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Abstract

Dynamical systems, such as electric circuits, mechanical systems, and chemical plants, can be modeled by mixed matrix pencils, i.e., matrix pencils having two kinds of nonzero coefficients: fixed constants that account for conservation laws and independent parameters that represent physical characteristics. Based on dimension analysis of dynamical systems, Murota (1985) introduced a physically meaningful subclass of mixed polynomial matrices. For this class of mixed matrix pencils, we provide a combinatorial characterization of the sums of the minimal row/column indices of the Kronecker canonical form. The characterization leads to an efficient algorithm for computing them. This is an extension of the result by Iwata and Shimizu (2007) on matrix pencils whose nonzero entries are all independent parameters.

1 Introduction

A *matrix pencil* is a polynomial matrix in which the degree of each entry is at most one. Each matrix pencil is known to be strictly equivalent to its *Kronecker canonical form*, which is in a block-diagonal form that consists of nilpotent blocks, rectangular blocks, and the residual square block, where rectangular blocks appear only in the singular case.

The Kronecker canonical form of matrix pencils plays an important role in many fields such as systems control [2, 27] and differential-algebraic equations [5, 12, 24]. The problem of computing the Kronecker canonical form has been studied especially for singular matrix pencils, because the singularity makes this problem much more complicated than the regular case. Several algorithms are designed for numerically stable computation of the Kronecker canonical form [1, 3, 4, 10, 28].

An alternative method for the Kronecker canonical form is based on the structural approach, which extracts zero/nonzero pattern of each coefficient in the matrix pencil, ignoring the numerical values. The structural approach has been adopted in control theory [13] and in

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theory of differential-algebraic equations [23]. Such a simplification enables us to compute the Kronecker canonical form of regular matrix pencils efficiently by exploiting graph-algorithmic techniques under the genericity assumption that all the nonzero coefficients are independent parameters, which do not cause any numerical cancellation. A recent work [7] has extended the structural approach to deal with singular matrix pencils. Under the genericity assumption, it provides a combinatorial characterization of the sizes of the nilpotent blocks as well as the sum of the sizes of the rectangular blocks.

The size of each rectangular block in the Kronecker canonical form is called the *minimal row/column indices*. They are also referred to as the left/right Kronecker indices in control theory [11, 29]. For a linear time-invariant dynamical system

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),$$

the minimal column indices of the matrix pencil $D(s) = \begin{pmatrix} sI - A \mid B \end{pmatrix}$ provides the so-called controllability indices [6, 11, 25, 30, 31], and the sum of the minimal column indices corresponds to the dimension of the controllable subspace.

An advantage of the structural approach is that it is supported by efficient combinatorial algorithms that are free from errors in numerical computation. On the other hand, however, the genericity assumption is often invalid when we set up a faithful model of a physical system. This is partly because structural equations such as the conservation laws can be described with specific numbers. This natural observation led Murota and Iri [20] to introduce the notion of a mixed matrix, which is a constant matrix that consists of two kind of numbers as follows.

Accurate Numbers (Fixed Constants) Numbers that account for conservation laws are precise in values. These numbers should be treated numerically.

Inaccurate Numbers (Independent Parameters) Numbers that represent physical characteristics are not precise in values. These numbers should be treated combinatorially as nonzero parameters without reference to their nominal values. Since each such nonzero entry often comes from a single physical device, the parameters are assumed to be independent.

A matrix consisting only of independent parameters is called a *generic matrix*, which is a special type of a mixed matrix.

The polynomial matrix version of a mixed matrix is called a *mixed polynomial matrix*. To be more specific, a mixed polynomial matrix is a polynomial matrix with each coefficient matrix being a mixed matrix. In other words, a mixed polynomial matrix is a polynomial matrix $D(s) = Q(s) + T(s)$ such that the nonzero entries in the coefficient matrices of $Q(s)$ are fixed constants and those of $T(s)$ are independent parameters.

The concept of mixed polynomial matrices may be too broad as a mathematical tool for describing dynamical systems in practice. Taking the consistency of physical dimensions in structural equations into account, Murota [14] introduced a class of mixed polynomial matrices that satisfy the following condition.

(DC) Every nonvanishing subdeterminant of $Q(s)$ is a monomial in s.

This subclass of mixed polynomial matrices has played an important role in matroid-theoretic structural approach to dynamical systems [16, 17, 22].

The results in [7] on the nilpotent blocks have been successfully extended to the framework of mixed matrix pencils, i.e., mixed polynomial matrices with degree at most one, without imposing the assumption (DC) on dimensional consistency [8]. Extending the remaining results on rectangular blocks has remained to be done. In this paper, we extend the characterization on the sum of the minimal row/column indices to the framework of mixed matrix pencils satisfying (DC). This characterization leads to an efficient matroid-theoretic algorithm for computing them.

In the derivation of our result, we have two difficulties to be overcome. In mixed matrix theory, a problem for a mixed matrix pencil is generally reduced to that for a certain *layered mixed matrix pencil*, but this straightforward approach does not work well for the minimal column indices as discussed in [8, Section 8]. This is the first difficulty, which is resolved by Theorem 6.2. The second one occurs in using the *Combinatorial Canonical Form (CCF)* decomposition [21], which is a generalized version of the Dulmage-Mendelsohn decomposition utilized in [7]. When we transform a mixed matrix pencil $D(s)$ into the CCF, the resulting matrix is not necessarily a matrix pencil. We resolve this problem by showing in Section 8 that a part of the CCF, called the *horizontal tail*, remains to be a matrix pencil and has the same minimal column indices as $D(s)$.

The rest of this paper is organized as follows. In Section 2, we recapitulate the Kronecker canonical form and its relation to the ranks of expanded matrices. Section 3 gives characterizations of square blocks of the Kronecker canonical form, and Section 4 discusses which blocks are invariant under equivalence transformations with unimodular matrices. Sections 5 and 6 are devoted to mixed polynomial matrices and mixed matrix pencils. After explaining the CCF in mixed matrix theory in Section 7, we give a combinatorial characterization of the sums of the minimal row/column indices in Section 8. Section 9 describes an application of our result to controllability analysis of dynamical systems. Finally, Section 10 concludes this paper.

2 The Kronecker Canonical Form of Matrix Pencils

Let $D(s) = sX + Y$ be an $m \times n$ matrix pencil with row set R and column set C. We denote by $D[I, J]$ the submatrix of $D(s)$ determined by $I \subseteq R$ and $J \subseteq C$. A matrix pencil $D(s)$ is said to be *regular* if $D(s)$ is square and $\det D(s) \neq 0$ as a polynomial in s. It is *strictly regular* if both X and Y are nonsingular. The rank of $D(s)$ is the maximum size of its submatrix that is a regular matrix pencil. A matrix pencil $\bar{D}(s)$ is said to be *strictly equivalent* to $D(s)$ if there exists a pair of nonsingular constant matrices U and V such that $\bar{D}(s) = UD(s)V$.

For a positive integer μ, we consider $\mu \times \mu$ matrix pencils K_μ and N_μ defined by

$$K_\mu = \begin{pmatrix} s & 1 & 0 & \cdots & 0 \\ 0 & s & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & s & 1 \\ 0 & \cdots & \cdots & 0 & s \end{pmatrix}, \qquad N_\mu = \begin{pmatrix} 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & s \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

For a positive integer ϵ, we denote by L_ϵ an $\epsilon \times (\epsilon+1)$ matrix pencil

$$L_\epsilon = \begin{pmatrix} s & 1 & 0 & \cdots & 0 \\ 0 & s & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s & 1 \end{pmatrix}.$$

We also denote by $L_\eta^\top$ the transpose matrix of L_η.

Let us denote by block-diag$(D_1, \ldots, D_b)$ the block-diagonal matrix with diagonal blocks $D_1, \ldots, D_b$. A matrix pencil is known to be strictly equivalent to a block-diagonal form called the Kronecker canonical form as follows.

Theorem 2.1. By a strict equivalence transformation, a matrix pencil $D(s)$ can be brought into a block-diagonal form $\bar{D}(s)$ with

$$\bar{D}(s) = \text{block-diag}(H_\nu, K_{\rho_1}, \ldots, K_{\rho_c}, N_{\mu_1}, \ldots, N_{\mu_d}, L_{\epsilon_1}, \ldots, L_{\epsilon_p}, L_{\eta_1}^\top, \ldots, L_{\eta_q}^\top),$$

where

$$\rho_1 \geq \cdots \geq \rho_c \geq 1, \quad \mu_1 \geq \cdots \geq \mu_d \geq 1, \quad \epsilon_1 \geq \cdots \geq \epsilon_p \geq 0, \quad \eta_1 \geq \cdots \geq \eta_q \geq 0,$$

and H_ν is a strictly regular matrix pencil of size ν. The numbers ν, c, d, p, q, $\rho_1, \ldots, \rho_c$, $\mu_1, \ldots, \mu_d$, $\epsilon_1, \ldots, \epsilon_p$, $\eta_1, \ldots, \eta_q$ are uniquely determined.

The matrix $\bar{D}(s)$ is called the *Kronecker canonical form* of $D(s)$. The matrices $N_{\mu_1}, \ldots, N_{\mu_d}$ are called the *nilpotent blocks*, and the numbers $\mu_1, \ldots, \mu_d$ are called the *indices of nilpotency*. The numbers $\epsilon_1, \ldots, \epsilon_p$ and $\eta_1, \ldots, \eta_q$ are the *minimal column indices* and *minimal row indices*, respectively. In addition, we call

$$(\nu, \rho_1, \ldots, \rho_c, \mu_1, \ldots, \mu_d, \epsilon_1, \ldots, \epsilon_p, \eta_1, \ldots, \eta_q) \tag{1}$$

the *structural indices* of $D(s)$.

For the rank r of $D(s)$, it holds that

$$r = \nu + \sum_{i=1}^{c} \rho_i + \sum_{i=1}^{d} \mu_i + \sum_{i=1}^{p} \epsilon_i + \sum_{i=1}^{q} \eta_i. \tag{2}$$

Moreover, we have

$$p = n - r, \quad q = m - r. \tag{3}$$

We denote the degree of a polynomial $f(s)$ by $\deg f(s)$, where $\deg 0 = -\infty$ by convention. For a rational function $f(s) = g(s)/h(s)$ with polynomials $g(s)$ and $h(s)$, its degree is defined by $\deg f(s) = \deg g(s) - \deg h(s)$. Let $B(s)$ be a rational function matrix with row set R and column set C. For $k = 1, \ldots, \operatorname{rank} B$, we denote

$$\delta_k(B) = \max\{\deg \det B(s)[I, J] \mid |I| = |J| = k, I \subseteq R, J \subseteq C\},$$

where $\delta_0(B) = 0$ by convention.

A rational function $f(s)$ is called a *Laurent polynomial* if $s^N f(s)$ is a polynomial for some integer N. For a Laurent polynomial $f(s)$, we define

$$\operatorname{ord} f = -\min\{N \in \mathbb{Z} \mid s^N f(s) \text{ is a polynomial}\}.$$

Let $B(s)$ be a Laurent polynomial matrix. For $k = 1, \ldots, \operatorname{rank} B$, we denote

$$\zeta_k(B) = \min\{\operatorname{ord} \det B(s)[I, J] \mid |I| = |J| = k, I \subseteq R, J \subseteq C\},$$

where $\zeta_0(B) = 0$ by convention. Note that $\operatorname{ord} f(s) = -\deg f(1/s)$ holds for any Laurent polynomial $f(s)$, and thus we have

$$\zeta_k(B(s)) = -\delta_k(B(1/s)) \tag{4}$$

for any Laurent polynomial matrix $B(s)$.

For the indices of nilpotency of the Kronecker canonical form, it is known that

$$d = r - \max_{k \geq 0} \delta_k(D), \quad \mu_i = \delta_{r-i}(D) - \delta_{r-i+1}(D) + 1 \quad (i = 1, \ldots, d) \tag{5}$$

hold [19, Theorem 5.1.8]. We also have the following lemma.

Lemma 2.2 ([7])**.** Let $D(s)$ be a matrix pencil of rank r with the structural indices (1). Then we have

$$\nu + \sum_{i=1}^{p} \epsilon_i + \sum_{i=1}^{q} \eta_i = \delta_r(D) - \zeta_r(D).$$

For an $m \times n$ matrix pencil $D(s) = sX + Y$, we consider $km \times kn$ matrices $\Theta_k(D)$ and $\Omega_k(D)$ defined by

$$\Theta_k(D) = \begin{pmatrix} X & O & \cdots & \cdots & O \\ Y & X & \ddots & & \vdots \\ O & Y & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & X & O \\ O & \cdots & O & Y & X \end{pmatrix}, \quad \Omega_k(D) = \begin{pmatrix} Y & O & \cdots & \cdots & O \\ X & Y & \ddots & & \vdots \\ O & X & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & Y & O \\ O & \cdots & O & X & Y \end{pmatrix}.$$

We also construct a $(k+1)m \times kn$ matrix $\Psi_k(D)$ and a $km \times (k+1)n$ matrix $\Phi_k(D)$ defined by

$$\Psi_k(D) = \begin{pmatrix} X & O & \cdots & O \\ Y & X & \ddots & \vdots \\ O & Y & \ddots & O \\ \vdots & \ddots & \ddots & X \\ O & \cdots & O & Y \end{pmatrix}, \quad \Phi_k(D) = \begin{pmatrix} X & Y & O & \cdots & O \\ O & X & Y & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O \\ O & \cdots & O & X & Y \end{pmatrix}.$$

The ranks of these expanded matrices are denoted by

$$\theta_k(D) = \operatorname{rank} \Theta_k(D), \quad \omega_k(D) = \operatorname{rank} \Omega_k(D),$$
$$\psi_k(D) = \operatorname{rank} \Psi_k(D), \quad \varphi_k(D) = \operatorname{rank} \Phi_k(D).$$

The following theorem shows a close relationship between the ranks of the expanded matrices and the structural indices.

Theorem 2.3 ([7, Theorem 2.3])**.** Let $D(s)$ be a matrix pencil of rank r with the structural indices (1). Then we have

$$\theta_k(D) = rk - \sum_{i=1}^{d} \min\{k, \mu_i\}, \quad \omega_k(D) = rk - \sum_{i=1}^{c} \min\{k, \rho_i\},$$
$$\psi_k(D) = rk + \sum_{i=1}^{p} \min\{k, \epsilon_i\}, \quad \varphi_k(D) = rk + \sum_{i=1}^{q} \min\{k, \eta_i\}.$$

By Theorem 2.3, the ranks of the expanded matrices determine μ_i $(i = 1, \dots, d)$, ρ_i $(i = 1, \dots, c)$, ϵ_i $(i = 1, \dots, p)$, and η_i $(i = 1, \dots, q)$.

We generalize the definitions of $\Psi_k(D)$ and $\psi_k(D)$ for a matrix pencil $D(s)$ to those for a polynomial matrix as follows. Let $A(s) = \sum_{i=0}^{N} s^i A_i$ be an $m \times n$ polynomial matrix such that the maximum degree of entries is N. Given $A(s)$ and an integer l, we define a $(k+l)m \times kn$ matrix $\Psi_k^l(A)$ by

$$\Psi_k^l(A) = \begin{array}{c} \\ R_0 \\ R_1 \\ \vdots \\ R_{l-1} \\ R_l \\ R_{l+1} \\ \vdots \\ R_{k+l-1} \end{array} \begin{array}{c} \begin{array}{cccc} C_0 & C_1 & \cdots & C_{k-1} \end{array} \\ \begin{pmatrix} A_0 & O & \cdots & O \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & A_1 & \ddots & O \\ A_{l-1} & \vdots & \ddots & A_0 \\ A_l & A_{l-1} & \ddots & A_1 \\ O & A_l & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{l-1} \\ O & \cdots & O & A_l \end{pmatrix} \end{array}$$

with row set $\tilde{R} = R_0 \cup R_1 \cup \cdots \cup R_{k+l-1}$ and column set $\tilde{C} = C_0 \cup C_1 \cup \cdots \cup C_{k-1}$. We can check that $\Psi_k^1(A)$ coincides with $\Psi_k(A_0 + sA_1)$. The ranks of $\Psi_k^l(A)$ for $l \geq N$ attain the same value, which we denote by $\psi_k(A)$.

We conclude this section with the following lemma, which is a generalization of Corollary 2.4 in [7].

Lemma 2.4. If an $m \times n$ polynomial matrix $A(s)$ is of full-column rank, we have $\psi_k(A) = kn$ for each k.

Proof. Let N denote the maximum degree of entries in $A(s)$. We assume that $\psi_k(A) \neq kn$, which implies that $\Psi_k^N(A)$ is not of full-column rank. Let $\boldsymbol{h}_j^l$ denote the lth column vector of $\Psi_k^N(A)[\tilde{R}, C_j]$. Then we have

$$\sum_{j=0}^{k-1} \sum_{l \in C_j} \lambda_j^l \boldsymbol{h}_j^l = \boldsymbol{0} \tag{6}$$

for some λ_j^l such that scalars λ_j^l are not all zero.

Let C denote the column set of $A(s)$. By the definition of $\Psi_k^N(A)$, a part of vector $\boldsymbol{h}_j^l$ indexed by R_i is equal to the lth vector of A_{i-j}, denoted by A_{i-j}^l, where we set $A_{i-j} = O$ if

$i-j<0$ or $i-j>N$. Hence it follows from (6) that

$$\sum_{j=0}^{k-1}\sum_{l\in C}\lambda_j^l A_{i-j}^l=\mathbf{0}\quad (i=0,1,\ldots,k+N-1). \tag{7}$$

We denote the lth vector of $A(s)$ by $\boldsymbol{a}_l(s)$. Consider a linear combination

$$\sum_{l\in C}\left(\sum_{j=0}^{k-1}\lambda_j^l s^j\right)\boldsymbol{a}_l(s) \tag{8}$$

of vectors in $A(s)$, where each coefficient $\sum_{j=0}^{k-1}\lambda_j^l s^j$ is a polynomial in s. The coefficient of s^i in (8) is expressed as

$$\sum_{l\in C}\sum_{j=0}^{k-1}\lambda_j^l A_{i-j}^l,$$

which is equal to $\mathbf{0}$ by (7). Hence the value of (8) is also equal to $\mathbf{0}$. This implies that $A(s)$ is not of full-column rank. □

3 Characterization of Square Blocks in the Kronecker Canonical Form

In Section 2, we have explained that the nilpotent blocks are characterized by (5). In this section, we characterize the other square blocks $K_{\rho_1},\ldots,K_{\rho_c}$ and H_ν.

A rational function matrix $B(s)=(B_{ij}(s))$ is called *proper* if $\deg B_{ij}(s)\le 0$ for all (i,j). A square proper rational function matrix is called *biproper* if it is invertible and its inverse is also proper.

A polynomial matrix is called *unimodular* if it is square and its determinant is a nonvanishing constant. This implies that a square polynomial matrix is unimodular if and only if its inverse is a polynomial matrix. If a polynomial matrix $U(s)$ is unimodular, then $U(1/s)$ is a biproper Laurent polynomial matrix.

For a matrix pencil $D(s)$, the indices $\rho_1,\ldots,\rho_c$ are expressed by $\zeta_k(D)$ as shown in the following lemma.

Lemma 3.1. For a matrix pencil $D(s)=sX+Y$ of rank r, we have

$$c=r+\min_{k\ge 0}(\zeta_k(D)-k),\quad \rho_i=\zeta_{r-i+1}(D)-\zeta_{r-i}(D)\quad (i=1,\ldots,c).$$

Proof. Consider a matrix pencil $D'(s)=X+sY$ and its Kronecker canonical form with the structural indices $(\nu',\rho_1',\ldots,\rho_{c'}',\mu_1',\ldots,\mu_{d'}',\epsilon_1',\ldots,\epsilon_{p'}',\eta_1',\ldots,\eta_{q'}')$. Then ρ_i $(i=1,\ldots,c)$ of $D(s)$ coincides with μ_i' $(i=1,\ldots,d')$ of $D'(s)$.

It clearly holds that $\operatorname{rank}D'(s)=\operatorname{rank}D(s)=r$. Hence $\mu_i'=\delta_{r-i}(D')-\delta_{r-i+1}(D')+1$ holds by (5). It follows from (4) that

$$\delta_k(D')=-\zeta_k\left(X+\frac{1}{s}Y\right)=-\zeta_k\left(\frac{1}{s}(sX+Y)\right)=-\zeta_k(D)+k.$$

Thus we obtain $\rho_i = \mu'_i = -\zeta_{r-i}(D) + \zeta_{r-i+1}(D)$.

The former equation is given by

$$c = d' = r - \max_{k \geq 0} \delta_k(D') = r - \max_{k \geq 0}(-\zeta_k(D) + k) = r + \min_{k \geq 0}(\zeta_k(D) - k),$$

where the second step is due to (5). □

Let $A(s)$ be an $m \times n$ polynomial matrix. The kth determinantal divisor $d_k(A)$ is defined to be the greatest common divisor of all the subdeterminants of order k:

$$d_k(A) = \gcd\{\det A[I, J] \mid |I| = |J| = k\} \quad (k = 0, 1, \dots, \operatorname{rank} A), \tag{9}$$

where $d_k(A)$ is chosen to be monic and $d_0(s) = 1$ by convention. The value of $d_k(A)$ is invariant under unimodular equivalence transformations, that is, $d_k(A) = d_k(A')$ if $A'(s) = U(s)A(s)V(s)$ with unimodular matrices $U(s)$ and $V(s)$. The following lemma characterizes the sum of H_ν block and K_{ρ_i} blocks.

Lemma 3.2. For a matrix pencil $D(s)$ of rank r, we have $\nu + \sum_{i=1}^{c} \rho_i = \deg d_r(D)$.

Proof. Let $\bar{D}(s)$ be the Kronecker canonical form of $D(s)$. We now have

$$\begin{aligned} &d_\nu(H_\nu) = \det H_\nu, \quad d_\rho(K_\rho) = s^\rho, \quad d_\mu(N_\mu) = 1 \\ &d_\epsilon(L_\epsilon) = \gcd\{s^\epsilon, s^{\epsilon-1}, \dots, 1\} = 1, \quad d_\eta(L_\eta^\top) = \gcd\{s^\eta, s^{\eta-1}, \dots, 1\} = 1. \end{aligned}$$

Hence it holds that $d_r(D) = d_r(\bar{D}) = s^{\rho_1 + \cdots + \rho_c} \det H_\nu$. Since H_ν is strictly regular, this implies that

$$\deg d_r(D) = \rho_1 + \cdots + \rho_c + \nu$$

holds. □

4 Invariance under Unimodular Equivalence Transformations

For a rational function matrix $B(s)$, it is known that $\delta_k(B)$ $(k = 1, 2, \dots)$ are invariant under biproper equivalence transformations, that is, $\delta_k(B) = \delta_k(B')$ $(k = 1, 2, \dots)$ if $B'(s) = U(s)B(s)V(s)$ with biproper matrices $U(s)$ and $V(s)$. Hence, it follows from (5) that $\mu_1, \dots, \mu_d$ are invariant under biproper equivalence transformations. The following lemma shows that $\zeta_k(B)$ is invariant under unimodular equivalence transformations.

Lemma 4.1. Let $B(s)$ be a Laurent polynomial matrix. Then we have $\zeta_k(B) = \zeta_k(B')$ $(k = 1, 2, \dots)$ if $B'(s) = U(s)B(s)V(s)$ with unimodular matrices $U(s)$ and $V(s)$.

Proof. It follows from (4) that

$$\zeta_k(B'(s)) = -\delta_k(U(1/s)B(1/s)V(1/s)) = -\delta_k(B(1/s)) = \zeta_k(B(s)),$$

where the second step is due to the fact that $U(1/s)$ and $V(1/s)$ are biproper. □

Table 1: The invariance of structural indices under equivalence transformations with unimodular matrices and nonsingular constant matrices, where $\surd$ represents that the indices are invariant, and — represents that the indices can be different. Here, $U(s)$ and $V(s)$ are unimodular matrices and U and V are nonsingular constant matrices.

	H_ν	K_ρ	N_μ	L_ϵ	$L_\eta^\top$
(1) $D(s) \to U(s)D(s)V(s)$	$\surd$	$\surd$	—	—	—
(2) $D(s) \to UD(s)V(s)$	$\surd$	$\surd$	—	—	$\surd$
(3) $D(s) \to U(s)D(s)V$	$\surd$	$\surd$	—	$\surd$	—
(4) $D(s) \to UD(s)V$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$

Consider a matrix pencil $D'(s)$ obtained by $D'(s) = U(s)D(s)V(s)$ with some unimodular matrices $U(s)$ and $V(s)$. Let

$$(\nu', \rho'_1, \ldots, \rho'_{c'}, \mu'_1, \ldots, \mu'_{d'}, \epsilon'_1, \ldots, \epsilon'_{p'}, \eta'_1, \ldots, \eta'_{q'})$$

be the structural indices of $D'(s)$. We have $p = p'$ and $q = q'$ by (3) and $c = c'$ by Lemmas 3.1 and 4.1. These lemmas also indicate $\rho_i = \rho'_i$ $(i = 1, \ldots, c)$. Since $d_r(D) = d_r(D')$ with $r = \operatorname{rank} D(s) = \operatorname{rank} D'(s)$, it then follows from Lemma 3.2 that $\nu = \nu'$.

Table 1 shows whether the size of each block is invariant or not under the following four kinds of transformations from $D(s)$ into another matrix pencil $D'(s)$.

(1) $D'(s) = U(s)D(s)V(s)$ with unimodular matrices $U(s)$ and $V(s)$

(2) $D'(s) = UD(s)V(s)$ with a nonsingular constant matrix U and a unimodular matrix $V(s)$

(3) $D'(s) = U(s)D(s)V$ with a unimodular matrix $U(s)$ and a nonsingular constant matrix V

(4) $D'(s) = UD(s)V$ with nonsingular constant matrices U and V

The results of (1) in Table 1 follow from the above discussion. The block N_μ is invariant under biproper equivalence transformations, but not under (1)–(3).

We now consider the L_ϵ block in Table 1. Let $A(s) = \sum_{i=0}^{N} A_i s^i$ be a polynomial matrix, $U(s) = \sum_i U_i s^i$ be a unimodular matrix and V be a nonsingular constant matrix. We denote the maximum degree of entries in $U(s)A(s)V$ by $N'(\geq N)$. Then we have

$$\Psi_k^{N'}(U(s)A(s)V) = \tilde{U}_{k+N'-1}\Psi_k^{N'}(A(s))\tilde{V}_k,$$

where $\tilde{U}_{k+N'-1}$ is a $(k+N')m \times (k+N')m$ matrix and $\tilde{V}_k$ is a $kn \times kn$ matrix defined by

$$\tilde{U}_{k+N'-1} = \begin{pmatrix} U_0 & O & \cdots & O \\ U_1 & U_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ U_{k+N'-1} & \cdots & U_1 & U_0 \end{pmatrix}, \quad \tilde{V}_k = \begin{pmatrix} V & O & \cdots & O \\ O & V & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & V \end{pmatrix}.$$

We note that $U(s)A(s)V$ does not have entries with degree $N'+1, N'+2, \ldots, N'+k-1$, because N' is the maximum degree of entries in $U(s)A(s)V$. Since $U(s)$ is unimodular, U_0 is nonsingular, which implies that $\tilde{U}_{k+N'-1}$ is nonsingular. In addition, $\tilde{V}$ is also nonsingular by the nonsingularity of V. Hence we obtain

$$\psi_k(U(s)A(s)V) = \psi_k(A(s)). \tag{10}$$

Let $D'(s) = U(s)D(s)V$ be a matrix pencil described in (3) in Table 1. Then it follows from (10) that

$$\psi_k(D') = \psi_k(U(s)D(s)V) = \psi_k(D).$$

Thus, $D(s)$ and $D'(s)$ have the same minimal column indices by Theorem 2.3.

For (2) in Table 1, we can prove that $D(s)$ and $UD(s)V(s)$ have the same minimal row indices in a similar way. The results of (4) are obvious because $D(s)$ and $D'(s)$ have the same Kronecker canonical form. Thus, we complete Table 1.

5 Mixed Polynomial Matrices

Let $\mathbf{K}$ be a subfield of a field $\mathbf{F}$. A matrix A over $\mathbf{F}$ is called a *mixed matrix* with respect to $(\mathbf{K}, \mathbf{F})$ if A is given by $A = Q + T$, where Q and T satisfy the following two conditions.

(M-Q) Q is a matrix over $\mathbf{K}$.

(M-T) T is a matrix over $\mathbf{F}$ such that the set of nonzero entries is algebraically independent over $\mathbf{K}$.

A typical setting of $(\mathbf{K}, \mathbf{F})$ is that $\mathbf{K}$ and $\mathbf{F}$ are the fields of rational and real numbers.

A matrix $A(s)$ is called a *mixed polynomial matrix* if $A(s)$ is given by $A(s) = Q(s) + T(s)$ with a pair of polynomial matrices $Q(s)$ over $\mathbf{K}$ and $T(s)$ over $\mathbf{F}$ that satisfy the following two conditions.

(MP-Q) The coefficients of nonzero entries of $Q(s)$ belong to $\mathbf{K}$.

(MP-T) The coefficients of nonzero entries of $T(s)$ belong to $\mathbf{F}$, and the set of nonzero coefficients of $T(s)$ is algebraically independent over $\mathbf{K}$.

A *layered mixed polynomial matrix* (or an *LM-polynomial matrix* for short) is defined to be a mixed polynomial matrix such that $Q(s)$ and $T(s)$ have disjoint nonzero rows. An LM-polynomial matrix $A(s)$ is expressed by $A(s) = \binom{Q(s)}{T(s)}$.

In order to reflect the dimensional consistency in conservation laws of dynamical systems, Murota [14] introduces the following condition on $Q(s)$, which is a formal version of (DC) in Section 1.

(MP-DC) Every nonvanishing subdeterminant of $Q(s)$ is a monomial in s over $\mathbf{K}$.

We call a mixed polynomial matrix and an LM-polynomial matrix satisfying (MP-DC) a *dimensionally consistent mixed polynomial matrix* (a *DCM-polynomial matrix*) and a *dimensionally consistent LM-polynomial matrix* (a *DCLM-polynomial matrix*), respectively. It is known [14, 16] that an $m \times n$ matrix $Q(s)$ satisfies (MP-DC) if and only if

$$Q(s) = \begin{pmatrix} s^{p_1} & 0 & \cdots & 0 \\ 0 & s^{p_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{p_m} \end{pmatrix} Q(1) \begin{pmatrix} s^{-q_1} & 0 & \cdots & 0 \\ 0 & s^{-q_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{-q_n} \end{pmatrix} \tag{11}$$

for some integers p_i $(i = 1, \dots, m)$ and q_j $(j = 1, \dots, n)$.

A mixed polynomial matrix $A(s) = Q(s)+T(s)$ is called a *mixed matrix pencil* if the degree of each entry is at most one. If in addition $Q(s)$ and $T(s)$ have disjoint nonzero rows, $A(s)$ is called a *layered mixed matrix pencil* (or an *LM-matrix pencil* for short). A mixed matrix pencil and an LM-matrix pencil satisfying (MP-DC) are called a *dimensionally consistent mixed matrix pencil* (a *DCM-matrix pencil*) and a *dimensionally consistent LM-matrix pencil* (a *DCLM-matrix pencil*), respectively.

6 Mixed Matrix Pencils and LM-matrix Pencils

This section reveals the relation between a mixed matrix pencil and its associated LM-matrix pencil. Let $D_{\mathrm{M}}(s) = s(X_Q + X_T) + (Y_Q + Y_T)$ be an $m \times n$ mixed matrix pencil with $Q(s) = sX_Q + Y_Q$ and $T(s) = sX_T + Y_T$. We construct an LM-matrix pencil

$$D(s) = s \begin{pmatrix} O & X_Q \\ O & X_T \end{pmatrix} + \begin{pmatrix} I & Y_Q \\ -Z & Y_T \end{pmatrix}, \tag{12}$$

where Z is a diagonal matrix with the (i,i) entry being a new parameter $t_i \in \mathbf{F}$. We transform $D(s)$ into its strictly equivalent matrix

$$\begin{pmatrix} I & O \\ O & Z^{-1} \end{pmatrix} D(s) = s \begin{pmatrix} O & X_Q \\ O & Z^{-1}X_T \end{pmatrix} + \begin{pmatrix} I & Y_Q \\ -I & Z^{-1}Y_T \end{pmatrix}.$$

Each entry of $Z^{-1}X_T$ and $Z^{-1}Y_T$ can be replaced by a new parameter belonging to $\mathbf{F}$. Thus, we regard $Z^{-1}X_T$ and $Z^{-1}Y_T$ as new matrices $\tilde{X}_T$ and $\tilde{Y}_T$ such that the set of nonzero entries of $\tilde{X}_T$ and $\tilde{Y}_T$ is algebraically independent over $\mathbf{K}$. Hence, $D(s)$ and $s \begin{pmatrix} O & X_Q \\ O & \tilde{X}_T \end{pmatrix} + \begin{pmatrix} I & Y_Q \\ -I & \tilde{Y}_T \end{pmatrix}$, as well as $\hat{D}(s) = s \begin{pmatrix} O & X_Q \\ O & X_T \end{pmatrix} + \begin{pmatrix} I & Y_Q \\ -I & Y_T \end{pmatrix}$, have the same Kronecker canonical form.

The following corollary shows that the minimal row indices of $D_{\mathrm{M}}(s)$ are the same as those of $D(s)$, which is derived from Table 1.

Corollary 6.1. Let $D_{\mathrm{M}}(s) = s(X_Q + X_T) + (Y_Q + Y_T)$ be an $m \times n$ mixed matrix pencil and $D(s)$ its associated LM-matrix pencil defined by (12). Then, the minimal row indices of $D_{\mathrm{M}}(s)$ coincide with those of $D(s)$.

Proof. As noted above, $D(s)$ has the same Kronecker canonical form as

$$\hat{D}(s) = s\begin{pmatrix} O & X_Q \\ O & X_T \end{pmatrix} + \begin{pmatrix} I & Y_Q \\ -I & Y_T \end{pmatrix}.$$

Let us define a nonsingular constant matrix U and a unimodular matrix $V(s)$ by

$$U = \begin{pmatrix} I & O \\ I & I \end{pmatrix}, \quad V(s) = \begin{pmatrix} I & -(sX_Q + Y_Q) \\ O & I \end{pmatrix}.$$

Then we have

$$U\hat{D}(s)V(s) = \begin{pmatrix} I & O \\ O & D_{\mathrm{M}}(s) \end{pmatrix}.$$

This transformation corresponds to (2) in Table 1. Hence, $D(s)$ and $U\hat{D}(s)V(s)$ have the same minimal row indices. The Kronecker canonical form of $U\hat{D}(s)V(s)$ consists of m copies of N_1 and the Kronecker canonical form of $D_{\mathrm{M}}(s)$. Therefore, $D(s)$ and $D_{\mathrm{M}}(s)$ have the same minimal row indices. □

According to (2) in Table 1, the indices of nilpotency and the minimal column indices of $D_{\mathrm{M}}(s)$ and $D(s)$ can be different. However, their sum has the following relation.

Theorem 6.2. Let $D_{\mathrm{M}}(s) = s(X_Q + X_T) + (Y_Q + Y_T)$ be an $m \times n$ mixed matrix pencil and $D(s)$ its associated LM-matrix pencil defined by (12). We denote the structural indices of $D_{\mathrm{M}}(s)$ and $D(s)$ by

$$(\nu', \rho'_1, \ldots, \rho'_{c'}, \mu'_1, \ldots, \mu'_{d'}, \epsilon'_1, \ldots, \epsilon'_{p'}, \eta'_1, \ldots, \eta'_{q'}),$$
$$(\nu, \rho_1, \ldots, \rho_c, \mu_1, \ldots, \mu_d, \epsilon_1, \ldots, \epsilon_p, \eta_1, \ldots, \eta_q),$$

respectively. Then we have

$$m + \sum_{i=1}^{d'} {\mu'}_i + \sum_{i=1}^{p'} {\epsilon'}_i = \sum_{i=1}^{d} \mu_i + \sum_{i=1}^{p} \epsilon_i. \tag{13}$$

Proof. As shown in the proof of Corollary 6.1, the transformation from a mixed matrix pencil into the associated LM-matrix pencil is regarded as the transformation (2) in Table 1. Hence we have

$$c' = c, \quad q' = q,$$
$$\nu' = \nu, \quad \rho'_i = \rho_i \ (i = 1, \ldots, c), \quad \eta'_i = \eta_i \ (i = 1, \ldots, q).$$

Let r' and r denote the ranks of $D_{\mathrm{M}}(s)$ and $D(s)$, respectively. Due to the proof of Corollary 6.1,

$$r = \operatorname{rank} \hat{D}(s) = \operatorname{rank} \begin{pmatrix} I & O \\ O & D_{\mathrm{M}}(s) \end{pmatrix} = m + r'$$

holds. It follows from (2) that

$$\sum_{i=1}^{d} \mu_i + \sum_{i=1}^{p} \epsilon_i = m + \sum_{i=1}^{d'} {\mu'}_i + \sum_{i=1}^{p'} {\epsilon'}_i.$$

Thus we obtain (13). □

7 The Combinatorial Canonical Form in Mixed Matrix Theory

In this section, we expound the combinatorial canonical form (CCF) in mixed matrix theory [21]. In particular, we describe the CCF for a DCLM-polynomial matrix $D(s) = \binom{Q(s)}{T(s)}$.

An *LM-admissible transformation* is defined to be an equivalence transformation in the form of

$$P_{\mathrm{r}} \begin{pmatrix} W(s) & O \\ O & I \end{pmatrix} \begin{pmatrix} Q(s) \\ T(s) \end{pmatrix} P_{\mathrm{c}}, \tag{14}$$

where P_{r} and P_{c} are permutation matrices, and $W(s)$ is a unimodular matrix. Remark that the resulting matrix is an LM-polynomial matrix but is not necessarily a matrix pencil even if $D(s) = \binom{Q(s)}{T(s)}$ is an LM-matrix pencil.

We denote the row set and the column set of $D(s) = \binom{Q(s)}{T(s)}$ by R and C, and the row sets of $Q(s)$ and $T(s)$ by R_Q and R_T. Consider a set function $\sigma : 2^C \to \mathbb{Z}$ defined by

$$\sigma(J) = \operatorname{rank} Q(s)[R_Q, J] + \left| \bigcup_{j \in J} \{ i \in R_T \mid T_{ij}(s) \neq 0 \} \right| - |J|, \tag{15}$$

where $T_{ij}(s)$ denotes the (i, j) entry of $T(s)$. Then the set function σ is known to be submodular, and the family of minimizers

$$\mathcal{L}_{\min}(\sigma) = \{ J \subseteq C \mid \sigma(J) \leq \sigma(J'), \forall J' \subseteq C \}$$

forms a sublattice of 2^C. Let $\mathcal{C} : J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_b$ be a maximal chain in $\mathcal{L}_{\min}(\sigma)$. Put $C_0 = J_0$, $C_k = J_k \setminus J_{k-1}$ for $k = 1, \ldots, b$, and $C_\infty = C \setminus J_b$ to obtain a partition $(C_0; C_1, \ldots, C_b; C_\infty)$ of C. Based on this partition, $D(s)$ can be brought into the CCF by an LM-admissible transformation as follows.

Theorem 7.1 ([17, Lemma 3.1])**.** Let $D(s) = \binom{Q(s)}{T(s)}$ be a DCLM-polynomial matrix. By an LM-admissible transformation, $D(s)$ can be brought into another LM-polynomial matrix $\tilde{D}(s) = \binom{\tilde{Q}(s)}{\tilde{T}(s)}$ with the following properties.

(B1) $\tilde{D}(s)$ is block-triangularized, i.e.,

$$\tilde{D}[R_k, C_l] = O \qquad \text{if} \quad 0 \leq l < k \leq \infty,$$

with respect to partitions $(R_0; R_1, \ldots, R_b; R_\infty)$ and $(C_0; C_1, \ldots, C_b; C_\infty)$ of the row set and the column set of $\tilde{D}(s)$, where $b \geq 0$, $R_k \neq \emptyset$ and $C_k \neq \emptyset$ for $k = 1, \ldots, b$, and R_0, R_∞, C_0, and C_∞ can be empty.

(B2) The sizes of the diagonal blocks satisfy:

$$\begin{aligned} &|R_0| < |C_0| \qquad \text{or} \qquad |R_0| = |C_0| = 0, \\ &|R_k| = |C_k| > 0 \quad \text{for} \quad k = 1, \ldots, b, \\ &|R_\infty| > |C_\infty| \qquad \text{or} \qquad |R_\infty| = |C_\infty| = 0. \end{aligned}$$

(B3) The diagonal blocks are of full-rank, i.e.,

$$\begin{aligned}
&\operatorname{rank} \tilde{D}[R_0, C_0] = |R_0|, \\
&\operatorname{rank} \tilde{D}[R_k, C_k] = |R_k| = |C_k| \quad \text{for} \quad k = 1, \ldots, b, \\
&\operatorname{rank} \tilde{D}[R_\infty, C_\infty] = |C_\infty|.
\end{aligned}$$

(B4) The diagonal blocks satisfy:

$$\begin{aligned}
&\operatorname{rank} \tilde{D}[R_0, C_0 \setminus \{j\}] = |R_0| \quad (j \in C_0), \\
&\operatorname{rank} \tilde{D}[R_k \setminus \{i\}, C_k \setminus \{j\}] = |R_k| - 1 = |C_k| - 1 \quad (i \in R_k, j \in C_k) \\
&\hspace{20em} \text{for} \quad k = 1, \ldots, b, \\
&\operatorname{rank} \tilde{D}[R_\infty \setminus \{i\}, C_\infty] = |C_\infty| \quad (i \in R_\infty).
\end{aligned}$$

(B5) $\tilde{D}(s)$ is the finest proper block-triangular matrix among LM-polynomial matrices connected by an LM-admissible transformation.

The LM-polynomial matrix $\tilde{D}(s)$ in Theorem 7.1 is called the *combinatorial canonical form (CCF)* of $D(s)$. We call $D_0(s) := \tilde{D}[R_0, C_0]$ and $D_\infty(s) := \tilde{D}[R_\infty, C_\infty]$ the *horizontal tail* and the *vertical tail*, respectively. Their ranks are denoted by $r_0 := |R_0|$ and $r_\infty := |C_\infty|$.

In the special case of $D(s) = T(s)$, the LM-admissible transformations reduce to permutations, and the CCF decomposition reduces to the Dulmage-Mendelsohn decomposition (DM-decomposition). In the case of $D(s) = Q$, the transformation reduces to $P_{\mathrm{r}} F Q P_{\mathrm{c}}$, and the CCF decomposition agrees with the ordinary Gauss-Jordan elimination in matrix computation. Thus, we may interpret the CCF decomposition as a generalized DM-decomposition with possible numerical computation of accurate numbers.

Recall the definition of $d_k(A)$ in (9). We now have the following lemma.

Lemma 7.2 ([19, Theorem 6.3.4 and Remark 6.3.7])**.** Let $D(s) = \binom{Q(s)}{T(s)}$ be a DCLM-polynomial matrix of rank r. The rth monic determinantal divisor $d_r(D)$ can be expressed by

$$d_r(D) = \alpha_r \cdot g(s) \cdot \prod_{l=1}^{b} \det \tilde{D}(s)[R_l, C_l], \tag{16}$$

where $\alpha_r \in \mathbf{F}$ is a constant, $g(s)$ is a monomial in s, and $\tilde{D}(s)[R_l, C_l]$ $(l = 1, \ldots, b)$ are the square blocks which appear in the CCF of $D(s)$.

If $D(s)$ is a DCLM-matrix pencil, we can construct a CCF such that the horizontal tail $D_0(s)$ is also a DCLM-matrix pencil. In the expression (11) of $Q(s)$, we can assume that

$$p_1 \leq p_2 \leq \cdots \leq p_m, \quad q_1 \leq q_2 \leq \cdots \leq q_n$$

without loss of generality. We now briefly describe the algorithm for computing $D_0(s)$, which is given in [17, §3.2].

Step 1 Determine the partition $(C_0; C_1, \ldots, C_b; C_\infty)$ of C with reference to the set function σ defined by (15).

Step 2 Find a basis of the row vectors of the submatrix $Q(1)[R_Q, C_0]$ by collecting independent row vectors according to the ordering with reference to p_i in such a manner that $p_1 \leq p_2 \leq \cdots \leq p_m$. This ordering guarantees that $W(s)$ of (14) is a unimodular matrix. We denote the basis by R_{Q0}.

Step 3 Output $R_0 = R_{Q0} \cup R_{T0}$ and C_0, where

$$R_{T0} = \{i \in R_T \mid T_{ij}(s) \neq 0, j \in C_0\}.$$

In Step 2, we have assumed that an ordering of rows h and h' with $p_h = p_{h'}$ is arbitrary. By determining this ordering based on $q_1, q_2, \ldots, q_n$, we prove the following lemma.

Lemma 7.3. If $D(s) = \binom{Q(s)}{T(s)}$ is a DCLM-matrix pencil, one can construct a CCF of $D(s)$ such that the horizontal tail $D_0(s)$ is also a DCLM-matrix pencil.

Proof. Since $Q(s)$ is a matrix pencil, $Q(1)[R_Q, C_0]$ is in the form of

$$Q^0 = \begin{array}{c} \\ \mathrm{Row}(0) \\ \mathrm{Row}(1) \\ \mathrm{Row}(2) \\ \vdots \\ \vdots \\ \mathrm{Row}(\gamma) \end{array} \begin{array}{c} \begin{array}{ccccc} \mathrm{Col}(0) & \mathrm{Col}(1) & \cdots & \mathrm{Col}(\gamma-2) & \mathrm{Col}(\gamma-1) \end{array} \\ \begin{pmatrix} * & O & \cdots & \cdots & O \\ ** & * & \ddots & & \vdots \\ O & ** & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * & O \\ \vdots & & \ddots & ** & * \\ O & \cdots & \cdots & O & ** \end{pmatrix} \end{array}$$

for some γ, where $\mathrm{Row}(h) = \{i \in R_Q \mid p_i = h\}$ and $\mathrm{Col}(h) = \{j \in C_0 \mid q_j = h\}$. Here, $*$ and $**$ denote a constant matrix and a coefficient matrix of s, respectively.

We can find a basis of the row vectors of the submatrix $Q^0[R_Q, C_0]$ by collecting independent row vectors from the top row to the bottom row, as explained below. We first find $R_*^0 \subseteq \mathrm{Row}(0)$ satisfying $\mathrm{rank}\, Q^0[R_*^0, \mathrm{Col}(0)] = \mathrm{rank}\, Q^0[\mathrm{Row}(0), \mathrm{Col}(0)]$, which means that R_*^0 is a basis of $\mathrm{Row}(0)$. By row transformations, we obtain Q^1 from Q^0 such that

$$Q^1[\mathrm{Row}(0) \cup \mathrm{Row}(1), C_0] = \begin{array}{c} \\ R_*^0 \\ \mathrm{Row}(0) \setminus R_*^0 \\ \mathrm{Row}(1) \end{array} \begin{array}{c} \begin{array}{ccccc} \leftarrow & \mathrm{Col}(0) & \rightarrow & \mathrm{Col}(1) & \cdots \end{array} \\ \begin{pmatrix} I & * & * & O & O \\ O & O & O & O & O \\ O & ** & ** & * & O \end{pmatrix} \end{array},$$

because the row vectors of $Q^0[\mathrm{Row}(0) \setminus R_*^0, C_0]$ can be expressed as linear combinations of those of $Q^0[R_*^0, C_0]$.

Next, we find $R_{**}^1 \subseteq \mathrm{Row}(1)$ satisfying $\mathrm{rank}\, Q^1[R_{**}^1, \mathrm{Col}(0)] = \mathrm{rank}\, Q^1[\mathrm{Row}(1), \mathrm{Col}(0)]$. Then we obtain Q^2 from Q^1 such that

$$Q^2[\mathrm{Row}(0) \cup \mathrm{Row}(1), C_0] = \begin{array}{c} \\ R_*^0 \\ \mathrm{Row}(0) \setminus R_*^0 \\ R_{**}^1 \\ \mathrm{Row}(1) \setminus R_{**}^1 \end{array} \begin{array}{c} \begin{array}{ccccc} \leftarrow & \mathrm{Col}(0) & \rightarrow & \mathrm{Col}(1) & \cdots \end{array} \\ \begin{pmatrix} I & * & * & O & O \\ O & O & O & O & O \\ O & I & ** & * & O \\ O & O & O & * & O \end{pmatrix} \end{array}$$

by row transformations. Then, we apply the same procedure to $Q^2[R_Q \setminus (\mathrm{Row}(0) \cup R^1_{**}), C_0 \setminus \mathrm{Col}(0)]$.

As a result, we obtain

$$
Q' = \begin{array}{c} \\ \mathrm{Row}(0) \\ \\ \mathrm{Row}(1) \\ \\ \mathrm{Row}(2) \\ \\ \vdots \\ \\ \vdots \\ \\ \mathrm{Row}(\gamma) \end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathrm{Col}(0) & \mathrm{Col}(1) & \cdots & \mathrm{Col}(\gamma-2) & \mathrm{Col}(\gamma-1) \end{array} \\
\left(\begin{array}{ccccc}
\begin{matrix} I & * & * \\ O & O & O \end{matrix} & O & \cdots & \cdots & O \\
\begin{matrix} O & I & ** \\ O & O & O \\ O & O & O \end{matrix} & \begin{matrix} * & * & * \\ I & * & * \\ O & O & O \end{matrix} & \ddots & & \vdots \\
O & \begin{matrix} O & I & ** \\ O & O & O \end{matrix} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \begin{matrix} * & * & * \\ I & * & * \\ O & O & O \end{matrix} & O \\
\vdots & & \ddots & \begin{matrix} O & I & ** \\ O & O & O \\ O & O & O \end{matrix} & \begin{matrix} * & * & * \\ I & * & * \\ O & O & O \end{matrix} \\
O & \cdots & \cdots & O & \begin{matrix} O & I & ** \\ O & O & O \end{matrix}
\end{array}\right)
\end{array},
$$

where the row sets of I in $Q'[\mathrm{Row}(h), \mathrm{Col}(h)]$ and $Q'[\mathrm{Row}(h), \mathrm{Col}(h-1)]$ correspond to R^h_* and R^h_{**}, respectively. This transformation preserves $Q'[\mathrm{Row}(h), \mathrm{Col}(l)] = O$ for any h, l satisfying $0 \le h \le \gamma$, $0 \le l \le \gamma - 1$, and $h - l \neq 0, 1$. We define

$$R_{Q0} = \bigcup_{i=1}^{\gamma} (R^{i-1}_* \cup R^i_{**}).$$

Then R_{Q0} is a basis of the row vectors of Q' as well as $Q(1)[R_Q, C_0] = Q^0$.

Let W be a nonsingular constant matrix such that $Q' = WQ^0$. We define $W(s)$ in (14) by

$$
W(s) = \begin{pmatrix} s^{p_1} & 0 & \cdots & 0 \\ 0 & s^{p_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{p_m} \end{pmatrix} W \begin{pmatrix} s^{-p_1} & 0 & \cdots & 0 \\ 0 & s^{-p_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{-p_m} \end{pmatrix}.
$$

The horizontal trail $D_0(s)$ is given by

$$
D_0(s) = \begin{pmatrix} W(s)Q(s)[R_{Q0}, C_0] \\ T(s)[R_{T0}, C_0] \end{pmatrix},
$$

where R_{T0} is defined in Step 3 in the algorithm for computing $D_0(s)$.

To prove that $D_0(s)$ is a matrix pencil, it suffices to show that $W(s)Q(s)[R_{Q0}, C_0]$ is also a matrix pencil, because $T(s)$ is a matrix pencil. We now have

$$
W(s)Q(s) = \begin{pmatrix} s^{p_1} & 0 & \cdots & 0 \\ 0 & s^{p_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{p_m} \end{pmatrix} WQ(1) \begin{pmatrix} s^{-q_1} & 0 & \cdots & 0 \\ 0 & s^{-q_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{-q_n} \end{pmatrix}
$$

and $WQ(1)[R_Q, C_0] = WQ^0 = Q'$. Hence $W(s)Q(s)[R_Q, C_0]$ is a matrix pencil, which implies that the submatrix $W(s)Q(s)[R_{Q0}, C_0]$ of $W(s)Q(s)[R_Q, C_0]$ is also a matrix pencil. Moreover, $D_0(s)$ satisfies (MP-DC), because $W(s)Q(s)$ satisfies (MP-DC). □

8 The Kronecker Canonical Form via CCF

In this section, we investigate the Kronecker canonical form of DCLM-matrix pencils via the CCF decomposition. For a DCLM-matrix pencil $D(s) = \binom{Q(s)}{T(s)}$ of rank r, we construct its CCF $\tilde{D}(s)$ so that the horizontal tail $D_0(s)$ is also a DCLM-matrix pencil. The existence of such CCF is assured by Lemma 7.3. The rank of $D_0(s)$ is denoted by r_0.

Lemma 8.1. We have $\psi_k(D) = \psi_k(D_0) + k(r - r_0)$.

Proof. We define $D_*(s) = \tilde{D}(s)[R \setminus R_0, C \setminus C_0]$. Since $D_*(s)$ is of full-column rank, it holds that

$$\psi_k(D_*) = k|C \setminus C_0| = k(r - r_0)$$

by Lemma 2.4. We also have $\psi_k(\tilde{D}) = \psi_k(D_0) + \psi_k(D_*)$ by $\tilde{D}(s)[R \setminus R_0, C_0] = O$. By (10) and the definition of an LM-admissible transformation (14), $\psi_k(\tilde{D}) = \psi_k(D)$ holds. Thus we obtain $\psi_k(D) = \psi_k(D_0) + k(r - r_0)$. □

We now investigate the Kronecker canonical form of $D_0(s)$.

Lemma 8.2. The monic determinantal divisor $d_{r_0}(D_0)$ is a monomial in s.

Proof. By $\operatorname{rank} D_0(s) = r_0$, we can apply Lemma 7.2 to $d_{r_0}(D_0)$. Since the CCF of $D_0(s)$ has no square blocks, $d_{r_0}(D_0)$ is a monomial in s by Lemma 7.2. □

We now obtain the following theorem on the sum of the minimal column indices.

Theorem 8.3. The sum of the minimal column indices of a DCLM-matrix pencil $D(s)$ is obtained by

$$\sum_{i=1}^{p} \epsilon_i = \delta_{r_0}(D_0) - \zeta_{r_0}(D_0). \tag{17}$$

Proof. The horizontal tail $D_0(s)$ has the Kronecker canonical form, because $D_0(s)$ is a matrix pencil by Lemma 7.3. By Theorem 2.3 and Lemma 8.1, the minimal column indices of $D(s)$ coincide with those of the horizontal tail $D_0(s)$. Let $\bar{D}_0(s)$ be the Kronecker canonical form of $D_0(s)$. Since $D_0(s)$ is of full-row rank, $\bar{D}_0(s)$ contains no rectangular blocks $L_\eta^\top$. In addition, $\bar{D}_0(s)$ does not contain a strictly regular block by Lemma 8.2. Hence we obtain (17) by Lemma 2.2. □

Theorem 8.3 indicates that the computation of the sum of the minimal column indices for a DCLM-matrix pencil $D(s)$ reduces to that of $\delta_{r_0}(D_0)$ and $\zeta_{r_0}(D_0)$. Here, $D_0(s)$ is smaller (and sometimes much smaller) than $D(s)$. Several efficient algorithms for δ_k and ζ_k have been developed [9, 18, 26]. Furthermore, since $D_0(s)$ satisfies (MP-DC), the computation of δ_k and

ζ_k is reduced to a weighted matroid intersection problem [19, Remark 6.2.10]. Thus, one can compute efficiently the sum of the minimal column indices of a DCLM-matrix pencil.

Theorem 8.3 combined with Theorem 6.2 enables us to compute the sum of the minimal column indices of a DCM-matrix pencil, as explained below. Let $D_{\mathrm{M}}(s) = s(X_Q + X_T) + (Y_Q + Y_T)$ be an $m \times n$ DCM-matrix pencil and $D(s)$ its associated LM-matrix pencil defined by (12). We denote the structural indices of $D_{\mathrm{M}}(s)$ and $D(s)$ by

$$(\nu', \rho'_1, \ldots, \rho'_{c'}, \mu'_1, \ldots, \mu'_{d'}, \epsilon'_1, \ldots, \epsilon'_{p'}, \eta'_1, \ldots, \eta'_{q'}),$$
$$(\nu, \rho_1, \ldots, \rho_c, \mu_1, \ldots, \mu_d, \epsilon_1, \ldots, \epsilon_p, \eta_1, \ldots, \eta_q),$$

respectively. It follows from Theorem 6.2 that

$$\sum_{i=1}^{p'} {\epsilon'}_i = \sum_{i=1}^{p} \epsilon_i + \sum_{i=1}^{d} \mu_i - \sum_{i=1}^{d'} {\mu'}_i - m.$$

In the right-hand side, $\sum_{i=1}^{p} \epsilon_i$ of the LM-matrix pencil $D(s)$ can be computed by Theorem 8.3. We can also find $\sum_{i=1}^{d} \mu_i$ of $D(s)$ and $\sum_{i=1}^{d'} {\mu'}_i$ of $D_{\mathrm{M}}(s)$ based on (5), because δ_k is computed efficiently as already mentioned. It should be noted that, in the computation of δ_k, the transformation from a mixed matrix pencil into an LM-matrix pencil is different from (12). Thus we can obtain $\sum_{i=1}^{p'} {\epsilon'}_i$ of the DCM-matrix pencil $D_{\mathrm{M}}(s)$.

In order to compute the sum of the minimal row indices, we apply Theorems 6.2 and 8.3 to $D(s)^\top$, because the minimal row indices of $D(s)$ coincide with the minimal column indices of $D(s)^\top$.

We conclude this section with an example.

Example 8.4. Consider a DCM-matrix pencil

$$D_{\mathrm{M}}(s) = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & s & t_2 s + 1 \end{pmatrix}.$$

Its associated LM-matrix pencil are given by

$$D(s) = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & s & 1 \\ \hline -t_3 & 0 & t_1 & 0 & 0 \\ 0 & -t_4 & 0 & 0 & t_2 s \end{array}\right).$$

The Kronecker canonical forms of $D_{\mathrm{M}}(s)$ and $D(s)$ are given by

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & s & 1 \end{array}\right), \quad \left(\begin{array}{c|c|ccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & s & 1 & 0 \\ 0 & 0 & 0 & s & 1 \end{array}\right), \tag{18}$$

respectively. As described below, we can compute the sums of the minimal column indices efficiently based on Theorems 6.2 and 8.3, even if they are different between $D_{\mathrm{M}}(s)$ and $D(s)$.

Since the CCF of $D(s)$ is

$$\left(\begin{array}{ccc|cc} 1 & s & 1 & 0 & 0 \\ -t_4 & 0 & t_2 s & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -t_3 & t_1 \end{array}\right),$$

we have the horizontal tail $D_0(s) = \begin{pmatrix} 1 & s & 1 \\ -t_4 & 0 & t_2 s \end{pmatrix}$. It follows from Theorem 8.3 that

$$\sum_i \epsilon_i = \delta_2(D_0) - \zeta_2(D_0) = 2 - 0 = 2.$$

We can obtain $\sum_i \mu_i = 2$ and $\sum_i {\mu'}_i = 1$ by executing any of the algorithms given in [9, 18, 26] or reducing to a weighted matroid intersection problem [19, Remark 6.2.10]. Thus we have

$$\sum_i {\epsilon'}_i = 2 + 2 - 1 - 2 = 1$$

by Theorem 6.2. We can check the correctness of these values by (18).

9 Application to Controllable Subspace

In this section, we present an application of our main result to controllability analysis of dynamical systems.

Consider a linear time-invariant dynamical system in a descriptor form

$$F\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t), \tag{19}$$

where F and A are $n \times n$ matrices and B is an $n \times l$ matrix. For the unique solvability, we assume that $A - sF$ is a regular matrix pencil.

Van Dooren [29] introduced the controllable subspace of the system (19) defined by

$$\mathcal{C} = \inf\{\mathcal{S} \mid \dim(F\mathcal{S} + A\mathcal{S}) = \dim \mathcal{S}, \operatorname{im} B \subseteq F\mathcal{S} + A\mathcal{S}\},$$

where the infimum can be proven to exist. In fact, the controllable subspace $\mathcal{C}$ is obtained as follows. With an appropriate nonsingular constant matrix S, one can transform $\begin{pmatrix} A - sF \mid B \end{pmatrix}$ into

$$S\begin{pmatrix} A - sF \mid B \end{pmatrix} = \begin{pmatrix} A_0 - sF_0 & O \\ * & B_0 \end{pmatrix}$$

so that B_0 is of full-row rank. Since $A_0 - sF_0$ is of full-row rank, its Kronecker canonical form does not contain a rectangular block $L_\eta^\top$. Therefore one can further transform $A_0 - sF_0$ with an appropriate pair of nonsingular constant matrices U and V into

$$U\begin{pmatrix} A_0 - sF_0 \end{pmatrix} V = \begin{pmatrix} A_1 - sF_1 & O \\ O & A_2 - sF_2 \end{pmatrix},$$

where $A_1 - sF_1$ is a regular matrix pencil and the Kronecker canonical form of $A_2 - sF_2$ consists only of rectangular blocks L_ϵ. Then the column set of $A_2 - sF_2$ corresponds to the controllable subspace $\mathcal{C}$, and the number of columns is equal to $\dim \mathcal{C}$.

The system (19) is controllable iff $\dim \mathcal{C} = n$. Murota [15] presented a matroid-theoretic algorithm for testing the controllability of a dynamical system (19) described by a DCM-matrix pencil $\begin{pmatrix} A - sF \mid B \end{pmatrix}$. The algorithm, however, does not provide the dimension of the controllable subspace.

The dimension of $\mathcal{C}$ is characterized by the rank of the $(n+1)n \times (n^2 + nl + l)$ matrix

$$\Sigma(F, A, B) = \begin{pmatrix} B & -F & O & O & \cdots & O & O & O & O \\ O & A & B & -F & \ddots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & A & \ddots & O & O & O & O \\ O & O & O & O & \ddots & -F & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & A & B & -F & O \\ O & O & O & O & \cdots & O & O & A & B \end{pmatrix},$$

as shown in the following lemma.

Lemma 9.1. It holds that

$$\dim \mathcal{C} = \operatorname{rank} \Sigma(F, A, B) - n^2.$$

Proof. We denote the row sets of $A_1 - sF_1$, $A_2 - sF_2$, and B_0 by R_1, R_2, and R_3. Since $A_1 - sF_1$ is a regular matrix pencil, we have $\dim \mathcal{C} = n - |R_1|$.

The rank of $\Sigma(F, A, B)$ is invariant under the above equivalence transformation. By putting

$$\hat{A} = \begin{pmatrix} A_1 & O \\ O & A_2 \\ * & * \end{pmatrix}, \quad \hat{F} = \begin{pmatrix} F_1 & O \\ O & F_2 \\ * & * \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} O \\ O \\ B_0 \end{pmatrix},$$

we have $\operatorname{rank} \Sigma(F, A, B) = \operatorname{rank} \Sigma(\hat{F}, \hat{A}, \hat{B})$. Since B_0 is of full-row rank, we have

$$\begin{aligned} \operatorname{rank} \Sigma(\hat{F}, \hat{A}, \hat{B}) &= (n+1)|R_3| + \operatorname{rank} \Psi_n \left(\begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} - s \begin{pmatrix} F_1 & O \\ O & F_2 \end{pmatrix} \right) \\ &= (n+1)|R_3| + \operatorname{rank} \begin{pmatrix} \Psi_n(A_1 - sF_1) & O \\ O & \Psi_n(A_2 - sF_2) \end{pmatrix}. \end{aligned}$$

Since $A_1 - sF_1$ is regular, it follows from Lemma 2.4 that $\psi_n(A_1 - sF_1) = n|R_1|$. By Theorem 2.3, we have

$$\psi_n(A_2 - sF_2) = n|R_2| + \sum_{i=1}^{p'} \min\{n, \epsilon'_i\} = n|R_2| + \sum_{i=1}^{p'} \epsilon'_i,$$

where $\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_{p'}$ denote the minimal column indices of $A_2 - sF_2$. Since the Kronecker canonical form of $A_2 - sF_2$ consists only of rectangular blocks L_ϵ, it holds that $\sum_{i=1}^{p'} \epsilon'_i = |R_2|$. Thus we obtain

$$\operatorname{rank} \Sigma(F, A, B) = (n+1)|R_3| + n|R_1| + (n+1)|R_2| = n^2 + \dim \mathcal{C}$$

by $n = |R_1| + |R_2| + |R_3|$ and $\dim \mathcal{C} = n - |R_1|$. □

The following theorem states that if F is nonsingular, the computation of $\dim \mathcal{C}$ is reduced to the computation of the sum of the minimal column indices of $\begin{pmatrix} A - sF \mid B \end{pmatrix}$.

Theorem 9.2. Let $\mathcal{C}$ be the controllable subspace of the system (19), and $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$ be the minimal column indices of a matrix pencil $D(s) = \begin{pmatrix} A - sF \mid B \end{pmatrix}$. If F is nonsingular, the dimension of $\mathcal{C}$ is given by

$$\dim \mathcal{C} = \sum_{i=1}^{p} \epsilon_i.$$

Proof. Since

$$\Psi_{n+1}(D) = \begin{pmatrix} -F & O \quad \cdots \quad O \\ A & \\ O & \Sigma(F, A, B) \\ \vdots & \\ O & \end{pmatrix},$$

we have $\psi_{n+1}(D) = n + \operatorname{rank} \Sigma(F, A, B)$ by the nonsingularity of F. Then it follows from Theorem 2.3 and $\operatorname{rank} D(s) = n$ that

$$\psi_{n+1}(D) = n(n+1) + \sum_{i=1}^{p} \min\{n+1, \epsilon_i\} = n^2 + n + \sum_{i=1}^{p} \epsilon_i$$

holds. Thus we obtain

$$\dim \mathcal{C} = \operatorname{rank} \Sigma(F, A, B) - n^2 = \psi_{n+1}(D) - n - n^2 = \sum_{i=1}^{p} \epsilon_i$$

by Lemma 9.1. □

By Theorem 9.2, if F is nonsingular, the computation of the dimension of the controllable subspace $\mathcal{C}$ is reduced to that of the sum of the minimal column indices of $D(s) = \begin{pmatrix} A - sF \mid B \end{pmatrix}$. If in addition $D(s)$ is a DCM-matrix pencil, one can obtain the dimension of $\mathcal{C}$ by solving a weighted matroid intersection problem as described in Section 8.

10 Conclusion

For mixed matrix pencils satisfying the assumption on dimensional consistency, we have characterized the sum of the minimal row/column indices of the Kronecker canonical form. An efficient matroid-theoretic algorithm for computing them is derived from this characterization. As an application example of our results, we describe the dimension of the controllable subspace of dynamical systems. In analysis of the controllable subspace, we have assumed that a coefficient matrix F of $\dot{\boldsymbol{x}}(t)$ is nonsingular. The computation of the dimension of the controllable subspace with singular F is left for future work.

The computation of the minimal row/column indices is difficult even under the genericity assumption, as discussed in [7, Section 7]. Our ultimate target is to present an algorithm based on structural approach for computing the minimal row/column indices. We anticipate that the characterization of the sums is useful for design of such algorithms.

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