mathcal{J}))$ the *lower extension* of $(S, \mathcal{J})$.

Lemma 2.7. The pair $(S, \mathrm{L}(\mathcal{J}))$ is a matroid.

Proof. By definition, the family $(S, \mathrm{L}(\mathcal{J}))$ clearly satisfies (I1).

To show (I2), assume $X, Y \in \mathrm{L}(\mathcal{J})$ and $|X| < |Y|$. Then, there are $\tilde{X}$ and $\tilde{Y}$ s.t. $X \subseteq \tilde{X} \in \mathcal{J}$ and $Y \subseteq \tilde{Y} \in \mathcal{J}$. Let $Z_1 \in \mathcal{J}$ be an independent set s.t. $Z_1 \subseteq X \cup \tilde{Y}$, $|Z_1| \geq |\tilde{Y}|$, and $|Z_1 \cap X|$ is maximal. Then, we have $X \subseteq Z_1$ since otherwise, by the exchange axiom (J1) for $Z_1, \tilde{X} \in \mathcal{J}$ and any $e \in X \setminus Z_1 \subseteq \tilde{X} \setminus Z_1$, we obtain Z_2 with $|Z_2 \cap X| > |Z_1 \cap X|$, which contradicts the maximality. Then, $(Y \setminus X) \cap Z_1 \neq \emptyset$ since otherwise $X \subseteq Z_1 \subseteq (X \cup \tilde{Y}) \setminus (Y \setminus X)$, and hence $|Z_1| \leq |\tilde{Y}| + |X \setminus \tilde{Y}| - |Y \setminus X| \leq |\tilde{Y}| + |X \setminus Y| - |Y \setminus X| < |\tilde{Y}|$ since $|X| < |Y|$. This contradicts $|Z_1| \geq |\tilde{Y}|$.

Take any $e \in (Y \setminus X) \cap Z_1$. If $e \in \tilde{X}$, then $X + e \subseteq \tilde{X} \in \mathcal{J}$, and we are done. Otherwise, we can apply the exchange axiom (J1) for $\tilde{X}, Z_1 \in \mathcal{J}$ and $e \in Z_1 \setminus \tilde{X}$, and hence $\tilde{X} + e \in \mathcal{J}$ or $\tilde{X} + e - e' \in \mathcal{J}$ for some $e' \in \tilde{X} \setminus Z_1$. Note that $e' \notin X$ since $X \subseteq Z_1$. Then, we have $X + e \subseteq \tilde{X} + e \in \mathcal{J}$ or $X + e \subseteq \tilde{X} + e - e' \in \mathcal{J}$, and both imply $X + e \in \mathrm{L}(\mathcal{J})$ for $e \in Y \setminus X$. □

Lemma 2.8. Let $\mathbf{M} := (S, \mathrm{L}(\mathcal{J}))$. For two independent sets $X, Y \in \mathrm{L}(\mathcal{J})$, suppose $\mathrm{span}_{\mathbf{M}}(X) = \mathrm{span}_{\mathbf{M}}(Y)$. Then, $X \in \mathcal{J}$ if and only if $Y \in \mathcal{J}$.

Proof. We show that $X \in \mathcal{J}$ implies $Y \in \mathcal{J}$, which is enough for the proof. As $X, Y \in \mathrm{L}(\mathcal{J})$ and $\mathrm{span}_{\mathbf{M}}(X) = \mathrm{span}_{\mathbf{M}}(Y)$, Observation 2.1 implies

$$Y + e \notin \mathrm{L}(\mathcal{J}) \quad (\forall e \in X \setminus Y). \tag{1}$$

Suppose, to the contrary, $X \in \mathcal{J}$ and $Y \in \mathrm{L}(\mathcal{J}) \setminus \mathcal{J}$. Let $\tilde{Y}$ be such that $Y \subsetneq \tilde{Y} \in \mathcal{J}$ and $|\tilde{Y} \setminus Y|$ is minimal. By (1), we have $Y + e \not\subseteq \tilde{Y}$ for any $e \in X \setminus Y$, which means $X \setminus Y \subseteq X \setminus \tilde{Y}$. By $Y \subseteq \tilde{Y}$, this gives $X \setminus Y = X \setminus \tilde{Y}$ and so $X \cap Y = X \cap \tilde{Y}$. Therefore, $\tilde{Y} \setminus X = \tilde{Y} \setminus (X \cap Y) \supseteq \tilde{Y} \setminus Y$.

Take $e \in \tilde{Y} \setminus Y \subseteq \tilde{Y} \setminus X$ and apply the exchange axiom (J2) for $X, \tilde{Y} \in \mathcal{J}$ and e. Then, we have $\tilde{Y} - e \in \mathcal{J}$ or $\tilde{Y} - e + e' \in \mathcal{J}$ for some $e' \in X \setminus \tilde{Y}$. Since $e \in \tilde{Y} \setminus Y$, in the former case, $\tilde{Y}_1 := \tilde{Y} - e$ satisfies $Y \subseteq \tilde{Y}_1 \in \mathcal{J}$ and contradicts the minimality. In the latter case, $\tilde{Y}_2 := \tilde{Y} - e + e'$ satisfies $Y + e' \subseteq \tilde{Y}_2 \in \mathcal{J}$ and $e' \in X \setminus \tilde{Y} = X \setminus Y$, which contradicts (1). □

3 Preferences on Generalized Matroids

We call a triple $(S, \mathcal{J}, \succ)$ an *ordered g-matroid* (*ordered matroid*) on S if $(S, \mathcal{J})$ is a g-matroid (matroid) and $\succ$ is a total order on S. In the stable matching model in Section 4, profiles of agents are represented by ordered g-matroids. This section provides some properties of ordered g-matroids.

3.1 Dominance Relation

For an ordered g-matroid $(S, \mathcal{J}, \succ)$, we say that a subset $X \in \mathcal{J}$ *dominates* an element $e \in S \setminus X$ w.r.t. $(S, \mathcal{J}, \succ)$ if the following two conditions hold:

$$X + e \notin \mathcal{J},$$
$$\forall e' \in X \colon [X + e - e' \in \mathcal{J} \implies e' \succ e].$$

Let $(S, \mathcal{J}, \succ)$ be an ordered g-matroid. Then, $(S, \mathrm{L}(\mathcal{J}), \succ)$ is an ordered matroid which we call the *lower extension* of $(S, \mathcal{J}, \succ)$.

Lemma 3.1. Let $X \in \mathcal{J}$ and $e \in S \setminus X$. Then, X dominates e w.r.t. $(S, \mathcal{J}, \succ)$ if and only if X dominates e w.r.t. $(S, \mathrm{L}(\mathcal{J}), \succ)$.

Proof. We show the following two claims, which complete the proof.

(i) For $X \in \mathcal{J}$ and $e \in S \setminus X$, $X + e \notin \mathcal{J}$ if and only if $X + e \notin \mathrm{L}(\mathcal{J})$.

(ii) For $X \subseteq S$ and $e \in S \setminus X$, assume $X \in \mathcal{J}$ and $X + e \notin \mathcal{J}$. Then, for any $e' \in X$, we have $X + e - e' \in \mathcal{J}$ if and only if $X + e - e' \in \mathrm{L}(\mathcal{J})$.

(i): The "if" part is obvious since $\mathcal{J} \subseteq \mathrm{L}(\mathcal{J})$. For the "only if" part, assume $X+e \in \mathrm{L}(\mathcal{J})$. Then, there is Y with $X+e \subseteq Y \in \mathcal{J}$. By Observation 2.4, $X \subseteq X + e \subseteq Y$ and $X, Y \in \mathcal{J}$ imply $X + e \in \mathcal{J}$.

(ii): The "only if" part is obvious. For the "if" part, let $X+e-e' \in \mathrm{L}(\mathcal{J})$. Then, there is Y with $X + e - e' \subseteq Y \in \mathcal{J}$. Apply the exchange axiom (J1) for $X, Y \in \mathcal{J}$ and $e \in Y \setminus X$. Since we have $X + e \notin \mathcal{J}$ and $X \setminus Y = \{e'\}$, there should holds $X + e - e' \in \mathcal{J}$. □

3.2 Choice Functions Induced by Ordered Matroids

Let $\mathscr{M} = (S, \mathcal{I}, \succ)$ be an ordered matroid with $S = \{e_1, e_2, \dots, e_n\}$ and $e_1 \succ e_2 \succ \cdots \succ e_n$. For any $X \subseteq S$, let $C_{\mathscr{M}}(X)$ be the subset of X defined by the following algorithm. Let $Y^0 := \emptyset$ and, define Y^l for each $l \in [n]$ by

$$\boldsymbol{Y}^l := \begin{cases} Y^{l-1} + e_l & \text{if } e_l \in X \text{ and } \ Y^{l-1} + e_l \in \mathcal{I}, \\ Y^{l-1} & \text{otherwise,} \end{cases} \tag{2}$$

and then let $C_{\mathscr{M}}(X) := Y^n$.

Then $C_{\mathscr{M}} : 2^S \to 2^S$ is a function such that $C_{\mathscr{M}}(X) \subseteq X$ ($\forall X \subseteq S$). We call $C_{\mathscr{M}}$ the *choice function induced from* $\mathscr{M}$. By definition, we can compute $C_{\mathscr{M}}(X)$ in $O(|E|)$, for any $X \subseteq S$, provided the membership oracle of $\mathcal{I}$.

Next, consider the case where $\mathscr{M}$ is the lower extension of some g-matroid, i.e., $\mathscr{M} := (S, \mathrm{L}(\mathcal{J}), \succ)$ for an ordered g-matroid $(S, \mathcal{J}, \succ)$. We now show that, for any $X \subseteq S$, we can compute $C_{\mathscr{M}}(X)$ with the membership oracle of $\mathcal{J}$ instead of that of $\mathrm{L}(\mathcal{J})$. We use the following lemma.

Lemma 3.2. Assume $Y \subseteq Z \in \mathcal{J}$. For any $e \in S \setminus Y$, we have $Y + e \in \mathrm{L}(\mathcal{J})$ if and only if $e \in Z$ or $Z + e \in \mathcal{J}$ or $Z + e - e' \in \mathcal{J}$ for some $e' \in Z \setminus Y$.

Proof. The "if" part is clear by the definition of $\mathrm{L}(\mathcal{J})$. To show the "only if" part, let $Y + e \in \mathrm{L}(\mathcal{J})$. Then, there is $Z' \subseteq S$ with $Y + e \subseteq Z' \in \mathcal{J}$. If $e \notin Z$, then by the exchange axiom (J1) for $Z, Z' \in \mathcal{J}$ and $e \in Z' \setminus Z$, we have $Z + e \in \mathcal{J}$ or $Z + e - e' \in \mathcal{J}$ for some $e' \in Z \setminus Z' \subseteq Z \setminus Y$. □

Proposition 3.3. Let $Z \in \mathcal{J}$ be an arbitrary independent set. For any $X \subseteq S$, let $(Y^0, Z^0) := (\emptyset, Z)$ and define (Y^l, Z^l) for each $l \in [n]$ by

$$(Y^l, Z^l) := \begin{cases} (Y^{l-1} + e_l,\ Z^{l-1}) & \text{if } e_l \in X \cap Z^{l-1}, \\ (Y^{l-1} + e_l,\ Z^{l-1} + e_l) & \text{if } e_l \in X \setminus Z^{l-1},\ Z^{l-1} + e_l \in \mathcal{J}, \\ (Y^{l-1} + e_l,\ Z^{l-1} + e_l - e) & \text{if } e_l \in X \setminus Z^{l-1},\ Z^{l-1} + e_l \notin \mathcal{J}, \text{ and} \\ & \exists e \in Z^{l-1} \setminus Y^{l-1} \colon Z^{l-1} + e_l - e \in \mathcal{J}, \\ (Y^{l-1},\ Z^{l-1}) & \text{otherwise.} \end{cases}$$

Then, Y^n coincides with $C_{\mathscr{M}}(X)$, where $\mathscr{M} = (S, \mathrm{L}(\mathcal{J}), \succ)$.

Proof. For each $l \in [n]$, we can observe $Y^l \subseteq Z^l \in \mathcal{J}$ and hence $Y^l \in \mathrm{L}(\mathcal{J})$. In the above definition of Y^l, Lemma 3.2 implies that $Y^l = Y^{l-1} + e_l$ if and only if $e_l \in X$ and $Y^{l-1} + e_l \in \mathrm{L}(\mathcal{J})$. Then, this definition of Y^l corresponds to (2) with $\mathcal{I}$ replaced by $\mathrm{L}(\mathcal{J})$, and hence Y^n coincides with $C_{\mathscr{M}}(X)$. □

We see that one can define (Y^l, Z^l) from (Y^{l-1}, Z^{l-1}) using the membership oracle of $\mathcal{J}$ at most $|S|$ times, which implies the following fact.

Corollary 3.4. Let $(S, \mathcal{J}, \succ)$ be an ordered g-matroid and $\mathscr{M}$ be its lower extension. Provided a membership oracle and an initial independent set of $\mathcal{J}$, we can compute $C_{\mathscr{M}}(X)$ in $O(|S|^2)$ time for any $X \subseteq S$. ■

3.3 Matroid Kernels

Here we introduce the notion of matroid kernels and provide their properties, which were given by Fleiner [5, 6].

Let $\mathscr{M}_1 = (S, \mathcal{I}_1, \succ_1)$ and $\mathscr{M}_2 = (S, \mathcal{I}_2, \succ_2)$ be two ordered matroids on the same ground set S. A subset $X \subseteq S$ is called an *$\mathscr{M}_1\mathscr{M}_2$-kernel* if it satisfies the following two conditions.

1. $X \in \mathcal{I}_1 \cap \mathcal{I}_2$.

2. Every $e = E \setminus X$ is dominated by X w.r.t. $\mathscr{M}_1$ or w.r.t. $\mathscr{M}_2$.

Theorem 3.5 (Fleiner [5, 6])**.** For any ordered matroids $\mathscr{M}_1$ and $\mathscr{M}_2$ on the same ground set S, there is an $\mathscr{M}_1\mathscr{M}_2$-kernel and one can find an $\mathscr{M}_1\mathscr{M}_2$-kernel in $O(|S| \cdot \mathrm{EO}_{\mathscr{M}_1\mathscr{M}_2})$ time, where $\mathrm{EO}_{\mathscr{M}_1\mathscr{M}_2}$ is the time required to compute $C_{\mathscr{M}_1}(X)$, $C_{\mathscr{M}_2}(X)$ for any subset X of S. ∎

Let $\mathfrak{K}_{\mathscr{M}_1\mathscr{M}_2}$ be the set of all $\mathscr{M}_1\mathscr{M}_2$-kernels.

Theorem 3.6 (Fleiner [5, 6])**.** Any two kernels $X, Y \in \mathfrak{K}_{\mathscr{M}_1\mathscr{M}_2}$ satisfies $\mathrm{span}_{\mathbf{M}_1}(X) = \mathrm{span}_{\mathbf{M}_1}(Y)$ and $\mathrm{span}_{\mathbf{M}_2}(X) = \mathrm{span}_{\mathbf{M}_2}(Y)$, where $\mathbf{M}_1$, $\mathbf{M}_2$ are matroids of $\mathscr{M}_1$, $\mathscr{M}_2$. ∎

For any $X, Y \subseteq S$, define subsets $X \vee_1 Y$ and $X \wedge_1 Y$ of S by

$$\begin{aligned} X \vee_1 Y &:= C_{\mathscr{M}_1}(X \cup Y), \\ X \wedge_1 Y &:= C_{\mathscr{M}_2}(X \cup Y). \end{aligned} \tag{3}$$

Theorem 3.7 (Fleiner [5, 6])**.** The triple $(\mathfrak{K}_{\mathscr{M}_1\mathscr{M}_2}, \vee_1, \wedge_1)$ is a lattice. ∎

4 Stable Matchings on Generalized Matroids

In this section, we formulate the matching model on g-matroids. Then, we give an algorithm to find a stable matching (or report the nonexistence), which runs in polynomial time provided a membership oracle and an initial independent set for each g-matroid.

4.1 Model Formulation

Consider two disjoint finite sets I and J of agents. Let $E = I \times J$ be the set of all pairs, and define $E_i = \{\, (i, j) \mid j \in J \,\}$ for each $i \in I$ and $E_j = \{\, (i, j) \mid i \in I \,\}$ for each $j \in J$. Each agent can have multiple partnerships, and any subset X of $E = I \times J$ is called a *matching*.

The *profile* of each agent $k \in I \cup J$ is an ordered g-matroid $(E_k, \mathcal{J}_k, \succ_k)$, where $\mathcal{J}_k \subseteq 2^{E_k}$ represents the feasible sets and $\succ_k$ represents the preference. The set $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ of profiles is called an *instance*.

Definition 4.1. A set $X \subseteq E$ is a *stable matching of* $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ (or, *stable w.r.t.* $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$) if the following two conditions hold.

1. For every $k \in I \cup J$, $\quad X \cap E_k \in \mathcal{J}_k$.

2. For every $e = (i, j) \in E \setminus X$, $\ X \cap E_i$ dominates e w.r.t. $(E_i, \mathcal{J}_i, \succ_i)$ or $X \cap E_j$ dominates e w.r.t. $(E_j, \mathcal{J}_j, \succ_j)$. ∎

We call this model *Generalized Matroid Stable Matching* (*GMSM*). By Example 2.2, we see that GMSM model includes the laminar classified stable matching (LCSM) model [7, 14]. Then, like LCSM model, an instance of GMSM model may have no stable matching. Then, *GMSM problem* is to find a stable matching if it exists, and otherwise to report the nonexistence.

4.2 Characterization through Lower Extension

Let $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ be an instance of GMSM model.

Lemma 4.2. For $X \subseteq E$, the following two conditions are equivalent:

1. X is a stable matching of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$.
2. X is a stable matching of $\{(E_k, \mathrm{L}(\mathcal{J}_k), \succ_k)\}_{k \in I \cup J}$ and satisfies $X \cap E_k \in \mathcal{J}_k$ for every $k \in I \cup J$.

Proof. Both conditions require $X \cap E_k \in \mathcal{J}_k$ ($\forall k \in I \cup J$). Then, Lemma 3.1 implies that, for any $k \in I \cup J$ and $e \in E_k \setminus X$, the subset $X \cap E_k$ dominates e w.r.t. $(E_k, \mathcal{J}_k, \succ_k)$ if and only if $X \cap E_k$ dominates e w.r.t. $(E_k, \mathrm{L}(\mathcal{J}_k), \succ_k)$. Hence, by the definition of the stability, two conditions are equivalent. □

For each agent $k \in I \cup J$, let $\mathscr{M}_k := (E_k, \mathrm{L}(\mathcal{J}_k), \succ_k)$. Then, each $\mathscr{M}_k$ is an ordered matroid. Since $\{E_i\}_{i \in I}$ is a partition of E, let $(E, \mathcal{I}_I)$ be the direct sum of matroids $\{(E_k, \mathrm{L}(\mathcal{J}_k))\}_{i \in I}$. Also, let $\succ_I$ be an arbitrary total order on E which is consistent with $\{\succ_i\}_{i \in I}$, i.e.,

$$e \succ_i e' \implies e \succ_I e' \quad (\forall i \in I,\ \forall e, e' \in E_i).$$

Then, $\mathscr{M}_I := (E, \mathcal{I}_I, \succ_I)$ is an ordered matroid and the following fact holds.

Observation 4.3. Let $X \in \mathcal{I}_I$, $i \in I$, and $e \in E_i \setminus X$. Then, X dominates e w.r.t. $\mathscr{M}_I$ if and only if $X \cap E_i$ dominates e w.r.t. $\mathscr{M}_i$. ■

Similarly, we define an ordered matroid $\mathscr{M}_J = (E, \mathcal{I}_J, \succ_J)$ as the direct sum of ordered matroids $\{\mathscr{M}_j = (E_j, \mathrm{L}(\mathcal{J}_j), \succ_j)\}_{j \in J}$. Then, Observation 4.3, implies the following lemma.

Lemma 4.4. A set $X \subseteq E$ is a stable matching of $\{(E_k, \mathrm{L}(\mathcal{J}_k), \succ_k)\}_{k \in I \cup J}$, if and only if X is an $\mathscr{M}_I\mathscr{M}_J$-kernel. ■

Combining Lemmas 4.2 and 4.4, we obtain the following theorem.

Theorem 4.5. A set $X \subseteq E$ is a stable matching of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ if and only if X is an $\mathscr{M}_I\mathscr{M}_J$-kernel satisfying $X \cap E_k \in \mathcal{J}_k$ ($\forall k \in I \cup J$). ■

4.3 Structure of Stable Matchings

Let $\mathfrak{S}$ be the set of stable matchings of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ and let $\mathfrak{K}_{\mathscr{M}_I\mathscr{M}_J}$ be the set of $\mathscr{M}_I\mathscr{M}_J$-kernels. Then, Theorem 4.5 is rephrased as

$$\mathfrak{S} = \{ X \in \mathfrak{K}_{\mathscr{M}_I\mathscr{M}_J} \mid X \cap E_k \in \mathcal{J}_k \ (\forall k \in I \cup J) \}. \tag{4}$$

This says that $\mathfrak{S}$ is a subset of $\mathfrak{K}_{\mathscr{M}_I\mathscr{M}_J}$. Moreover, members of $\mathfrak{K}_{\mathscr{M}_I\mathscr{M}_J}$ satisfy the following significant property.

Lemma 4.6. For any $\mathscr{M}_I\mathscr{M}_J$-kernels $X, Y \in \mathfrak{K}_{\mathscr{M}_I\mathscr{M}_J}$ and any $k \in I \cup J$, $X \cap E_k \in \mathcal{J}_k$ if and only if $Y \cap E_k \in \mathcal{J}_k$.

Proof. Without loss of generality, we assume $k = i \in I$. By Theorem 3.6, we have $\mathrm{span}_{\mathbf{M}_I}(X) = \mathrm{span}_{\mathbf{M}_I}(Y)$ where $\mathbf{M}_I = (E, \mathcal{I}_I)$. By the definition of $\mathcal{I}_I$, this implies $\mathrm{span}_{\mathbf{M}_i}(X \cap E_i) = \mathrm{span}_{\mathbf{M}_i}(Y \cap E_i)$ where $\mathbf{M}_i = (E_i, \mathrm{L}(\mathcal{J}_i))$. Then, Lemma 2.8 implies that $X \cap E_i \in \mathcal{J}_i$ if and only if $Y \cap E_i \in \mathcal{J}_i$. □

Combining (4) and Lemma 4.6 yields the following dichotomy theorem.

Theorem 4.7. There holds $\mathfrak{S} = \mathfrak{K}_{\mathscr{M}_I\mathscr{M}_J}$ or $\mathfrak{S} = \emptyset$. ■

Corollary 4.8. If $X \cap E_k \notin \mathcal{J}_k$ for some $X \in \mathfrak{K}_{\mathscr{M}_I\mathscr{M}_J}$ and $k \in I \cup J$, then $\mathfrak{S} = \emptyset$, i.e., $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ has no stable matching. ■

Recall Theorem 3.7, with which Theorem 4.7 implies the following fact.

Corollary 4.9. If the set of stable matchings of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ is nonempty, then it forms a lattice under the operations $\vee_1$ and $\wedge_1$, which are defined by (3) with $C_{\mathscr{M}_1}$, $C_{\mathscr{M}_2}$ replaced by $C_{\mathscr{M}_I}$, $C_{\mathscr{M}_J}$, respectively. ■

4.4 Algorithm for Finding a Stable Matching

For an instance $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ of GMSM problem, let us consider the following algorithm.

Algorithm: GMSM

Step1. Find an $\mathscr{M}_I\mathscr{M}_J$-kernel X.

Step2. If $X \cap E_k \in \mathcal{J}_k$ for every $k \in I \cup J$, then return X. Otherwise, report "There is no stable matching."

Theorem 4.5 and Corollary 4.8 guarantee the correctness of this algorithm.

Proposition 4.10. If the algorithm GMSM returns a matching $X \subseteq E$, then X is a stable matching of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$. Otherwise, there is no stable matching of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$. ■

We now check the time complexity of this algorithm. For an instance $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$, we assume that each $\mathcal{J}_k$ is given by a membership oracle, which answers whether a given subset of E_k is in $\mathcal{J}_k$ or not. Also, we assume that one independent set of $\mathcal{J}_k$ is given for each $k \in I \cup J$.

We first evaluate the time required for Step 1 of GMSM.

Proposition 4.11. An $\mathscr{M}_I\mathscr{M}_J$-kernel can be found in $O(|E|^3)$ time.

Proof. By Theorem 3.5, an $\mathscr{M}_I\mathscr{M}_J$-kernel can be found in $O(|E| \cdot \mathrm{EO}_{\mathscr{M}_I\mathscr{M}_J})$ time, where $\mathrm{EO}_{\mathscr{M}_I\mathscr{M}_J}$ is the time required to compute $C_{\mathscr{M}_I}(X)$, $C_{\mathscr{M}_J}(X)$ for any $X \subseteq E$. Note that, $C_{\mathscr{M}_I}(X)$ is the direct sum of $\{C_{\mathscr{M}_i}(X \cap E_i)\}_{i \in I}$. Since $C_{\mathscr{M}_i}(X \cap E_i)$ is computed in $O(|E_i|^2) = O(|J|^2)$ time by Corollary 3.4, $C_{\mathscr{M}_I}(X)$ is obtained in $O(|I| \cdot |J|^2) = O(|E|^2)$ time. Similar arguments apply to $C_{\mathscr{M}_J}(X)$. Then, $\mathrm{EO}_{\mathscr{M}_I\mathscr{M}_J}$ is $O(|E|^2)$, and the proof is completed. □

Note that Step 2 of GMSM requires only $O(|I| + |J|)$ time.

Proposition 4.12. The algorithm GMSM runs in $O(|E|^3)$ time. ■

Propositions 4.10 and 4.12 imply the following main theorem.

Theorem 4.13. For any instance of GMSM problem, one can determine whether a stable matching exists or not in $O(|E|^3)$ time. Also, one can obtain a stable matching simultaneously if exists. ■

5 Polymatroids and Generalized Polymatroids

In this section, we introduce the notion of polymatroids and generalized polymatroids, the polyhedral versions of matroids and g-matroids.

We introduce some notations. Let S be a nonempty finite set. For a real vector $x = (x(e) \mid e \in S) \in \mathbf{R}^S$ and a subset $A \subseteq S$, we write $x(A)$ for $\sum_{e \in A} x(e)$ and let $x(\emptyset) = 0$. We also write $|x|$ for $x(S)$. For vectors $x, y \in (\mathbf{R} \cup \{\pm\infty\})^S$, the notation $x \leq y$ means $x(e) \leq y(e)$ ($\forall e \in S$), and $x \wedge y$ and $x \vee y$ are vectors in $(\mathbf{R}^S \cup \{\pm\infty\})^S$ whose e-th components are respectively $\min\{x(e), y(e)\}$ and $\max\{x(e), y(e)\}$. For each $e \in S$, denote by $\chi_e \in \mathbf{R}^S$ the vector whose e-th component is 1 and others are 0.

5.1 Polymatroids

A family $\mathcal{F} \subseteq 2^S$ is called a *ring family* if $A, B \in \mathcal{F}$ implies $A \cup B, A \cap B \in \mathcal{F}$. A function $f \colon \mathcal{F} \to \mathbf{R}$ is called *submodular* if its domain $\mathcal{F}$ is a ring family and the *submodular inequality*

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

holds for every $A, B \in \mathcal{F}$. Also, f is called *supermodular* if $-f$ is submodular. We say that f is *monotone* if $A \subseteq B$ implies $f(A) \leq f(B)$ for every $A, B \in \mathcal{F}$.

Throughout this section, we suppose that any set function $f\colon \mathcal{F} \to \mathbf{R}$ satisfies $\emptyset \in \mathcal{F}$ and $f(\emptyset) = 0$. For any $f\colon \mathcal{F} \to \mathbf{R}$, we associate three kinds of polyhedra

$$\begin{aligned}
&\mathbf{P}(f) = \{\, x \in \mathbf{R}^S \mid x(A) \le f(A) \ (\forall A \in \mathcal{F}) \,\}, \\
&\mathbf{B}(f) = \{\, x \in \mathbf{R}^S \mid x(A) \le f(A) \ (\forall A \in \mathcal{F}),\ x(S) = f(S) \,\}, \\
&\mathbf{P}_+(f) = \mathbf{P}(f) \cap \mathbf{R}^S_+.
\end{aligned}$$

Here is a basic property of submodular functions.

Proposition 5.1 (cf. [10, Theorem 2.3])**.** If a set function $f : \mathcal{F} \to \mathbf{R}$ is submodular, then $\mathbf{B}(f)$ is nonempty and coincides with the set of all maximal elements of $\mathbf{P}(f)$ (i.e., $\mathbf{B}(f) = \{\, x \in \mathbf{P}(f) \mid \nexists y \in \mathbf{P}(f) : y \ge x,\ y \ne x \,\}$). ■

Corollary 5.2. If a set function $f : \mathcal{F} \to \mathbf{R}$ is submodular, then

$$\mathbf{P}(f) = \{\, x \in \mathbf{R}^S \mid \exists y : x \le y \in \mathbf{B}(f) \,\}$$

holds. ■

A polyhedron $P \subseteq \mathbf{R}^S$ is called a *polymatroid* if $P = \mathbf{P}_+(f)$ for some monotone submodular function $f : 2^S \to \mathbf{R}$. In fact, such a function is uniquely determined as follows.

Proposition 5.3 (Edmonds[4])**.** For a polymatroid $P \subseteq \mathbf{R}^S$, a function $f : 2^S \to \mathbf{R}$ defined by $f(A) = \max\{\, x(A) \mid x \in P \,\}$ is a unique monotone submodular function which satisfies $P = \mathbf{P}_+(f)$. ■

We call this unique function f the *defining function* of P. Polymatroids can be regarded as the polyhedral versions of matroids. Indeed, for a matroid $\mathbf{M} = (S, \mathcal{I})$, the convex hull of characteristic vectors of $\mathcal{I}$ is a polymatroid defined by the rank function of $\mathbf{M}$.

Next, we introduce the notion of intersecting-submodularity, which is weaker than submodularity but still yields polymatroids. We say that subsets $A, B \subseteq S$ are *intersecting* if none of $A \cap B$, $A \setminus B$, and $B \setminus A$ is empty. A family $\mathcal{F} \subseteq 2^S$ is called an *intersecting family* if every intersecting $A, B \in \mathcal{F}$ satisfy $A \cup B, A \cap B \in \mathcal{F}$. A function $f\colon \mathcal{F} \to \mathbf{R}$ is called *intersecting-submodular* if $\mathcal{F}$ is an intersecting family and f satisfies the submodular inequality for every intersecting $A, B \in \mathcal{F}$. Also, f is called *intersecting-supermodular* if $-f$ is intersecting-submodular.

Proposition 5.4 (Edmonds [4])**.** Let $f\colon \mathcal{F} \to \mathbf{R}$ be an intersecting-submodular function. Then, $\mathbf{P}_+(f)$ is a polymatroid if it is nonempty and bounded. ■

5.2 Generalized Polymatroids

For families $\mathcal{F}, \mathcal{G} \subseteq 2^S$ and set functions $f : \mathcal{F} \to \mathbf{R}$ and $g : \mathcal{G} \to \mathbf{R}$, we call the pair (f, g) a *strong pair* if

- f is submodular,
- g is supermodular, and
- for any $A \in \mathcal{F}$ and $B \in \mathcal{G}$, there hold $A \setminus B \in \mathcal{F}$, $B \setminus A \in \mathcal{G}$, and the following *cross inequality* holds:

$$f(A) - g(B) \geq f(A \setminus B) - g(B \setminus A).$$

For any set functions $f : \mathcal{F} \to \mathbf{R}$ and $g : \mathcal{G} \to \mathbf{R}$, we associate polyhedra

$$\mathbf{P}(f,g) = \left\{ x \in \mathbf{R}^E \;\middle|\; \begin{array}{l} x(A) \leq f(A) \quad (\forall A \in \mathcal{F}), \\ x(A) \geq g(A) \quad (\forall A \in \mathcal{G}) \end{array} \right\},$$
$$\mathbf{P}_+(f,g) = \mathbf{P}(f,g) \cap \mathbf{R}^S_+.$$

A polyhedron $Q \subseteq \mathbf{R}^S$ is called a *generalized polymatroid* (*g-polymatroid*) if $Q = \mathbf{P}(f, g)$ for some strong pair (f, g).

Proposition 5.5 (Frank and Tardos [9])**.** For a g-polymatroid $Q \subseteq \mathbf{R}^S$, define set functions f and g respectively by $f(A) = \max\{\, x(A) \mid x \in Q \,\}$ and $g(A) = \min\{\, x(A) \mid x \in Q \,\}$, where they are defined only on subsets which make the right-hand side finite. Then, the pair (f, g) is a unique strong pair which satisfies $Q = \mathbf{P}(f, g)$. ■

For such a unique strong pair (f, g), we call f the *upper bound function* and g the *lower bound function* of Q. Generalized polymatroids are the polyhedral versions of g-matroids. One can show that, for any g-matroid $(S, \mathcal{J})$, the convex hull of characteristic vectors of $\mathcal{J}$ is a g-polymatroid. Note that a g-polymatroid is identical with a polymatroid if it contains the zero vector as the minimum point, and then the defining function coincides with the upper bound function.

As shown in Proposition 5.4, a function f yields a polymatroid if it is intersecting-submodular. We give a g-polymatroid version of such a condition. For two set functions $f : \mathcal{F} \to \mathbf{R}$ and $g : \mathcal{G} \to \mathbf{R}$, the pair (f, g) is called a *weak pair* if

- f is intersecting-submodular,
- g is intersecting-supermodular, and
- for each intersecting $A \in \mathcal{F}$ and $B \in \mathcal{G}$, there hold $A \setminus B \in \mathcal{F}$, $B \setminus A \in \mathcal{G}$ and the cross inequality holds for A and B.

Proposition 5.6 (Frank [8])**.** Let (f, g) be a weak pair. If $\mathbf{P}(f, g) \neq \emptyset$, then $\mathbf{P}(f, g)$ is a g-polymatroid. The same holds for $\mathbf{P}_+(f, g)$. ■

G-polymatroids are characterized by the following *exchange axiom* (see [[10, 17, 18]]). For a real $\alpha_0 > 0$, we write $[0, \alpha_0] := \{ \alpha \in \mathbf{R} \mid 0 \le \alpha \le \alpha_0 \}$.

Theorem 5.7. A set $Q \subseteq \mathbf{R}^S$ is a g-polymatroid if and only if it satisfies the following property. For all $x, y \in Q$ and $e \in S$ with $x(e) < y(e)$, either of the following holds for some positive real $\alpha_0 > 0$:

1. $x + \alpha\chi_e,\ y - \alpha\chi_e \in Q \quad (\forall \alpha \in [0, \alpha_0])$.
2. There exists $e' \in S$ with $x(e') > y(e')$ such that $x + \alpha(\chi_e - \chi_{e'}),\ y - \alpha(\chi_e - \chi_{e'}) \in Q \quad (\forall \alpha \in [0, \alpha_0])$. ■

The reduction of a g-polymatroid is also a g-polymatroid as follows. For any vector $a \in (\mathbf{R} \cup \{\infty\})^S$, we write $[-\infty, a] := \{ x \in \mathbf{R}^S \mid x \le a \}$.

Proposition 5.8 (Frank ans Tardos [9])**.** For any vector $a \in (\mathbf{R} \cup \{\infty\})^S$ with $Q \cap [-\infty, a] \neq \emptyset$, the set $Q \cap [-\infty, a]$ is a g-polymatroid whose upper and lower bound functions are respectively given by

$$f^a(A) = \min \{ f(B) + a(A \setminus B) \mid B \subseteq A \},$$
$$g^a(A) = \max \{ g(B) - a(B \setminus A) \mid A \subseteq B \subseteq S \},$$

where they are defined only on subsets which make the right-hand side finite. ■

5.3 Lower Extension of Generalized Polymatroids

Here, we introduce the lower extension of g-polymatroids. This is the vector version of the lower extension of g-matroids.

Let $Q \subseteq \mathbf{R}^S_+$ be a g-polymatroid which is bounded and included in the nonnegative region. Also, let $f, g : 2^S \to \mathbf{R}$ be the upper and lower bound functions of Q, respectively. Then, (f, g) is a strong pair.

Lemma 5.9. There holds $\mathbf{B}(f) \subseteq Q$.

Proof. Since $Q = \mathbf{P}(f, g) = \{ x \in \mathbf{P}(f) \mid x(A) \ge g(A)\ (\forall A \subseteq S) \}$, it suffices to show that $x \in \mathbf{B}(f)$ implies $x(A) \ge g(A)$ for any $A \subseteq S$.

Take any $x \in \mathbf{B}(f)$ and $A \subseteq S$. As (f, g) is a strong pair, we have the crossing inequality $f(S) - g(A) \ge f(S \setminus A) - g(\emptyset)$. Substituting $x(S) = f(S)$, $x(S \setminus A) \le f(S \setminus A)$, and $g(\emptyset) = 0$, we obtain $x(A) \ge g(A)$. □

Define a superset $\mathrm{L}(Q) \subseteq \mathbf{R}^S_+$ of $Q \subseteq \mathbf{R}^S_+$ by

$$\mathrm{L}(Q) = \{ x \in \mathbf{R}^S_+ \mid \exists y : x \le y \in Q \}.$$

We call $\mathrm{L}(Q)$ the *lower extension* of Q.

Lemma 5.10. The lower extension $\mathrm{L}(Q)$ of Q is a polymatroid whose defining function is f. i.e., $\mathrm{L}(Q)$ and Q have the same upper bound function.

Proof. By $Q = \mathbf{P}(f,g) \subseteq \mathbf{P}(f)$ and $Q \subseteq \mathbf{R}^S_+$, we have $Q \subseteq \mathbf{P}_+(f)$. With Lemma 5.9, we obtain $\mathbf{B}(f) \subseteq Q \subseteq \mathbf{P}_+(f)$, which implies

$$\{\, x \in \mathbf{R}^S_+ \mid \exists y : x \le y \in \mathbf{B}(f) \,\} \subseteq \mathrm{L}(Q) \subseteq \{\, x \in \mathbf{R}^S_+ \mid \exists y : x \le y \in \mathbf{P}_+(f) \,\}.$$

By Corollary 5.2, the left-hand side coincides with $\mathbf{P}_+(f)$. The right-hand side also coincides with $\mathbf{P}_+(f)$ by definition. Thus, we obtain $\mathrm{L}(Q) = \mathbf{P}_+(f)$. Also, by Proposition 5.5, $Q \subseteq \mathbf{R}^S_+$ implies the monotonicity of the submodular function f, which completes the proof. □

Combining Proposition 5.8 and Lemma 5.10 yields the following lemma.

Lemma 5.11. For a g-polymatroid $Q \subseteq \mathbf{R}^S_+$ and a vector $a \in (\mathbf{R}_+ \cup \{\infty\})^E$, assume $Q \cap [-\infty, a] \neq \emptyset$. Then, $Q \cap [-\infty, a]$ and $\mathrm{L}(Q) \cap [-\infty, a]$ are g-polymatroids whose upper bound functions are the same. ■

6 Preferences on Generalized Polymatroids

We introduce a partial order on vectors, which is used to represent a preference of each agent in the stable allocation model in Section 7.

6.1 Optimal Points of Generalized Polymatroids

Let $S = \{e_1, e_2, \dots, e_n\}$ be a finite set and $\succ$ be a total order on S such that $e_1 \succ e_2 \succ \cdots \succ e_n$. Let $S_0 = \emptyset$ and $S_l = \{e_1, e_2, \cdots, e_l\}$ for each $l \in [n]$. For vectors $x, y \in \mathbf{R}^S$, we say that x is *preferable* to y w.r.t. $\succ$ if

$$x(S_l) \ge y(S_l) \quad (\forall l \in [n]).$$

Note that this is a partial order on $\mathbf{R}^S$, and call it the *preference order based on* $\succ$. For a set $Q \subseteq \mathbf{R}^S$, we say that $x \in \mathbf{R}^S$ is an *optimal point* of Q w.r.t. $\succ$ (or, *optimal* in Q w.r.t. $\succ$) if $x \in Q$ and x is preferable to all $y \in Q$.

Observation 6.1. If $x \in Q_2 \subseteq Q_1 \subseteq \mathbf{R}^S$ and x is optimal in Q_1 w.r.t. $\succ$, then x is optimal in Q_2 w.r.t. $\succ$. ■

Since the preference order is a partial order, a general set of vectors may have no optimal point. A bounded g-polymatroid, however, does have an optimal point. Moreover, it is obtained by the following greedy algorithm.

Proposition 6.2. Let $Q \subseteq \mathbf{R}^S$ be a bounded g-polymatroid whose upper bound function is $f\colon 2^S \to \mathbf{R}$. Define a vector $x^* \in \mathbf{R}^S$ by

$$x^*(e_l) = f(S_l) - f(S_{l-1}) \tag{5}$$

for each $l \in [n]$. Then, x^* is optimal in Q w.r.t. $\succ$.

Proof. Since this construction of x^* is the greedy algorithm of Edmonds [4] and Shapley [21], we have $x^* \in \mathbf{P}(f)$. Also, (5) implies $x^*(S_l) = f(S_l)$ for each $l \in [n]$. Since every $x \in \mathbf{P}(f)$ satisfies $x(S_l) \leq f(S_l) = x^*(S_l)$ $(\forall l \in [n])$, the vector x^* is optimal in $\mathbf{P}(f)$ w.r.t. $\succ$. Also, $x^*(S) = f(S)$ implies $x^* \in Q$ by Lemma 5.9. Then, by Observation 6.1, x^* is optimal in Q w.r.t. $\succ$. □

The important fact observed from Proposition 6.2 is that the optimal point of Q depends only on the upper bound function and the total order. Then, Lemma 5.11 implies the following lemma.

Lemma 6.3. Let $Q \subseteq \mathbf{R}_+^S$ be a g-polymatroid in the nonnegative region. For any vector $a \in (\mathbf{R}_+ \cup \{\infty\})^E$ with $Q \cap [-\infty, a] \neq \emptyset$, g-polymatroids $\mathrm{L}(Q) \cap [-\infty, a]$ and $Q \cap [-\infty, a]$ have the same optimal point w.r.t. $\succ$. ■

6.2 Choice Functions Induced from Ordered Polymatroids

A *choice function* (on vectors) is a function $C\colon (\mathbf{R}_+ \cup \{\infty\})^S \to \mathbf{R}_+^S$ such that $C(x) \leq x$ for every $x \in (\mathbf{R}_+ \cup \{\infty\})^S$. In the work of Alkan and Gale [1], the following three conditions of choice functions play a central rôle.

- *Consistency* : $C(x) \leq y \leq x$ implies $C(y) = C(x)$.
- *Persistence* : $x \leq y$ implies $C(y) \wedge x \leq C(x)$.
- *Size-monotonicity* : $x \leq y$ implies $|C(x)| \leq |C(y)|$.

An *ordered polymatroid* is a triple $(S, P, \succ)$ such that $P \subseteq \mathbf{R}_+^S$ is a polymatroid and $\succ$ is a total order on S. Here, we show how choice functions arise from ordered polymatroids.

For an ordered polymatroid $\mathscr{P} = (S, P, \succ)$, define a choice function $C_{\mathscr{P}}\colon (\mathbf{R}_+ \cup \{\infty\})^S \to \mathbf{R}_+^S$ letting $C_{\mathscr{P}}(x)$ be the optimal point of $P \cap [-\infty, x]$ w.r.t. $\succ$ for each $x \in (\mathbf{R}_+ \cup \{\infty\})^S$. Note that, for any x, $P \cap [-\infty, x]$ is a nonempty polymatroid and has the optimal point, and hence $C_{\mathscr{P}}(x)$ is well-defined. We call $C_{\mathscr{P}}$ the *choice function induced from* $\mathscr{P}$.

Observation 6.4. $C_{\mathscr{P}}(x) = x \iff x \in P \quad (\forall x \in \mathbf{R}_+^S)$. ■

Lemma 6.5. Let $f\colon 2^S \to \mathbf{R}$ be the defining function of P. Then, for every $x \in \mathbf{R}_+^S$, there holds $|C_{\mathscr{P}}(x)| = \min\{\, f(A) + x(S \setminus A) \mid A \subseteq S \,\}$.

Proof. By Proposition 5.8, the upper bound function of $P \cap [-\infty, x]$ satisfies $f^x(S) = \min\{\, f(A) + x(S \setminus A) \mid A \subseteq S \,\}$. Also, since $C_{\mathscr{P}}(x)$ is the optimal point of $P \cap [-\infty, x]$, Proposition 6.2 implies $|C_{\mathscr{P}}(x)| = f^x(S)$. □

We now show that choice functions induced from ordered polymatroids are consistent, persistent, and size-monotone.

Lemma 6.6. $C_{\mathscr{P}}$ is consistent, i.e., $C_{\mathscr{P}}(x) \leq y \leq x$ implies $C_{\mathscr{P}}(y) = C_{\mathscr{P}}(x)$.

Proof. Assume $C_{\mathscr{P}}(x) \leq y \leq x$. Then, $P \cap [-\infty, y] \subseteq P \cap [-\infty, x]$ and $C_{\mathscr{P}}(x) \leq y$ implies $C_{\mathscr{P}}(x) \in P \cap [-\infty, y]$. Then by Observation 6.1, $C_{\mathscr{P}}(x)$ is optimal in $P \cap [-\infty, y]$ w.r.t. $\succ$, and hence $C_{\mathscr{P}}(y) = C_{\mathscr{P}}(x)$. □

Lemma 6.7. $C_{\mathscr{P}}$ is persistent, i.e., $x \leq y$ implies $C_{\mathscr{P}}(y) \wedge x \leq C_{\mathscr{P}}(x)$.

Proof. For $x \leq y$, set $x' = C_{\mathscr{P}}(x)$ and $y' = C_{\mathscr{P}}(y)$. Suppose, to the contrary, $y' \wedge x \not\leq x'$. Then, there is $e \in S$ with $y'(e) > x'(e)$ and $x(e) > x'(e)$. Apply the exchange axiom of Theorem 5.7 for $x', y' \in P$ and $e \in S$. Then, there is $\alpha_0 > 0$ such that (i) $x' + \alpha\chi_e \in P$ ($\forall \alpha \in [0, \alpha_0]$), or (ii) there is $e' \in S$ with $x'(e') > y'(e')$ s.t. $x' + \alpha(\chi_e - \chi_{e'}),\ y' - \alpha(\chi_e - \chi_{e'}) \in P$ ($\forall \alpha \in [0, \alpha_0]$).

In Case (i), let $\beta = \min\{\alpha_0, x(e) - x'(e)\} > 0$. Then, $x' + \beta\chi_e$ is in $P \cap [-\infty, x]$ and preferable to x', a contradiction. In Case (ii), let $\beta_1 := \min\{\alpha_0, x(e) - x'(e), x'(e')\}$. As $x(e) > x'(e)$ and $x'(e') > y'(e') \geq 0$,

$$\beta_1 > 0 \quad \text{and} \quad x'' := x' + \beta_1(\chi_e - \chi_{e'}) \in P \cap [-\infty, x].$$

Similarly, let $\beta_2 := \min\{\alpha_0, y'(e), y(e') - y'(e')\}$. As $y'(e) > x'(e) \geq 0$ and $y(e') \geq x(e') \geq x'(e') > y'(e')$, we have

$$\beta_2 > 0 \quad \text{and} \quad y'' := y' - \beta_2(\chi_e - \chi_{e'}) \in P \cap [-\infty, y].$$

If $e \succ e'$, then x'' is preferable to x' which contradicts $x' = C_{\mathscr{P}}(x)$. Otherwise, y'' is preferable to y' which contradicts $y' = C_{\mathscr{P}}(y)$. □

Lemma 6.8. $C_{\mathscr{P}}$ is size-monotone, i.e., $x \leq y$ implies $|C_{\mathscr{P}}(x)| \leq |C_{\mathscr{P}}(y)|$.

Proof. This follows from Lemma 6.5 immediately. □

The following Lemma plays an important rôle in Section 7.

Lemma 6.9. For $x, y \in P$, suppose $|C_{\mathscr{P}}(x \vee y)| = |x| = |y|$. Let $Q \subseteq \mathbf{R}_+^S$ be any g-polymatroid s.t. $\mathrm{L}(Q) = P$. Then, $x \in Q$ if and only if $y \in Q$.

Proof. Let $f \colon 2^S \to \mathbf{R}$ be the defining function of P. Then, by Lemma 6.5, the condition $|x| = |C_{\mathscr{P}}(x \vee y)|$ implies

$$x(S) = \min\{\, f(A) + (x \vee y)(S \setminus A) \mid A \subseteq S \,\}.$$

Let $A^* \subseteq S$ be the minimizer of the right-hand side. Then,

$$x(S) = x(A^*) + x(S \setminus A^*) = f(A^*) + (x \vee y)(S \setminus A^*). \tag{6}$$

Since $x \in P = \mathbf{P}_+(f)$ implies $x(A^*) \leq f(A^*)$ and $x \leq (x \vee y)$ implies $x(S \setminus A^*) \leq (x \vee y)(S \setminus A^*)$, the condition (6) leads to $x(A^*) = f(A^*)$ and $x(S \setminus A^*) = (x \vee y)(S \setminus A^*)$. Similarly we obtain $y(A^*) = f(A^*)$ and $y(S \setminus A^*) = (x \vee y)(S \setminus A^*)$. As $x(S \setminus A^*) = (x \vee y)(S \setminus A^*) = y(S \setminus A^*)$ implies $x(e) = y(e)$ ($\forall e \in S \setminus A^*$), we obtain

$$x(A^*) = y(A^*) = f(A^*) \quad \text{and} \quad x(e) = y(e)\ (\forall e \in S \setminus A^*). \tag{7}$$

For a g-polymatroid Q with $\mathrm{L}(Q) = P$, we show $x \in Q \implies y \in Q$ which completes the proof. By Lemma 5.10, the upper bound function of Q is f. Denote the lower bound function of Q by $g : 2^S \to \mathbf{R}$. Then (f, g) is a strong pair with $Q = \mathbf{P}(f, g)$. Since $x, y \in P = \mathbf{P}_+(f)$, it suffices to show $y(B) \geq g(B)$ $(\forall B \subseteq S)$ assuming $x(B) \geq g(B)$ $(\forall B \subseteq S)$. For any $B \subseteq S$,

$$f(A^*) - g(B) \geq f(A^* \setminus B) - g(B \setminus A^*) \tag{8}$$

by the cross inequality of (f, g). By (7) and $B \setminus A^* \subseteq S \setminus A^*$, we have $x(B \setminus A^*) = y(B \setminus A^*)$, and hence $-g(B \setminus A^*) \geq -y(B \setminus A^*)$ by assumption. Also, we have $f(A^*) = y(A^*)$ by (7) and $f(A^* \setminus B) \geq y(A^* \setminus B)$ by $y \in \mathbf{P}_+(f)$. Substituting these three to (8), we obtain $y(B) \geq g(B)$. □

For an arbitrary choice function $C\colon (\mathbf{R}_+ \cup \{\infty\})^S \to \mathbf{R}_+^S$ and $e \in S$, a vector $x \in \mathbf{R}_+^S$ is said to be *e-satiated* if $(C(y))(e) \leq x(e)$ for all $y \geq x$. Let x^e be the vector s.t. $x^e(e) = \infty$ and $x^e(e') = x(e')$ $(e' \in S \setminus \{e\})$.

Lemma 6.10. For the choice function $C_{\mathscr{P}}$ induced from $\mathscr{P} = (S, P, \succ)$, $x \in P$ is e-satiated if and only if x is optimal in $P \cap [-\infty, x^e]$ w.r.t. $\succ$.

Proof. By the definition of $C_{\mathscr{P}}$, x is optimal in $P \cap [-\infty, x^e]$ w.r.t. $\succ$ if and only if $C_{\mathscr{P}}(x^e) = x$. Then, the "only if" part follows immediately. To show the "if" part, assume $C_{\mathscr{P}}(x^e) = x$ and take any $y \geq x$. Let $z \in \mathbf{R}_+^S$ be the vector s.t. $z(e) = y(e)$ and $z(e') = x(e')$ $(e' \in S \setminus \{e\})$. Since $C_{\mathscr{P}}(x^e) = x \leq z \leq x^e$, the consistency of $C_{\mathscr{P}}$ implies $C_{\mathscr{P}}(z) = x$. Also, the persistence of $C_{\mathscr{P}}$ and $z \leq y$ imply $C_{\mathscr{P}}(y) \wedge z \leq C_{\mathscr{P}}(z) = x$, which yields $(C_{\mathscr{P}}(y))(e) \leq x(e)$ since $(C_{\mathscr{P}}(y))(e) \leq y(e) = z(e)$. □

7 Stable Allocations on Generalized Polymatroids

In this section, we formulate and analyse the stable allocation model on generalized polymatroids.

7.1 Model Formulation

Consider two disjoint agent sets I and J, which are interpreted as workers and firms. Let $E = I \times J$ be the set of all worker-firm pairs, and let $E_i = \{\, (i, j) \mid j \in J \,\}$ for each $i \in I$ and $E_j = \{\, (i, j) \mid i \in I \,\}$ for each $j \in J$.

An *allocation* is a vector $x = (x(i, j) \mid (i, j) \in E) \in \mathbf{R}_+^E$, where $x(i, j)$ means the amount of contracted labor time of i at j. For an allocation x, we write $x_i = x|_{E_i} := (x(i, j) \mid j \in J)$ for each $i \in I$ and $x_j = x|_{E_j}$ for each $j \in J$. The profile of each $k \in I \cup J$ is given by an ordered g-polymatroid $(E_k, Q_k \succ_k)$ with $Q_k \subseteq \mathbf{R}_+^{E_k}$. Here, Q_k means the set of acceptable vectors of k, and his preference on Q_k is defined based on $\succ$ as in Section 6.1. The set $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$ of profiles is called an *instance*.

Definition 7.1. A vector $x \in \mathbf{R}_+^E$ is a *stable allocation of* $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$ (or, *stable w.r.t.* $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$) if the following two conditions hold.

1. For every $k \in I \cup J, \quad x_k \in Q_k$.
2. For every pair $e = (i, j) \in E, \quad x_i$ is optimal in $Q_i \cap [-\infty, x_i^e]$ w.r.t. $\succ_i$ or x_j is optimal in $Q_j \cap [-\infty, x_j^e]$ w.r.t. $\succ_j$,

where $x_i^e(e) = \infty$ and $x_i^e(e') = x_k(e')$ $(e' \in E_i \setminus \{e\})$, and x_j^e is similar. ■

Condition 1 requires x to be feasible for everyone. Note that $Q_i \cap [-\infty, x_i^e]$ in Condition 2 coincides with $\{ y_i \in Q_i \mid y_i(e') \leq x_i(e') \ (e' \in E_i \setminus \{e\}) \}$. Then, the optimality of x_i in $Q_i \cap [-\infty, x_i^e]$ means that i has no incentive to increase $x(e)$. Hence, Condition 2 guarantees that there is no pair s.t. both of them have incentives to increase the amount of the contract between them.

7.2 Choice Function Model

Here, we briefly introduce the results of Alkan and Gale [1], which will be used in the subsequent sections.

In their choice function model, each agent $k \in I \cup J$ has a choice function $C_k \colon (\mathbf{R}_+ \cup \{\infty\})^{E_k} \to \mathbf{R}_+^{E_k}$ instead of a profile, and hence an instance is given in the form $\{C_k\}_{k \in I \cup J}$. A vector $x \in \mathbf{R}_+^E$ is said to be a stable allocation of $\{C_k\}_{k \in I \cup J}$ if x satisfies the following two conditions.

1. For every $k \in I \cup J, \ C_k(x_k) = x_k$.
2. For every $e = (i, j) \in E, \ x_i$ is e-satiated for C_i or x_j is e-satiated for C_j.

For two vectors $x, y \in \mathbf{R}^E$, define vectors $x \vee_I y$ and $x \wedge_I y$ in R^E by

$$\begin{aligned} (x \vee_I y)_i &:= C_i(x_i \vee y_i) \quad (i \in I), \\ (x \wedge_I y)_j &:= C_j(x_j \vee y_j) \quad (j \in J). \end{aligned} \tag{9}$$

That is, $x \vee_I y$ is the direct sum of $\{C_i(x_i \vee y_i)\}_{i \in I}$, and $x \wedge_I y$ is the direct sum of $\{C_j(x_j \vee y_j)\}_{j \in J}$.

Theorem 7.2 (Alkan and Gale [1])**.** Let $\mathfrak{L}$ be the set of stable allocations of $\{C_k\}_{k \in I \cup J}$. If C_k is consistent, persistent, and size-monotone for every $k \in I \cup J$, then $\mathfrak{L}$ satisfies the following properties:

(a) $\mathfrak{L} \neq \emptyset$, i.e., there exists a stable allocation.

(b) The triple $(\mathfrak{L}, \vee_I, \wedge_I)$ is a distributive lattice.

(c) For every $x, y \in \mathfrak{L}$ and every $k \in I \cup J$, there holds $|x_k| = |y_k|$. ■

7.3 Characterization through Lower Extension

In this section, we connect our ordered g-polymatroid model to the choice function model of Alkan and Gale [1] by using the lower extension of g-matroids and the induction of choice functions.

Let $\{(E_k, Q_k, \succ_k)\}_{k\in I\cup J}$ be an instance of our model, i.e., $(E_k, Q_k, \succ_k)$ is an ordered g-polymatroid with $Q_k \subseteq \mathbf{R}_+^{E_k}$ for each $k \in I \cup J$.

Lemma 7.3. For $x \in \mathbf{R}_+^E$, the following two conditions are equivalent:

1. x is a stable allocation of $\{(E_k, Q_k, \succ_k)\}_{k\in I\cup J}$.
2. x is a stable allocation of $\{(E_k, \mathrm{L}(Q_k), \succ_k)\}_{k\in I\cup J}$ and satisfies $x_k \in Q_k$ for every $k \in I \cup J$.

Proof. For each $k \in I \cup J$, both conditions require $x_k \in Q_k$, which implies $Q_k \cap [-\infty, x_k^e] \neq \emptyset$. Then, by Lemma 6.3, the optimal points of $Q_k \cap [-\infty, x_k^e]$ and $\mathrm{L}(Q_k) \cap [-\infty, x_k^e]$ coincide with each other, and the lemma follows. □

For each $k \in I \cup J$, the triple $\mathscr{P}_k := (E_k, \mathrm{L}(Q_k), \succ_k)$ is an ordered polymatroid by Lemma 5.10. Hence, let $C_{\mathscr{P}_k} \colon (\mathbf{R}_+ \cup \{\infty\})^{E_k} \to \mathbf{R}_+^{E_k}$ be the choice function induced from $\mathscr{P}_k$ for each $k \in I \cup J$. Then, by Observation 6.4 and Lemma 6.10, we obtain the following lemma.

Lemma 7.4. $x \in \mathbf{R}_+^E$ is a stable allocation of $\{\mathscr{P}_k = (E_k, \mathrm{L}(Q_k), \succ_k)\}_{k\in I\cup J}$ if and only if x is a stable allocation of $\{C_{\mathscr{P}_k}\}_{k\in I\cup J}$. ■

Combining Lemmas 7.3 and 7.3 gives the following theorem.

Theorem 7.5. A vector $x \in \mathbf{R}_+^E$ is a stable allocation of $\{(E_k, Q_k, \succ_k)\}_{k\in I\cup J}$ if and only if x is a stable allocation of $\{C_{\mathscr{P}_k}\}_{k\in I\cup J}$ satisfying $x_k \in Q_k$ for every $k \in I \cup J$. ■

7.4 Structure of Stable Allocations

Let $\mathfrak{S}$ be the set of stable allocations of $\{(E_k, Q_k, \succ_k)\}_{k\in I\cup J}$ and let $\mathfrak{L}$ be that of $\{C_{\mathscr{P}_k}\}_{k\in I\cup J}$, where $\mathscr{P}_k = (E_k, \mathrm{L}(Q_k), \succ_k)$. Then, Theorem 7.5 is rephrased as

$$\mathfrak{S} = \{\, x \in \mathfrak{L} \mid x_k \in Q_k \ (\forall k \in I \cup J) \,\}. \tag{10}$$

Lemma 7.6. The set $\mathfrak{L}$ is nonempty, and $(\mathfrak{L}, \vee_I, \wedge_I)$ is a distributive lattice, where $\vee_I$ and $\wedge_I$ are defined by (9) with C_i and C_j replaced by $C_{\mathscr{P}_i}$ and $C_{\mathscr{P}_j}$, respectively. Also, for any $x, y \in \mathfrak{L}$ and $k \in I \cup J$, there holds $|x_k| = |y_k|$.

Proof. By Lemmas 6.6, 6.7, and 6.8, each choice function $C_{\mathscr{P}_k}$ is consistent, persistent, and size-monotone. Then, Theorem 7.2 implies the lemma. □

Lemma 7.7. For any $x, y \in \mathfrak{L}$ and $k \in I \cup J$, $x_k \in Q_k$ if and only if $y_k \in Q_k$.

Proof. Take any $x, y \in \mathfrak{L}$ and $k \in I \cup J$. By Lemma 7.6, we have $x \vee_I y \in \mathfrak{L}$ and hence $|(x \vee_I y)_k| = |x_k| = |y_k|$. By the definition of $\vee_I$, this implies $|C_{\mathscr{P}_k}(x_k \vee y_k)| = |x_k| = |y_k|$. Then, the claim follows from Lemma 6.9. □

Combining (10) and Lemma 7.7 yields the following dichotomy theorem.

Theorem 7.8. There holds $\mathfrak{S} = \mathfrak{L}$ or $\mathfrak{S} = \emptyset$. ■

Corollary 7.9. If the set $\mathfrak{S}$ of stable allocations of $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$ is nonempty, then $(\mathfrak{S}, \vee_I, \wedge_I)$ is a distributive lattice, where $\vee_I$ and $\wedge_I$ are defined by (9) with C_i and C_j replaced by $C_{\mathscr{P}_i}$ and $C_{\mathscr{P}_j}$, respectively. Also, for any $x, y \in \mathfrak{L}$ and $k \in I \cup J$, there holds $|x_k| = |y_k|$. ■

Corollary 7.10. If the feasible region Q_k is a polymatroid (i.e., $\mathbf{0} \in Q_k$) for each agent $k \in I \cup J$, then $\mathfrak{S} = \mathfrak{L} \neq \emptyset$, and hence stable allocations form a nonempty distributive lattice. ■

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A Basics on Generalized Matroids

A.1 Operations on Generalized Matroids

G-matroids are closed under the following basic operations.

Proposition A.1 (**Restriction** [22, Lemma 2.7])**.** For a g-matroid $(S, \mathcal{J})$ and any subset $T \subseteq S$, define

$$\mathcal{J}_T = \{\, X \cap T \mid X \in \mathcal{J} \,\} .$$

Then, $(T, \mathcal{J}_T)$ is a g-matroid if $\mathcal{J}_T \neq \emptyset$. ■

Proposition A.2 (**Truncation** [22, Corollary 2.7])**.** For a g-matroid $(S, \mathcal{J})$ and any $k, l \in \mathbf{Z}_+$ with $k \leq l$, define

$$\mathcal{J}_k^l = \{\ X\ :\ X \in \mathcal{J},\ k \leq |X| \leq l\ \}.$$

Then, $(S, \mathcal{J}_k^l)$ is a g-matroid if $\mathcal{J}_k^l \neq \emptyset$. ■

Proposition A.3 (**Direct Sum**)**.** For g-matroids $(S_1, \mathcal{J}_1), \dots, (S_k, \mathcal{J}_k)$ on disjoint ground sets $S_1, \dots, S_k$, define

$$\begin{aligned} &S = S_1 \cup \cdots \cup S_k, \\ &\mathcal{J} = \{\, X \subseteq S \mid X \cap S_i \in \mathcal{J}_i \ (\forall i \in [k]) \,\} . \end{aligned}$$

Then, $(S, \mathcal{J})$ is a g-matroid. ■

Next, we show that a g-matroid induces another g-matroid via matchings on a bipartite graph. We use the following lemmas.

Lemma A.4 ([22, Theorem 2.9])**.** A pair $(S, \mathcal{J})$ is a g-matroid if and only if $\mathcal{J}$ is written in the form

$$\mathcal{J} = \{\, B \cap S \mid B \in \mathcal{B} \}$$

for the base family $\mathcal{B}$ of some matroid whose ground set is a superset of S. ■

Lemma A.5 ([19])**.** Let $G = (S, T; E)$ be a bipartite graph with vertex classes S and T, and edge set E. Let $(T, \mathcal{I}_T)$ be a matroid on T and define a family $\mathcal{I}_S$ of subsets of S by

$$\mathcal{I}_S = \{\, \partial M \cap S \mid M : \text{matching in } G,\ \partial M \cap T \in \mathcal{I}_T \,\} ,$$

where ∂M is the set of end vertices of M. Then, $(S, \mathcal{I}_S)$ is a matroid. ■

In the above lemma, if a matching $M \subseteq E$ satisfies $\partial M \cap T \in \mathcal{B}_T$, then $\partial M \cap S$ is a maximal member of $\mathcal{I}_S$. This implies the following fact.

Lemma A.6. Under the conditions in Lemma A.5, denote by $\mathcal{B}_T$ the base family of $(T, \mathcal{I}_T)$. Then, the family defined by

$$\mathcal{B}_S = \{\, \partial M \cap S \mid M : \text{matching in } G,\ \partial M \cap T \in \mathcal{B}_T \,\},$$

is a base family of a matroid $(S, \mathcal{I}_S)$ if $\mathcal{B}_S \neq \emptyset$. ■

Proposition A.7 (Induction by Bipartite Graphs). Let $G = (S, T; E)$ be a bipartite graph with vertex classes S and T, and an edge set E. Let $(T, \mathcal{J}_T)$ be a g-matroid and define a family $\mathcal{J}_S$ of subsets of S by

$$\mathcal{J}_S = \{\, \partial M \cap S \mid M : \text{matching in } G,\ \partial M \cap T \in \mathcal{J}_T \,\}.$$

Then, $(S, \mathcal{J}_S)$ is a g-matroid if $\mathcal{J}_S \neq \emptyset$.

Proof. By Lemma A.4, there is a matroid $(\hat{T}, \mathcal{I}_{\hat{T}})$ with $\hat{T} = T \cup U$ $(T \cap U = \emptyset)$ whose base family $\mathcal{B}_{\hat{T}}$ satisfies $\mathcal{J}_T = \{\, B \cap T \mid B \in \mathcal{B}_{\hat{T}} \,\}$. Let U' be a copy of U and let $\hat{G}$ be a bipartite graph with vertex classes $\hat{S} = S \cup U'$ and $\hat{T} = T \cup U$, and edge set $\hat{E} = E \cup \{\, (u', u) \in U' \times U \mid u \in U \,\}$.

For any matching M in G with $\partial M \cap T \in \mathcal{J}_T$, there is a base $B \in \mathcal{B}_T$ with $B \cap T = \partial M \cap T$. Then $\hat{M} := M \cup \{\, (u', u) \mid u \in B \cap U \,\}$ is a matching in $\hat{G}$ with $\partial \hat{M} \cap \hat{T} \in \mathcal{B}_{\hat{T}}$ and $(\partial \hat{M} \cap \hat{S}) \cap S = \partial M \cap S$. On the other hand, for any matching $\hat{M}$ in $\hat{G}$ with $\partial \hat{M} \cap \hat{T} \in \mathcal{B}_T$, $M := \hat{M} \cap E$ satisfies $\partial M \cap T \in \mathcal{J}_T$ and $\partial M \cap S = B \cap S$. Therefore, $\mathcal{J}_S = \{\, B \cap S \mid B \in \mathcal{B}_{\hat{S}} \,\}$ where

$$\mathcal{B}_{\hat{S}} = \{\, \partial \hat{M} \cap \hat{S} \mid \hat{M} : \text{matching in } \hat{G},\ \partial \hat{M} \cap \hat{T} \in \mathcal{B}_{\hat{T}} \,\}.$$

Then, $\mathcal{J}_S \neq \emptyset$ implies $\mathcal{B}_{\hat{S}} \neq \emptyset$. By Lemma A.6, $\mathcal{B}_{\hat{S}}$ is the base family of a matroid on $\hat{S}$, and hence $(S, \mathcal{J}_S)$ is a g-matroid by Lemma A.4. □

A.2 Examples of Generalized Matroids

Here we construct two examples of g-matroids. Propositions A.8 and A.9 respectively show that Examples 2.2 and 2.3 indeed give g-matroids.

Proposition A.8. For a finite set S, let $\mathcal{F} \subseteq 2^S$ be a laminar family and $f, g \colon \mathcal{F} \to \mathbf{Z}_+$ be set functions. Define a family $\mathcal{J}(f, g)$ of subsets of S by

$$\mathcal{J}(f, g) = \{\ X \subseteq S :\ g(A) \leq |X \cap A| \leq f(A)\ (\forall A \in \mathcal{F})\ \}.$$

Then, $(S, \mathcal{J}(f, g))$ is a g-matroid if $\mathcal{J}(f, g) \neq \emptyset$.

Proof. For any member $A \in \mathcal{F}$, define $\mathcal{F}_A := \{\, A' \in \mathcal{F} \mid A' \subseteq A \,\}$ and

$$\mathcal{J}_A(f, g) := \{\ X \subseteq A :\ g(A') \leq |X \cap A'| \leq f(A')\ (\forall A' \in \mathcal{F}_A)\ \}.$$

Let us call a subset $B \in \mathcal{F}$ a *child* of $A \in \mathcal{F}$ if $B \subsetneq A$ and there is no $B' \in \mathcal{F}$ such that $B \subsetneq B' \subsetneq A$. We now show the following two claims, which leads to the proposition by induction.

(i) If $A \in \mathcal{F}$ has no child, then $(A, \mathcal{J}_A(f,g))$ is a g-matroid.

(ii) Assume that $A \in \mathcal{F}$ has just k children $B_1, B_2, \ldots, B_k \in \mathcal{F}_A$ and $(B_i, \mathcal{J}_{B_i}(f,g))$ is a g-matroid for each $i \in [k]$. Then, $(A, \mathcal{J}_A(f,g))$ is a g-matroid if $\mathcal{J}_A(f,g) \neq \emptyset$.

Note that $(A, 2^A)$ is a g-matroid for any $A \subseteq E$. Then, (i) is shown by Proposition A.2. We now show (ii). Let $B_{k+1} := A \setminus (B_1 \cup B_2 \cup \cdots \cup B_k)$. Define a family $\mathcal{J}$ of subsets of A by

$$\mathcal{J} = \{\, X \subseteq A \mid X \cap B_i \in \mathcal{J}_{B_i}(f,g) \quad (\forall i \in [k]) \,\}.$$

Then, $(A, \mathcal{J})$ is the direct sum of $\{(B_i, \mathcal{J}_{B_i}(f,g))\}_{i \in [k]}$ and $(B_{k+1}, 2^{B_{k+1}})$, and hence a g-matroid by Proposition A.3. Also, we see that

$$\mathcal{J}_A(f,g) = \{\ X \in \mathcal{J} :\ g(A) \leq |X| \leq f(A)\ \}.$$

Then, by Proposition A.2, $(A, \mathcal{J}_A(f,g))$ is a g-matroid if $\mathcal{J}_A(f,g) \neq \emptyset$. $\square$

Proposition A.9. Let S and D be finite sets. For each $d \in D$, let $\Gamma(d) \subseteq S$ and $u(d), l(d) \in \mathbf{Z}_+$ such that $l(d) \leq u(d)$. Define a family $\mathcal{J}(\Gamma, u, l)$ by

$$\mathcal{J}(\Gamma, u, l) = \left\{ X \subseteq S \;\middle|\; \begin{array}{l} \exists \pi : X \to D \ \text{ s.t. every } d \in D \text{ satisfies} \\ \pi^{-1}(d) \subseteq \Gamma(d),\ l(d) \leq |\pi^{-1}(d)| \leq u(d) \end{array} \right\}.$$

Then, $(S, \mathcal{J}(\Gamma, u, l))$ is a g-matroid if $\mathcal{J}(\Gamma, u, l) \neq \emptyset$.

Proof. Let $T_d := \{\, (d,s) \mid s \in S \,\}$ for each $d \in D$, and let T be the direct sum of $\{T_d\}_{d \in D}$. Define a family $\mathcal{J}_T$ of subsets of T by

$$\mathcal{J}_T = \{\ X \subseteq T\ :\ l(d) \leq |X \cap T_d| \leq u(d)\ (\forall d \in D)\ \}.$$

Then, $(T, \mathcal{J}_T)$ is a g-matroid by Proposition A.8. Let $G = (S, T; E)$ be a bipartite graph with edge set $E = \{\, (s, (d,s)) \in S \times T \mid s \in \Gamma(d) \,\}$ and define a family $\mathcal{J}_S$ of subsets of S by

$$\mathcal{J}_S = \{\, \partial M \cap S \mid M : \text{matching in } G,\ \partial M \cap T \in \mathcal{J}_T \,\}.$$

Then, by Proposition A.9, $(S, \mathcal{J}_S)$ is a g-matroid if $\mathcal{J}_S \neq \emptyset$. Also, we see that $\mathcal{J}_S$ coincides with $\mathcal{J}(\Gamma, u, l)$ as follows.

For $X \in \mathcal{J}(\Gamma, u, l)$, there is $\pi : X \to D$ with (i) $\pi^{-1}(d) \subseteq \Gamma(d)$, and (ii) $l(d) \leq |\pi^{-1}(d)| \leq u(d)$ for all $d \in D$. Let $M_\pi := \{\, (s, (d,s)) \mid s \in X,\ \pi(s) = d \,\}$. Then, (i) implies $M_\pi \subseteq E$ and and (ii) implies $l(d) \leq |\partial M_\pi \cap T_d| \leq u(d)$ for each $d \in D$, and hence $\partial M_\pi \cap T \in \mathcal{J}_T$. Thus, $X = \partial M_\pi \cap S \in \mathcal{J}_S$.

For $X \in \mathcal{J}_S$, there is a matching $M \subseteq E$ with $\partial M \cap S = X$ and $\partial M \cap T \in \mathcal{J}_T$. Define $\pi_M : X \to D$ by letting $\pi_M(s)$ be the unique $d \in D$ satisfying $(s, (d,s)) \in M$ for each $s \in X$. Then, conditions (i) and (ii) follow from $M \subseteq E$ and $\partial M \cap T \in \mathcal{J}_T$, respectively. Thus, $X \in \mathcal{J}(\Gamma, u, l)$. $\square$