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Abstract

In 1995, Galvin provided an elegant proof for the list edge coloring conjecture for bipartite graphs, utilizing the stable matching theorem of Gale and Shapley. In this paper, we generalize Galvin's result to the setting of supermodular coloring, introduced by Schrijver, with the aid of the monochromatic path theorem of Sands, Sauer and Woodrow. Our result derives a list coloring version of Gupta's theorem, which can also be proven directly from Galvin's theorem.

1 Introduction

A list coloring is a type of coloring in which each of the elements to be colored has its own list of permissible colors. One of the most celebrated results in the study of list coloring is the following theorem of Galvin on edge colorings of bipartite graphs. An *edge coloring* of an undirected graph is a function which assigns a color to each edge so that no two adjacent edges have the same color.

Theorem 1.1 (Galvin [8]). For a bipartite graph that admits an edge coloring with $k \in \mathbf{Z}_{>0}$ colors, if each edge e has a list $L(e)$ of k colors, then there exists an edge coloring such that every edge e is assigned a color in $L(e)$. ■

The existence of an edge coloring in a bipartite graph is characterized by Kőnig's theorem.

Theorem 1.2 (Kőnig [10]). A bipartite graph admits an edge coloring with k or less colors if and only if each vertex is incident to at most k edges. ■

In other words, the minimum number of colors required for a bipartite edge coloring is equal to the maximum degree. Combining Theorems 1.1 and 1.2 we see that, if the size of $L(e)$ for each edge e is at least the maximum degree, there is an edge coloring which assigns a color in $L(e)$ for each edge e.

In this paper, we generalize the above results of Galvin to the setting of supermodular coloring introduced by Schrijver [12]. Let U be a finite set. We say that $X, Y \subseteq U$ are *intersecting* if none of $X \cap Y$, $X \setminus Y$ and $Y \setminus X$ are empty. A family $\mathcal{F} \subseteq 2^U$ is called an *intersecting family* if every intersecting pair of $X, Y \in \mathcal{F}$ satisfies $X \cup Y, X \cap Y \in \mathcal{F}$. A function $g \colon \mathcal{F} \to \mathbf{R}$ is called *intersecting-supermodular* if $\mathcal{F}$ is an intersecting family and g satisfies the *supermodular inequality* $g(X) + g(Y) \leq g(X \cup Y) + g(X \cap Y)$ for every intersecting pair of $X, Y \in \mathcal{F}$.

For any $k \in \mathbf{Z}_{>0}$, we write $[k] := \{1, 2, \ldots, k\}$. A function $\pi : U \to [k]$ is called a k-coloring. We say that π *dominates* a function $g : \mathcal{F} \to \mathbf{Z}$ if $|\pi(X)| \geq g(X)$ holds for every $X \in \mathcal{F}$, where $\pi(X) := \{\, \pi(u) \mid u \in X \,\}$. For two intersecting-supermodular functions $g_1 : \mathcal{F}_1 \to \mathbf{Z}$ and $g_2 : \mathcal{F}_2 \to \mathbf{Z}$, a k-coloring π is called a *supermodular k-coloring* if π dominates both g_1 and g_2. Schrijver characterized the existence of a supermodular k-coloring in the following theorem, which generalizes Theorem 1.2.

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Theorem 1.3 (Schrijver [12]). Let $g_1 : \mathcal{F}_1 \to \mathbf{Z}$ and $g_2 : \mathcal{F}_2 \to \mathbf{Z}$ be intersecting-supermodular functions such that each g_i satisfies $|X| \geq g_i(X)$ for every $X \in \mathcal{F}_i$. Then, for any $k \in \mathbf{Z}_{>0}$, there exists a supermodular k-coloring for (g_1, g_2) if and only if both $k \geq \max\{\, g_1(X) \mid X \in \mathcal{F}_1 \,\}$ and $k \geq \max\{\, g_2(X) \mid X \in \mathcal{F}_2 \,\}$ hold. ■

Tardos [13] provided an alternative proof for this theorem using the generalized matroid intersection theorem. Theorem 1.3 has been extended in [6] to a more general framework including skew-supermodular coloring.

Let us consider the list coloring version of supermodular colorings. Let Σ be a finite set of colors and let each $u \in U$ have a color list $L(u) \subseteq \Sigma$, that is, L is a mapping from U to 2^{Σ}. For intersecting-supermodular functions $g_1 : \mathcal{F}_1 \to \mathbf{Z}$, $g_2 : \mathcal{F}_2 \to \mathbf{Z}$ and color lists $\{L(u)\}_{u \in U}$, a *list supermodular coloring* is a function $\varphi : U \to \Sigma$ such that every $u \in U$ satisfies $\varphi(u) \in L(u)$ and φ dominates both g_1 and g_2. The main result of this paper is as follows.

Theorem 1.4. For intersecting-supermodular functions $g_1 : \mathcal{F}_1 \to \mathbf{Z}$ and $g_2 : \mathcal{F}_2 \to \mathbf{Z}$ and $k \in \mathbf{Z}_{>0}$, assume that there exists a supermodular k-coloring for (g_1, g_2). If L satisfies $|L(u)| = k$ for each $u \in U$, then there exists a list supermodular coloring $\varphi : U \to \Sigma$ for (g_1, g_2, L). ■

An interesting special case of Schrijver's theorem is Gupta's [9] generalization of Kőnig's theorem. Theorem 1.4 then naturally derives its list coloring version, which can also be shown from Galvin's theorem. See Section 6.

The pair (g_1, g_2) of intersecting-supermodular functions is called *k-choosable* if, for every $L : U \to 2^{\Sigma}$ with $|L(u)| = k$ $(\forall u \in U)$, there exists a list supermodular coloring for (g_1, g_2, L). Combining Theorems 1.3 and 1.4 implies the following corollary.

Corollary 1.5. Let $g_1 : \mathcal{F}_1 \to \mathbf{Z}$ and $g_2 : \mathcal{F}_2 \to \mathbf{Z}$ be intersecting-supermodular functions such that each g_i satisfies $|X| \geq g_i(X)$ for every $X \in \mathcal{F}_i$. Then, for any $k \in \mathbf{Z}_{>0}$, the pair (g_1, g_2) is k-choosable if and only if both $k \geq \max\{\, g_1(X) \mid X \in \mathcal{F}_1 \,\}$ and $k \geq \max\{\, g_2(X) \mid X \in \mathcal{F}_2 \,\}$ hold. ■

A surprising aspect of Galvin's proof is that it utilizes a famous result of Gale and Shapley [7] on the existence of stable matchings in bipartite graphs. See also [1] for a beautiful exposition. To show Theorem 1.4, we utilize the theorem of Sands, Sauer and Woodrow [11], which is shown by Fleiner [3] to be a generalization of the result of Gale and Shapley. Here we introduce the theorem with the terminology of Fleiner and Jankó [4].

In a partially ordered set (poset) $P = (U, \preceq)$, two elements $u, v \in U$ are *comparable* if $u \preceq v$ or $v \preceq u$ holds, and otherwise they are *incomparable*. A *chain* is a subset in which each pair of elements is comparable. An *antichain* is a subset in which each pair of distinct elements is incomparable. Let $P_1 = (U, \preceq_1)$ and $P_2 = (U, \preceq_2)$ be two posets on the same ground set U. A subset $K \subseteq U$ is called a *kernel* if K is a common antichain and every element $u \in U \setminus K$ admits an element $v \in K$ such that $v \prec_1 u$ or $v \prec_2 u$. Moreover, for any subset $S \subseteq U$, we call its subset $K \subseteq S$ a *kernel of* S if K is a common antichain and every element $u \in S \setminus K$ admits an element $v \in K$ such that $v \prec_1 u$ or $v \prec_2 u$. We are now ready to describe the theorem of Sands et al.

Theorem 1.6 (Sands et al. [11]). Let $P_1 = (U, \preceq_1)$ and $P_2 = (U, \preceq_2)$ be posets on the same ground set U. For any subset $S \subseteq U$, there exists a kernel of S. ■

Their original statement was described in terms of directed graphs whose edges are colored with two colors, and the binary relation $v \prec u$ in Theorem 1.6 corresponds to the existence of a monochromatic path from a node u to another node v. Their statement can be applied to more general binary relations but only to the case of $S = U$. It is easy to see the equivalence between the two variants.

We also show that Theorem 1.3 can be extended to the setting of skew-supermodular coloring. The same statement has been conjectured in [2].

The rest of this paper is organized as follows. In Section 2, we introduce skeleton posets for colorings that dominate intersecting-supermodular functions. The existence proof of skeleton posets is postponed to Section 4. In Section 3, we give a proof of Theorem 1.4 using induction on the ground set. Each step of the induction applies Theorem 1.6 to skeleton posets. Section 5 extends Theorem 1.4 to skew-supermodular functions. Finally, in Section 6, we show the list coloring version of Gupta's theorem.

2 Skeleton Posets

Let $g : \mathcal{F} \to \mathbf{Z}$ be an intersecting-supermodular function on $\mathcal{F} \subseteq 2^U$. For a subset $K \subseteq S$, the *reduction* of g by K is the function $g_K : \mathcal{F}_K \to \mathbf{Z}$ defined by $\mathcal{F}_K = \{ Z \setminus K \mid Z \in \mathcal{F} \}$ and

$$g_K(X) = \max \{ \hat{g}_K(Z) \mid Z \in \mathcal{F},\ Z \setminus K = X \} \quad (X \in \mathcal{F}_K),$$

where $\hat{g}_K : \mathcal{F} \to \mathbf{Z}$ is defined by

$$\hat{g}_K(Z) = \begin{cases} g(Z) - 1 & (Z \in \mathcal{F},\ Z \cap K \neq \emptyset), \\ g(Z) & (Z \in \mathcal{F},\ Z \cap K = \emptyset). \end{cases}$$

Claim 2.1. The reduction $g_K : \mathcal{F}_K \to \mathbf{Z}$ is an intersecting-supermodular function.

Proof. For every $X, Y \in \mathcal{F}_K$ and $Z_X, Z_Y \in \mathcal{F}$ such that $Z_X \setminus K = X$ and $Z_Y \setminus K = Y$, we have $X \cup Y = (Z_X \setminus K) \cup (Z_X \setminus K) = (Z_X \cup Z_Y) \setminus K$, and the same holds for the intersection. These imply that $\mathcal{F}_K$ is an intersecting family. We now show the supermodular inequality of g_K for intersecting $X, Y \in \mathcal{F}_K$. Take $Z_X, Z_Y \in \mathcal{F}$ which attain $g_K(X) = \hat{g}_K(Z_X)$ and $g_K(Y) = \hat{g}_K(Z_Y)$. As easily confirmed, $\hat{g}_K : \mathcal{F} \to \mathbf{Z}$ is intersecting supermodular, and hence $\hat{g}_K(Z_X) + \hat{g}_K(Z_Y) \leq \hat{g}_K(Z_X \cup Z_Y) + \hat{g}_K(Z_X \cap Z_Y)$. Also, we have $\hat{g}_K(Z_X \cup Z_Y) \leq g_K(X \cup Y)$ because $Z_X \cup Z_Y \in \mathcal{F}$ and $(Z_X \cup Z_Y) \setminus K = X \cup Y$. Similarly, $\hat{g}_K(Z_X \cap Z_Y) \leq g_K(X \cap Y)$ holds. Combining these inequalities, we obtain $g_K(X) + g_K(Y) \leq g_K(X \cup Y) + g_K(X \cap Y)$. □

Let $\pi : U \to [k]$ be a k-coloring. We say that a poset $P = (U, \preceq)$ is *consistent* with π if $u \prec v$ implies $\pi(u) < \pi(v)$ for every $u, v \in U$. For a consistent poset P and a subset $K \subseteq U$, the *reduction* of π by K in P is the k-coloring $\pi_K : U \setminus K \to [k]$ defined by

$$\pi_K(u) = \begin{cases} \pi(u) - 1 & (\exists v \in K : v \prec u), \\ \pi(u) & (\text{otherwise}). \end{cases}$$

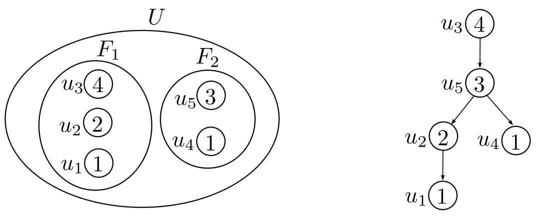
Note that every $u \in U \setminus K$ indeed satisfies $\pi_K(u) \geq 1$ because of the consistency of P.

Definition 2.2. A *skeleton poset* of (π, g) is a poset $P = (U, \preceq)$ which is consistent with π and satisfies the following condition: For every antichain K in P, the reduction of π by K in P dominates the reduction of g by K. ■

Here, we provide a sufficient condition for the existence of a skeleton poset. We call π a *dominating k-coloring* if it dominates g. A dominating k-coloring π is called *minimal* if there is no dominating k-coloring $\tilde{\pi} : U \to [k]$ such that $\tilde{\pi}(u) \leq \pi(u)$ for every $u \in U$ and $\tilde{\pi}(v) < \pi(v)$ for some $v \in U$.

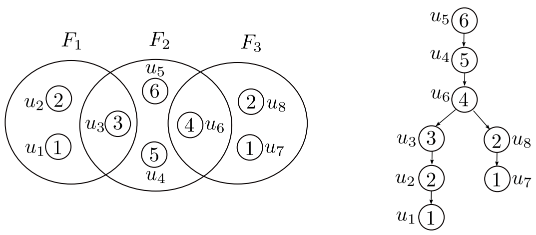
Proposition 2.3. For every intersecting-supermodular function $g : \mathcal{F} \to \mathbf{Z}$ and every minimal dominating k-coloring $\pi : U \to [k]$ for g, there exists a skeleton poset P of (π, g). ■

The proof of Proposition 2.3 is postponed to Section 4. Instead, we demonstrate some examples of skeleton posets below.



(a) (b)

Figure 1: (a) The family $\mathcal{F}$ in Example 2.5. The value $\pi(u)$ of each $u \in U$ is written in the circle corresponding to u. (b) The Hasse diagram which defines a skeleton poset of (π, g) in Example 2.5.



(a) (b)

Figure 2: (a) The family $\mathcal{F}$ in Example 2.6. The value $\pi(u)$ of each $u \in U$ is written in the circle corresponding to u. (b) The Hasse diagram which defines a skeleton poset of (π, g) in Example 2.6.

Example 2.4. Let $\mathcal{F}$ be a partition of U and π be a minimal dominating k-coloring for a function $g : \mathcal{F} \to \mathbf{Z}$. By the minimality, in each $X \in \mathcal{F}$, exactly one element $u \in X$ satisfies $\pi(u) = j$ for each $j \in \{2, 3, \ldots, g(X)\}$ and other elements u satisfy $\pi(u) = 1$.

Define $\prec$ so that $u, v \in U$ satisfy $u \prec v$ if and only if $u, v \in X$ for some $X \in \mathcal{F}$ and $\pi(u) < \pi(v)$. Let $u \preceq v$ mean $u \prec v$ or $u = v$. Then, $P = (U, \preceq)$ is a skeleton poset of (π, g).

Example 2.5. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $\mathcal{F} = \{F_1, F_2, U\}$, where $F_1 = \{u_1, u_2, u_3\}$, $F_2 = \{u_4, u_5\}$. Define $g : \mathcal{F} \to \mathbf{Z}$ by $g(F_1) = 3$, $g(F_2) = 2$, $g(U) = 4$ and $\pi : U \to [k]$ by $(\pi(u_1), \pi(u_2), \ldots, \pi(u_5)) = (1, 2, 4, 1, 3)$, where $k \geq 4$ (see Figure 1 (a)). Then, π is a minimal dominating k-coloring for g. Let $P = (U, \preceq)$ be a poset whose Hasse diagram is depicted in Figure 1 (b). We can check that P is a skeleton poset of (π, g).

Example 2.6. Let $U = \{\, u_1, u_2, \ldots, u_8 \,\}$. Let $F_1 = \{u_1, u_2, u_3\}$, $F_2 = \{u_3, u_4, u_5, u_6\}$, $F_3 = \{u_6, u_7, u_8\}$ and $\mathcal{F} = \{F_1, F_2, F_3, F_1 \cap F_2, F_2 \cap F_3, F_1 \cup F_2, F_2 \cup F_3, U\}$. Define $g : \mathcal{F} \to \mathbf{Z}$ by $g(F_1) = 3$, $g(F_2) = 2$, $g(F_3) = 3$, $g(F_1 \cap F_2) = g(F_2 \cap F_3) = 1$, $g(F_1 \cup F_2) = g(F_2 \cup F_3) = 4$, and $g(U) = 6$. Define $\pi : U \to [k]$ by $(\pi(u_1), \pi(u_2), \ldots, \pi(u_8)) = (1, 2, 3, 5, 6, 4, 1, 2)$, where $k \geq 6$ (see Figure 2 (a)). Then, π is a minimal dominating k-coloring for g. Let $P = (U, \preceq)$ be a poset whose Hasse diagram is depicted in Figure 2 (b). We see that P is a skeleton poset of (π, g).

3 Proof

In this section, we give a proof to Theorem 1.4 relying on Theorem 1.6 and Proposition 2.3. Let $g_1 : \mathcal{F}_1 \to \mathbf{Z}$ and $g_2 : \mathcal{F}_2 \to \mathbf{Z}$ be intersecting-supermodular functions on $\mathcal{F}_1, \mathcal{F}_2 \subseteq 2^U$ and let $k \in \mathbf{Z}_{>0}$.

Lemma 3.1. If $\pi_1, \pi_2 : U \to [k]$ dominate g_1 and g_2, respectively, then for any nonempty subset $S \subseteq U$, there exist nonempty $K \subseteq S$ and k-colorings $\pi_1', \pi_2' : U \setminus K \to [k]$ that satisfy the following conditions.

(a) For every $u \in U \setminus K$, we have $\pi'_1(u) + \pi'_2(u) \leq \pi_1(u) + \pi_2(u)$.
Moreover, $u \in S \setminus K$ implies $\pi'_1(u) + \pi'_2(u) < \pi_1(u) + \pi_2(u)$.

(b) For each $i \in \{1, 2\}$, π'_i dominates the reduction of g_i by K.

Proof. For each $i \in \{1, 2\}$, since π_i dominates g_i, there is a minimal dominating k-coloring $\hat{\pi}_i : U \to [k]$ with $\hat{\pi}_i \leq \pi_i$. By Proposition 2.3, there is a skeleton poset $P_i = (U, \preceq_i)$ of $(\hat{\pi}_i, g_i)$ for each i. Take any nonempty $S \subseteq U$ and apply Theorem 1.6 to P_1, P_2, and S. Then, we obtain a kernel K of S. That is, $K \subseteq S$ is a common antichain and every $u \in S \setminus K$ admits some $v \in K$ such that $v \prec_1 u$ or $v \prec_2 u$. Let $\pi'_i : U \setminus K \to [k]$ be a reduction of $\hat{\pi}_i$ by K in P_i. Then $\pi'_1(u) + \pi'_2(u) \leq \hat{\pi}_1(u) + \hat{\pi}_2(u)$ for every $u \in U \setminus K$ and the inequality holds for every $u \in S \setminus K$ by the definition of a kernel. As $\hat{\pi}_1(u) \leq \pi_1(u)$, $\hat{\pi}_2(u) \leq \pi_2(u)$ for every $u \in U$, condition (a) follows. Since P_i is a skeleton poset, π'_i dominates the reduction of g_i by K. $\square$

Recall that Σ is a set of colors and $L : U \to 2^\Sigma$ is an assignment of color lists to elements.

Proposition 3.2. For $L : U \to 2^\Sigma$, assume that there exist k-colorings $\pi_1, \pi_2 : U \to [k]$ satisfying the following conditions.

(i) For every $u \in U$, we have $\pi_1(u) + \pi_2(u) - 1 \leq |L(u)|$.

(ii) For each $i \in \{1, 2\}$, π_i dominates g_i.

Then there exists a list supermodular coloring $\varphi : U \to \Sigma$ for (g_1, g_2, L).

Proof. We show this by induction on $|U|$, i.e., the size of the ground set. If $|U| = 1$, the statement is obvious.

If $|U| > 1$, take some $l \in \bigcup \{ L(u) \mid u \in U \}$ and let $S := \{ u \in U \mid l \in L(u) \}$. By Lemma 3.1, there exist nonempty $K \subseteq S$ and $\pi'_1, \pi'_2 : U \setminus K \to [k]$ satisfying (a) and (b). For each $i \in \{1, 2\}$, let g'_i denote the reduction of g_i by K. Then, g'_i is intersecting supermodular by Claim 2.1, and π'_i dominates g'_i by (b). Define $L' : U \setminus K \to 2^\Sigma$ by $L'(u) = L(u) \setminus \{l\}$ for each $u \in U \setminus K$. It then follows from (a) that $\pi'_1(u) + \pi'_2(u) - 1 \leq |L'(u)|$ for every $u \in U \setminus K$. Thus, π'_1, π'_2 satisfy (i) and (ii) with $(U \setminus K, g'_1, g'_2, L')$ in place of (U, g_1, g_2, L). By the inductive assumption, there exists a list supermodular coloring $\varphi' : U \setminus K \to \Sigma$ for (g'_1, g'_2, L'). Define $\varphi : U \to \Sigma$ by

$$\varphi(u) = \begin{cases} \varphi'(u) & (u \in U \setminus K), \\ l & (u \in K). \end{cases}$$

Then, clearly $\varphi(u) \in L(u)$ for every $u \in U$. We see that every $X \in \mathcal{F}$ with $X \cap K \neq \emptyset$ satisfies $|\varphi(X)| = |\varphi'(X \setminus K)| + 1 \geq g'(X \setminus K) + 1 \geq g(X)$, and every $X \in \mathcal{F}$ with $X \cap K = \emptyset$ satisfies $|\varphi(X)| = |\varphi'(X \setminus K)| \geq g'(X \setminus K) \geq g(X)$. Thus φ is a list supermodular coloring for (g'_1, g'_2, L). $\square$

Proof of Theorem 1.4. Recall that L satisfies $|L(u)| = k$ for every $u \in U$. Also, we are provided a supermodular k-coloring $\pi : U \to [k]$ which dominates both g_1 and g_2. Let $\pi_1(u) := \pi(u)$ and $\pi_2(u) := k + 1 - \pi(u)$ for every $u \in U$. They satisfy the condition (i) of Proposition 3.2 as $\pi_1(u) + \pi_2(u) - 1 = k = |L(u)|$ for every u. Also (ii) holds as $|\pi(X)| = |\pi_1(X)| = |\pi_2(X)|$ for every $X \subseteq U$. Proposition 3.2 then implies the statement of Theorem 1.4. $\square$

4 Existence of Skeleton Posets

Let $g : \mathcal{F} \to \mathbf{Z}$ be an intersecting-supermodular function and $\pi : U \to [k]$ be a minimal dominating k-coloring for g. In this section, we prove Proposition 2.3 by constructing a skeleton poset $P = (U, \preceq)$ of (π, g). We first define the poset and then show that it is indeed a skeleton poset of (π, g).

4.1 Poset Construction

We call a subset $X \in \mathcal{F}$ *tight* if $|\pi(X)| = g(X)$ holds. Note that the function $|\pi(\cdot)| : 2^U \to \mathbf{Z}$ is *submodular*, that is, $|\pi(X)| + |\pi(Y)| \geq |\pi(X \cup Y)| + |\pi(X \cap Y)|$ for any $X, Y \subseteq 2^U$. This implies the following fact.

Claim 4.1. If $X, Y \in \mathcal{F}$ are tight and intersecting, then $X \cup Y$, $X \cap Y \in \mathcal{F}$ are also tight.

Proof. Since $|\pi(\cdot)|$ is submodular and π dominates g, we have

$$g(X) + g(Y) = |\pi(X)| + |\pi(Y)| \geq |\pi(X \cup Y)| + |\pi(X \cap Y)| \geq g(X \cup Y) + g(X \cap Y).$$

As g is intersecting-supermodular, $g(X \cup Y) + g(X \cap Y) \geq g(X) + g(Y)$ also holds. Then, all above equalities hold and we obtain $|\pi(X \cup Y)| = g(X \cup Y)$ and $|\pi(X \cap Y)| = g(X \cap Y)$. □

Claim 4.2. If $X, Y \in \mathcal{F}$ are tight and intersecting, then $\pi(X) \cap \pi(Y) = \pi(X \cap Y)$.

Proof. Clearly, $\pi(X \cap Y) \subseteq \pi(X) \cap \pi(Y)$. We then show $|\pi(X \cap Y)| = |\pi(X) \cap \pi(Y)|$ to complete the proof. As shown in the proof of Claim 4.1, $|\pi(X)| + |\pi(Y)| = |\pi(X \cup Y)| + |\pi(X \cap Y)|$. Also, we see $\pi(X) \cup \pi(Y) = \pi(X \cup Y)$. These imply $|\pi(X \cap Y)| = |\pi(X)| + |\pi(Y)| - |\pi(X \cup Y)| = |\pi(X)| + |\pi(Y)| - |\pi(X) \cup \pi(Y)| = |\pi(X) \cap \pi(Y)|$. □

Claim 4.3. For any $u \in U$ with $\pi(u) > 1$ and $j \in \{1, \ldots, \pi(u) - 1\}$, there exists $F_j \in \mathcal{F}$ which satisfies $u \in F_j$, $|\pi(F_j)| = g(F_j)$, $\pi(F_j - u) \not\ni \pi(u)$, and $\pi(F_j) \ni j$.

Proof. Let $\pi' : U \to [k]$ be a k-coloring such that $\pi'(v) = \pi(v)$ for every $v \in U \setminus \{u\}$ and $\pi'(u) = j$. Since π is a minimal dominating k-coloring, π' does not dominate g. Hence there exists F_j such that $|\pi'(F_j)| < g(F_j)$. As $|\pi(F_j)| \geq g(F_j)$ holds, we have $|\pi'(F_j)| < |\pi(F_j)|$, which implies the four conditions in the statement. □

Claim 4.4. For any $u \in U$ with $\pi(u) > 1$, there exist one or more $F \in \mathcal{F}$ such that

$$u \in F, \tag{4.1}$$
$$|\pi(F)| = g(F), \tag{4.2}$$
$$\pi(F - u) \not\ni \pi(u), \tag{4.3}$$
$$\pi(F) \supseteq \{1, 2, \ldots, \pi(u)\}. \tag{4.4}$$

Furthermore, among all such $F \in \mathcal{F}$, there exists a unique minimal one.

Proof. For each $j \in \{1, 2, \ldots, \pi(u) - 1\}$, let $F_j \in \mathcal{F}$ be a subset which satisfies four conditions in Claim 4.3. Then $F := \bigcup \{ F_j \mid j = 1, 2, \ldots, \pi(u) - 1 \}$ satisfies (4.1)–(4.4). Condition (4.2) follows from Claim 4.1 since all F_j contain u. Other three are clear by definition. To show the existence of the minimum, we show that, if both F and F' satisfy (4.1)–(4.4), then so does $F \cap F'$. By definition, (4.1) and (4.3) are clear. Claims 4.1 and 4.2 imply (4.2) and (4.4), respectively. □

For any $u \in U$ with $\pi(u) > 1$, denote by $D(u)$ the unique minimal $F \in \mathcal{F}$ satisfying (4.1)–(4.4). For $u \in U$ with $\pi(u) = 1$, let $D(u)$ be $\{u\}$. Define $\prec$ by

$$u \prec v \iff [D(u) \subseteq D(v),\ \pi(u) < \pi(v)]$$

and let $u \preceq v$ mean $u \prec v$ or $u = v$. Then, we see that $\preceq$ is a partial order. Let $P = (U, \preceq)$.

Claim 4.5. If $D(u) \cap D(v) \neq \emptyset$, then u and v are comparable.

Proof. Let $u \neq v$ since otherwise the claim is obvious. We assume $\pi(u) \leq \pi(v)$ without loss of generality. By Claims 4.1 and 4.2, $D(u) \cap D(v)$ is tight and satisfies $\pi(D(u) \cap D(v)) \supseteq \{1, 2, \dots, \pi(u)\}$. The latter implies $D(u) \cap D(v) \ni u$ since $D(u)$ satisfies $\pi(D(u) - u) \not\ni \pi(u)$. Thus, conditions (4.1)–(4.4) hold with $F = D(u) \cap D(v)$. By the minimality of $D(u)$, this implies $D(u) \cap D(v) = D(u)$, and hence $D(u) \subseteq D(v)$. Also, as $D(v)$ satisfies $\pi(D(v) - v) \not\ni \pi(v)$, the condition $u \in D(u) \subseteq D(v)$ implies $\pi(u) \neq \pi(v)$, and hence $\pi(u) < \pi(v)$. Thus, $u \prec v$ holds. $\square$

By Claim 4.5, $D(u) \cap D(v) \neq \emptyset$ implies $D(u) \subseteq D(v)$ or $D(u) \supseteq D(v)$, i.e., the family $\{\, D(u) \mid u \in U \,\}$ forms a *laminar family.*

Claim 4.6. For any $u \in U$ with $\pi(u) > 1$, there exists $v \in U$ with $\pi(v) = \pi(u) - 1$ and $v \prec u$.

Proof. Since (4.4) holds with $F = D(u)$, there is $v \in D(u)$ with $\pi(v) = \pi(u) - 1$. As $v \in D(v) \cap D(u) \neq \emptyset$ and $\pi(v) < \pi(u)$, Claim 4.5 implies $v \prec u$. $\square$

Claim 4.7. If $v \preceq u$, then $v \in D(u)$. Conversely, if $v \in D(u)$, then $v \preceq u$ or $u \prec v$.

Proof. The condition $v \preceq u$ implies $v \in D(v) \subseteq D(u)$, and the first claim holds. Also, $v \in D(u)$ implies $v \in D(v) \cap D(u) \neq \emptyset$, and hence v is comparable with u by Claim 4.5. $\square$

Claim 4.8. If $u \preceq v$ and $u \preceq w$, then v and w are comparable.

Proof. Since $D(u) \subseteq D(v) \cap D(w) \neq \emptyset$, Claim 4.5 implies the statement. $\square$

Claim 4.8 implies that the Hasse diagram of $P = (U, \preceq)$ forms a *branching*, i.e., a collection of rooted directed trees.

Claim 4.9. For each $u \in U$ with $\pi(u) > 1$, let $C(u)$ be a maximal chain included in $D(u)$. Then, the following statements hold:

- $\pi(C(u)) \supseteq \{1, 2, \dots, \pi(u)\}$,
- $\pi(D(u) \setminus C(u)) \subseteq \{1, 2, \dots, \pi(u) - 1\}$,
- $\pi(C(u)) = \pi(D(u))$ and $g(D(u)) = |C(u)|$.

Proof. The first statement follows from Claims 4.6 and 4.7. The second one follows from Claims 4.7, 4.8, and the maximality of $C(u)$. From these two, we have $\pi(C(u)) = \pi(D(u))$. As $C(u)$ is a chain, we have $|C(u)| = |\pi(C(u))| = |\pi(D(u))|$. Since $D(u)$ is tight, this equals $g(D(u))$. $\square$

The following fact will be useful later.

Claim 4.10. Assume that $u, v \in U$ satisfies $\pi(v) = \pi(u) - 1$. If $X \in \mathcal{F}$ is tight and $\{u, v\} \subseteq X$ holds, then we have $D(u) \setminus D(v) \subseteq X$.

Proof. Note that $D(v)$ is a singleton or a member of $\mathcal{F}$. Also, we have $v \in D(v) \cap X \neq \emptyset$. Then $D(v) \cup X$ is a member of $\mathcal{F}$ and tight. As $\pi(u) > 1$, the set $D(u)$ is also in $\mathcal{F}$ and tight. Then, the nonemptyset $F := (D(v) \cup X) \cap D(u) \ni u$ is also a member of $\mathcal{F}$ and tight. Note that then conditions (4.1)–(4.4) hold, where (4.4) follows from Claim 4.2 and the condition $\pi(v) = \pi(u) - 1$. Therefore, the minimality of $D(u)$ implies $D(u) \subseteq F$, which yields $D(u) \setminus D(v) \subseteq X$. $\square$

4.2 Reduction by an Antichain

We now show that $P = (U, \preceq)$ is indeed a skeleton poset of (π, g). Clearly P is consistent with π, i.e., $u \prec v$ implies $\pi(u) < \pi(v)$. We now show that, for any antichain K in P, the reduction of π by K dominates the reduction g_K of g by K.

Take an antichain $K \subseteq U$. Let $\pi_K : U \setminus K \to [k]$ be the reduction of π by K in P, i.e.,

$$\pi_K(u) = \begin{cases} \pi(u) - 1 & (\exists v \in K : v \prec u), \\ \pi(u) & (\text{otherwise}). \end{cases}$$

To prove that π_K dominates g_K, it suffices to show that $|\pi_K(X \setminus K)| \geq \hat{g}_K(X)$ holds for every $X \in \mathcal{F}$, where $\hat{g}_K : \mathcal{F} \to \mathbf{Z}$ is defined by $\hat{g}_K(X) = g(X) - 1$ for $X \in \mathcal{F}$ with $X \cap K \neq \emptyset$ and $\hat{g}_K(X) = g(X)$ for $X \in \mathcal{F}$ with $X \cap K = \emptyset$.

The definition of π_K implies the following observation.

Claim 4.11. For any chain $C \subseteq U$, exactly one of the following holds.

1. $|C \cap K| = 1$ and $\pi_K(u) \neq \pi_K(v)$ for every distinct $u, v \in C \setminus K$. Hence $|\pi_K(C \setminus K)| = |C| - 1$.
2. $|C \cap K| = 0$ and $\pi_K(u) \neq \pi_K(v)$ for every distinct $u, v \in C \setminus K$. Hence $|\pi_K(C \setminus K)| = |C|$.
3. $|C \cap K| = 0$ and just one pair of $u, v \in C$ satisfies $\pi_K(u) = \pi_K(v)$. Hence $|\pi_K(C \setminus K)| = |C| - 1$. If $\pi(v) \leq \pi(u)$ for such $u, v \in C$, then $\pi(v) = \pi(u) - 1$ and $(D(u) \setminus D(v)) \cap K \neq \emptyset$.

Proof. The only point to concern is the last statement in the third case. Since C is a chain, the condition $\pi_K(u) = \pi_K(v)$ and the definition of π_K imply $\pi(v) = \pi_K(v) = \pi_K(u) = \pi(u) - 1$. By $\pi_K(u) = \pi(u) - 1$, there exists $w \in K$ with $w \prec u$, which implies $w \in D(u)$ by Claim 4.7. We now prove $w \notin D(v)$ which completes the proof. Note that $w \prec u$ implies $\pi(w) < \pi(u)$, and hence $\pi(w) \leq \pi(u) - 1 = \pi(v)$. Suppose, to the contrary, $w \in D(v)$. If $\pi(w) = \pi(v)$, then $\pi(D(v) - v) \not\ni \pi(v)$ implies $w = v$, which contradicts $w \in K$, $v \in C$, and $C \cap K = \emptyset$. If $\pi(w) < \pi(v)$, then $w \prec v$ holds, which contradicts $\pi_K(v) = \pi(v)$ by the definition of π_K. Thus, we obtain $w \notin D(v)$. □

Lemma 4.12. For $X \in \mathcal{F}$, if there is a chain $C \subseteq X$ with $\pi(C) = \pi(X)$, then we have $|\pi_K(X \setminus K)| \geq \hat{g}_K(X)$.

Proof. Since C is a chain and $\pi(C) = \pi(X)$, we have $|C| = |\pi(C)| = |\pi(X)| \geq g(X)$. Let us consider the three cases described in Claim 4.11.

In the first case, we have $C \cap K \neq \emptyset$ and $|\pi_K(C \setminus K)| = |C| - 1$. Since $X \cap K \supseteq C \cap K \neq \emptyset$ implies $\hat{g}_K(X) = g(X) - 1$, we have $|\pi_K(X \setminus K)| \geq |\pi_K(C \setminus K)| = |C| - 1 \geq g(X) - 1 = \hat{g}_K(X)$.

In the second case, we have $C \cap K = \emptyset$ and $|\pi_K(C \setminus K)| = |C|$, which together with $g(X) \geq \hat{g}_K(X)$ imply $|\pi_K(X \setminus K)| \geq |\pi_K(C \setminus K)| = |C| \geq g(X) \geq \hat{g}_K(X)$.

In the third case, we have $|\pi_K(C \setminus K)| = |C| - 1$, $\pi(v) = \pi(u) - 1$, and $(D(u) \setminus D(v)) \cap K \neq \emptyset$. If X is not tight, then $|C| = |\pi(X)| > g(X)$, and hence $|\pi_K(X \setminus K)| \geq |\pi_K(C \setminus K)| = |C| - 1 \geq g(X) \geq \hat{g}_K(X)$. If X is tight, then Claim 4.10 and $\pi(v) = \pi(u) - 1$ imply $D(u) \setminus D(v) \subseteq X$. Combined with $(D(u) \setminus D(v)) \cap K \neq \emptyset$, this implies $X \cap K \neq \emptyset$, and hence $\hat{g}_K(X) = g(X) - 1$. Thus, we obtain $|\pi_K(X \setminus K)| \geq |\pi_K(C \setminus K)| = |C| - 1 \geq g(X) - 1 = \hat{g}_K(X)$. □

Lemma 4.13. For $u \in U$ with $\pi(u) > 1$, the set $D(u) \in \mathcal{F}$ satisfies $|\pi_K(D(u) \setminus K)| = \hat{g}_K(D(u))$.

Proof. By Claim 4.9, a maximal chain $C(u)$ in $D(u)$ satisfies $\pi(C(u)) = \pi(D(u))$ and $|C(u)| = g(D(u))$. Then Lemma 4.12 implies $|\pi_K(D(u) \setminus K)| \geq \hat{g}_K(D(u))$. We then intend to show $|\pi_K(D(u) \setminus K)| \leq \hat{g}_K(D(u))$.

Claim 4.9 says $\pi(D(u) \setminus C(u)) \subseteq \{1, 2, \ldots, \pi(u) - 1\}$ and $\pi(C(u)) \supseteq \{1, 2, \ldots, \pi(u)\}$, which imply $\pi_K((D(u) \setminus C(u)) \setminus K) \subseteq \{1, 2, \ldots, \pi(u) - 1\} \subseteq \pi_K(C(u) \setminus K)$ by the definition of π_K.

Hence, we have $\pi_K(D(u)\setminus K) = \pi_K(C(u)\setminus K)$, which yields $|\pi_K(D(u)\setminus K)| = |\pi_K(C(u)\setminus K)| \leq |C(u)| = g(D(u))$. In particular, if $D(u)\cap K = \emptyset$, then $|\pi_K(D(u)\setminus K)| \leq g(D(u)) = \hat{g}_K(D(u))$.

We now consider the case of $D(u)\cap K \neq \emptyset$. As we have $|\pi_K(D(u)\setminus K)| = |\pi_K(C(u)\setminus K)|$ and $g(D(u)) = |C(u)|$, it suffices to show $|\pi_K(C(u)\setminus K)| < |C(u)|$. If $C(u)\cap K \neq \emptyset$, this is clear. Assume $C(u)\cap K = \emptyset$, and then $D(u)\cap K \neq \emptyset$ implies $(D(u)\setminus C(u))\cap K \neq \emptyset$. As we have $D(u)\setminus C(u) \subseteq \{\, v \in U \mid v \prec u \,\}$ by Claims 4.7 and 4.9, we obtain $\{\, v \in U \mid v \prec u \,\} \cap K \neq \emptyset$, and hence $\pi_K(u) = \pi(u) - 1$. Since $v \preceq u$ implies $\pi_K(v) \leq \pi_K(u)$ for any v, the subset $C' := \{\, v \in C(u) \mid v \preceq u \,\}$ satisfies $\pi_K(C') \subseteq \{1, 2, \ldots, \pi(u)-1\}$. This implies $|\pi_K(C')| < \pi(u) = |C'|$, where the last equality follows from Claim 4.9. Therefore, some pair of distinct $v, w \in C' \subseteq C(u)$ satisfies $\pi_K(v) = \pi_K(w)$, and hence $|\pi_K(C(u)\setminus K)| = |\pi_K(C(u))| < |C(u)|$, which is the desired conclusion. □

Proposition 4.14. Every $X \in \mathcal{F}$ satisfies $|\pi_K(X\setminus K)| \geq \hat{g}_K(X)$.

Proof. The proof is by induction w.r.t. set inclusion.

First, consider the case in which $X \in \mathcal{F}$ is minimal, i.e., there is no $Y \in \mathcal{F}$ with $Y \subsetneq X$. Then, every $u \in X$ with $\pi(u) > 1$ satisfies $X \subseteq D(u)$ since otherwise we have $u \in X\cap D(u) \subsetneq X$ and $X \cap D(u) \in \mathcal{F}$, which contradict the minimality of X. Also $u \in X$ with $\pi(u) = 1$ satisfies $D(u) = \{u\} \subseteq X$. Then, every pair of $u, v \in X$ with $\pi(u) < \pi(v)$ satisfies $X \subseteq D(u)\cap D(v) \neq \emptyset$ or $\{u\} \subseteq X \subseteq D(v)$. In either case, we have $u \prec v$ by Claim 4.5. That is, every pair of elements is comparable if their values of π are different. Hence, there is a chain $C \subseteq X$ such that $\pi(C) = \pi(X)$, which implies $|\pi_K(X\setminus K)| \geq \hat{g}_K(X)$ by Lemma 4.12.

We now intend to show $|\pi_K(X\setminus K)| \geq \hat{g}_K(X)$, assuming inductively that $|\pi_K(Y\setminus K)| \geq \hat{g}_K(Y)$ holds for every $Y \in \mathcal{F}$ with $Y \subsetneq X$.

We start with the case in which every $u \in X$ satisfies $X \subseteq D(u)$. For $u, v \in X$ with $\pi(u) < \pi(v)$, we have $X \subseteq D(u)\cap D(v) \neq \emptyset$, which implies $u \prec v$ by Claim 4.5. Then, there is a chain $C \subseteq X$ such that $\pi(C) = \pi(X)$, and hence $|\pi_K(X\setminus K)| \geq \hat{g}_K(X)$ by Lemma 4.12.

We now consider the case in which some $u \in X$ satisfies $X \not\subseteq D(u)$. Among all such elements, let $u \in X$ maximize $\pi(u)$. Then, every $v \in X$ with $\pi(v) > \pi(u)$ satisfies $u \in X \subseteq D(v)$, and hence $v \succ u$ by Claim 4.7. Recall that every $v \in D(u)$ with $\pi(v) > \pi(u)$ also satisfies $v \succ u$. Then, $C := \{\, v \in X \cup D(u) \mid \pi(v) > \pi(u) \,\} \cup \{u\}$ forms a chain whose minimum is u. Let $\hat{C}$ be a maximal chain subject to $C \subseteq \hat{C} \subseteq X \cup D(u)$. The maximality and Claim 4.9 imply $\pi(\hat{C}) \supseteq \{1, 2, \ldots, \pi(u)\}$. Therefore, we have $\pi(\hat{C}) \supseteq \pi(C) \cup \{1, 2, \ldots, \pi(u)\} \supseteq \pi(X \cup D(u))$. Since $\hat{C} \subseteq X \cup D(u)$, this means $\pi(\hat{C}) = \pi(X \cup D(u))$. Lemma 4.12 then implies $|\pi_K((X \cup D(u))\setminus K)| \geq \hat{g}_K(X \cup D(u))$. We also have $|\pi_K(D(u)\setminus K)| = \hat{g}_K(D(u))$ by Lemma 4.13 and $|\pi_K((X\cap D(u))\setminus K)| \geq \hat{g}_K(X\cap D(u))$ by the inductive assumption. Since $|\pi_K(\,\cdot\setminus K)| : 2^U \to \mathbf{Z}$ is submodular and $\hat{g}_K$ is intersecting supermodular, we obtain

$$
\begin{aligned}
|\pi_K(X\setminus K)| &\geq |\pi_K((X \cup D(u))\setminus K)| + |\pi_K((X \cap D(u))\setminus K)| - |\pi_K(D(u)\setminus K)| \\
&\geq \hat{g}_K(X \cup D(u)) + \hat{g}_K(X \cap D(u)) - \hat{g}_K(D(u)) \\
&\geq \hat{g}_K(X),
\end{aligned}
$$

which completes the proof. □

5 Extension to Skew-supermodular Coloring

This section extends Theorem 1.4 to the setting of skew-supermodular coloring. A function $g\colon 2^U \to \mathbf{Z} \cup \{-\infty\}$ is called *skew-supermodular* if every pair of $X, Y \subseteq U$ satisfies either the supermodular inequality or the *negamodular inequality* $g(X) + g(Y) \leq g(X\setminus Y) + g(Y\setminus X)$. By definition, skew-supermodularity is a generalization of intersecting supermodularity. It is known that Theorem 1.3 remains true for a pair of skew-supermodular functions [5].

For a skew-supermodular function, define its reduction by a subset $K \subseteq U$ as in Section 2. Then, we can confirm that is again a skew-supermodular. Also, proofs in Section 3 do not depend on intersecting supermodularity. Therefore, to extend Theorem 1.4, it suffices to show the existence of skeleton posets for skew-supermodular functions. Let $g\colon 2^U \to \mathbf{Z} \cup \{-\infty\}$ be a skew-supermodular function and $\pi : U \to [k]$ be a minimal dominating k-coloring for g.

Claim 5.1. If two sets $X, Y \subseteq U$ are tight and satisfy the negamodular inequality of g, then $X \setminus Y$ and $Y \setminus X$ are also tight and we have $\pi(X \setminus Y) = \pi(X)$ and $\pi(Y \setminus X) = \pi(Y)$.

Proof. Since π dominates g and $\pi(X) \supseteq \pi(X \setminus Y)$, $\pi(Y) \supseteq \pi(Y \setminus X)$ clearly hold,

$$g(X) + g(Y) = |\pi(X)| + |\pi(Y)| \geq |\pi(X \setminus Y)| + |\pi(Y \setminus X)| \geq g(X \setminus Y) + g(Y \setminus X).$$

As we have $g(X \setminus Y) + g(Y \setminus X) \geq g(X) + g(Y)$, all the above equalities hold. Hence $|\pi(X)| = |\pi(X \setminus Y)| = g(X \setminus Y)$ and $|\pi(Y)| = |\pi(Y \setminus X)| = g(Y \setminus X)$, which imply the required formulas. □

Claim 5.2. If the conditions (4.1)–(4.3) hold with $F = X$ and $F = Y$, then X and Y satisfy the supermodular inequality of g.

Proof. Suppose, to the contrary, the supermodular inequality fails to hold. Since g is skew-supermodular, we then obtain the negamodular inequality. As X, Y are tight by (4.2), Claim 5.1 implies $\pi(X \setminus Y) = \pi(X)$. By (4.1), we have $u \in X \cap Y$, and hence $X \setminus Y \subseteq X - u$. Then (4.3) with $F = X$ implies $\pi(u) \notin \pi(X \setminus Y)$. Thus, we obtain $\pi(u) \in \pi(X) \setminus \pi(X \setminus Y)$, which contradicts $\pi(X \setminus Y) = \pi(X)$. □

Claim 5.2 enables us to extend Claim 4.4 for skew-supermodular functions. Therefore, we can define $D(u)$ for each $u \in U$ similarly to the case of intersecting-supermodular functions.

Claim 5.3. For any $u \in U$ and any tight set $X \subseteq U$, we see that $D(u)$ and X satisfy the supermodular inequality of g.

Proof. If $D(u) \subseteq X$, the claim is obvious. We assume $D(u) \not\subseteq X$, and hence $D(u) \cap X \subsetneq D(u)$. Suppose, to the contrary, the supermodular inequality fails to hold, and then the negamodular inequality holds. Since $D(u)$ and X are tight, Claim 5.1 implies $|\pi(D(u) \setminus X)| = g(D(u) \setminus X)$ and $\pi(D(u) \setminus X) = \pi(D(u))$. The latter implies $u \in D(u) \setminus X$ because we have $\pi(D(u) - u) \not\ni \pi(u)$. We then see that the conditions (4.1)–(4.4) holds with $F = D(u) \setminus X \subsetneq D(U)$, where (4.4) follows from $\pi(D(u) \setminus X) = \pi(D(u))$. This contradicts the minimality of $D(u)$. □

Claim 5.3 says that, for $D(u)$ and tight set X, the skew-supermodularity implies the supermodular inequality. Observe that, in the proofs after Claim 4.4, we apply the supermodular inequality only for such pairs of subsets. Thus, the same arguments work for skew-supermodular functions, and we obtain the following extension of Theorem 1.4.

Theorem 5.4. For skew-supermodular functions $g_1, g_2 : 2^U \to \mathbf{Z} \cup \{-\infty\}$ and $k \in \mathbf{Z}_{>0}$, assume that there exists a k-coloring which dominates both g_1 and g_2. If L satisfies $|L(u)| = k$ for each $u \in U$, then there exists a coloring $\varphi : U \to \Sigma$ such that every $u \in U$ satisfies $\varphi(u) \in L(u)$ and φ dominates both g_1 and g_2. ■

6 The List Coloring Version of Gupta's Theorem

While Kőnig showed that the minimum number of colors required for a bipartite edge coloring is equal to the maximum degree, Gupta showed that the maximum number of disjoint edge covers in a bipartite graph is equal to the minimum degree. More generally, Gupta showed the following theorem. We denote by $\delta_G(t)$ the set of edges incident to a vertex t in a graph G.

Theorem 6.1 (Gupta [9])**.** For a bipartite graph $G = (T, E)$ and $k \in \mathbf{Z}_{>0}$, there exists a function $\pi : E \to [k]$ such that $|\pi(\delta_G(t))| \geq \min\{k, |\delta_G(t)|\}$ for every $t \in T$. ∎

Theorem 6.1 is a special case of Schrijver's theorem as follows. Set $U := E$ and $\mathcal{F}_i := \{\, \delta_G(t) \mid t \in T_i \,\}$ for each $i \in \{1, 2\}$, where (T_1, T_2) is a bipartition of T into two independent sets. Set $g_i(\delta_G(t)) := \min\{k, |\delta_G(t)|\}$ for each $i \in \{1, 2\}$ and $t \in T_i$. We see that both g_1 and g_2 are intersecting-supermodular functions. Theorem 1.4 then naturally derives the list coloring version of Theorem 6.1. Here, however, we provide an alternative proof, which uses Theorem 1.1 rather than Theorem 1.4.

Corollary 6.2. For a bipartite graph $G = (T, E)$, assume that every edge e has a color list $L(e) \subseteq \Sigma$ with $|L(e)| = k$. Then, there exists a function $\varphi : E \to \Sigma$ such that $\varphi(e) \in L(e)$ for every $e \in E$ and $|\varphi(\delta_G(t))| \geq \min\{k, |\delta_G(t)|\}$ for every $t \in T$.

Proof. By Theorem 6.1, there is a k-coloring $\pi : E \to [k]$ such that $|\pi(\delta_G(t))| \geq \min\{k, |\delta_G(t)|\}$ for every $t \in T$. From G and π, we derive another bipartite graph $\hat{G} = (\hat{T}, \hat{E})$ and a k-coloring $\hat{\pi} : \hat{E} \to [k]$. First, set $\hat{G} := G$ and $\hat{\pi} := \pi$. Then, repeat the following procedure as long as $\hat{G}$ has a pair of adjacent edges whose values of $\hat{\pi}$ are the same.

> Take a vertex $t \in T$ and an edge $e \in \delta_{\hat{G}}(t)$ such that $\hat{\pi}(e) = \hat{\pi}(e')$ for some $e' \in \delta_{\hat{G}} \setminus \{e\}$. Add a new vertex $\hat{t}$ and replace $e = (t, s)$ by $\hat{e} = (\hat{t}, s)$, where s is another endpoint of e. Set $\hat{\pi}(\hat{e}) := \pi(e)$.

There is a one-to-one correspondence between the final edge set $\hat{E}$ and the original E, and the final vertex set $\hat{T}$ is a superset of T. By the definition of the procedure, $\hat{\pi}$ is an edge coloring of $\hat{G}$ and each $t \in T$ satisfies $|\delta_{\hat{G}}(t)| = |\pi(\delta_G(t))|$. Set $\hat{L}(\hat{e}) := L(e)$ for each corresponding edge pair $(\hat{e}, e) \in \hat{E} \times E$. We then have $|\hat{L}(\hat{e})| = k$ for every $\hat{e} \in \hat{E}$. Since $\hat{\pi}$ is an edge coloring of $\hat{G}$ with k colors, Theorem 1.1 implies that there is $\hat{\varphi} : \hat{E} \to \Sigma$ such that $\hat{\varphi}(\hat{e}) \in \hat{L}(\hat{e})$ for each $\hat{e} \in \hat{E}$ and $|\hat{\varphi}(\delta_{\hat{G}}(t))| = |\delta_{\hat{G}}(t)|$ for every $t \in \hat{T}$. Define $\varphi : E \to \Sigma$ by $\varphi(e) := \hat{\varphi}(\hat{e})$ for each corresponding edge pair $(\hat{e}, e) \in \hat{E} \times E$. For each $t \in T$, we have $\delta_G(t) \supseteq \delta_{\hat{G}}(t)$, where edges in E are identified with the corresponding edges in $\hat{E}$. Therefore, we obtain

$$|\varphi(\delta_G(t))| \geq |\hat{\varphi}(\delta_{\hat{G}}(t))| = |\delta_{\hat{G}}(t)| = |\pi(\delta_G(t))| \geq \min\{k, |\delta_G(t)|\}.$$

We also have $\varphi(e) = \hat{\varphi}(\hat{e}) \in \hat{L}(\hat{e}) = L(e)$ for each corresponding pair $(\hat{e}, e) \in \hat{E} \times E$. Thus, $\varphi : E \to \Sigma$ satisfies the required conditions. □

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