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Abstract

We propose a quadratically convergent algorithm for inverse symmetric eigenvalue problems based on matrix equations. The basic idea is seen in the latest study by Ogita and Aishima, while they derive an efficient iterative refinement algorithm for symmetric eigenvalue problems using special matrix equations. Since the proposed algorithm for the inverse eigenvalue problems can be viewed as the Newton method for the matrix equations, the quadratic convergence is naturally proved. Our algorithm is interpreted as an improved version of the Cayley transform method for the inverse eigenvalue problems. Although the Cayley transform method is one of the effective methods, the Cayley transform takes $\mathcal{O}(n^3)$ arithmetic operations to ensure the orthogonality of the iterative matrices. Our algorithm can refine the orthogonality without the Cayley transform, which reduces the operations in each iteration. It is worth noting that our approach overcomes the limitation of the Cayley transform method to the inverse standard eigenvalue problems, resulting in an extension to inverse generalized eigenvalue problems.

1 Introduction

Let $A_0, A_1, \ldots, A_n$ be real symmetric $n \times n$ matrices and $\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*$ be real numbers. In addition, let $\boldsymbol{c} = [c_1, \ldots, c_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\Lambda^* = \mathrm{diag}(\lambda_1^*, \ldots, \lambda_n^*)$. Define

$$A(\boldsymbol{c}) := A_0 + c_1 A_1 + \cdots + c_n A_n \tag{1}$$

and denote its eigenvalues by $\lambda_1(\boldsymbol{c}) \leq \lambda_2(\boldsymbol{c}) \leq \cdots \leq \lambda_n(\boldsymbol{c})$ in the ascending order. A typical inverse eigenvalue problem is to find $\boldsymbol{c}^* \in \mathbb{R}^n$ such that $\lambda_i(\boldsymbol{c}^*) = \lambda_i^*$ for all $1 \leq i \leq n$. Such inverse eigenvalue problems often arise in a variety of applications [5, 6, 7]. In this study, we focus on numerical algorithms for solving the inverse eigenvalue problems above. As in [1, 2, 3, 13], we assume that the prescribed eigenvalues are all distinct, i.e.,

$$\lambda_1^* < \lambda_2^* < \cdots < \lambda_n^*. \tag{2}$$

Let $X^* \in \mathbb{R}^{n\times n}$ denote an orthogonal matrix whose columns are the eigenvectors of $A(\boldsymbol{c}^*)$. Throughout the paper, I is an identity matrix and O is a zero matrix. For any matrix, let $\|\cdot\|$ denote the spectral norm and $[\cdot]_{ij}$ denote the (i,j) elements for $1 \le i,j \le n$.

In this paper, we propose a new iterative algorithm to solve the inverse eigenvalue problems. The basic idea to design the proposed algorithm is seen in the latest study by Ogita–Aishima [15], while they propose an efficient iterative refinement algorithm for symmetric eigenvalue problems. Similarly to [15], our algorithm for the inverse eigenvalue problems is derived as follows. For computed matrices $X^{(k)} \in \mathbb{R}^{n\times n}$ $(k = 0, 1, \ldots)$ in the iterative process, define $E^{(k)} \in \mathbb{R}^{n\times n}$ $(k = 0, 1, \ldots)$ such that $X^{(k)} = X^*(I + E^{(k)})$. Then we compute $\widetilde{E}^{(k)}$ approximating $E^{(k)}$ from the following relations:

$$\begin{cases} X^{*\mathrm{T}} X^* = I, \\ X^{*\mathrm{T}} A(\boldsymbol{c}^*) X^* = \Lambda^*. \end{cases} \tag{3}$$

After computing $\widetilde{E}^{(k)}$, we can update $X^{(k+1)} := X^{(k)}(I - \widetilde{E}^{(k)})$, where $I - \widetilde{E}^{(k)}$ is the first order approximation of $(I + \widetilde{E}^{(k)})^{-1}$ using the Neumann series. Under some conditions, we prove $E^{(k)} \to O$ and $X^{(k)} \to X^*$ as $k \to \infty$, where the convergence rates are quadratic. Our algorithm can be viewed as the Newton method for the matrix equations corresponding to (3).

In the research fields of the inverse eigenvalue problems, the strengths of the proposed algorithm are summarized as follows:

- The proposed algorithm is primarily comprising matrix multiplications, though a typical Newton method [13, Methd I] requires to solve eigenvalue problems in each iteration
- Although the proposed algorithm is similar to the Cayley transform method [13, Methd III], the proposed algorithm does not require the Cayley transform, which reduces the operations in each iteration.
- Our idea can be extended to inverse generalized eigenvalue problems in [8, 9, 10], while the Cayley transform method cannot be applied to such problems.

Moreover, we show a sufficient condition on the convergence of the proposed algorithm, and prove its quadratic convergence.

This paper is organized as follows. Section 2 is devoted to descriptions of the previous work relevant to this study, and to clarifications of the strengths of the proposed algorithm. In Section 3, we derive a new algorithm for the inverse eigenvalue problems. In Section 4, we provide a detailed convergence analysis of our algorithm. Moreover, we extend the proposed algorithm and its convergence analysis to inverse generalized eigenvalue problems in Section 5. In Section 6, we present some numerical results to illustrate the convergence theory and to compare the proposed algorithm with the existing algorithms.

2 Previous work

In this section, we explain some existing algorithms relevant to this study.

2.1 The Newton method and the Cayley transform method

There are many algorithms for solving the inverse eigenvalue problems. Among them, almost all the quadratically convergent methods are based on the Newton-like methods in [13] by Friedland, Nocedal, and Overton.

First, let us see a typical method, namely the standard Newton method as follows. For any $\boldsymbol{c} \in \mathbb{R}^n$, define $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\boldsymbol{f}(\boldsymbol{c}) = [\lambda_1(\boldsymbol{c}) - \lambda_1^*, \lambda_2(\boldsymbol{c}) - \lambda_2^*, \ldots, \lambda_n(\boldsymbol{c}) - \lambda_n^*]^{\mathrm{T}}, \tag{4}$$

where $\lambda_i(\boldsymbol{c}^*)$ $(1 \le i \le n)$ are the eigenvalues of $A(\boldsymbol{c})$ in (1) and λ_i^* $(1 \le i \le n)$ are the prescribed eigenvalues. Recall that we assume that all the prescribed eigenvalues are distinct. Then all the eigenvalues of $A(\boldsymbol{c})$ are also distinct in some neighborhood of $\boldsymbol{c}^*$. Hence, $\boldsymbol{f}$ is analytic in such a neighborhood and the Jacobi matrix is given by

$$[J(\boldsymbol{c})]_{ij} := \frac{\partial[\boldsymbol{f}(\boldsymbol{c})]_i}{\partial c_j} = \boldsymbol{x}_i(\boldsymbol{c})^{\mathrm{T}} A_j \boldsymbol{x}_i(\boldsymbol{c}), \tag{5}$$

where $\boldsymbol{x}_i(\boldsymbol{c})$ are the normalized eigenvectors of $A(\boldsymbol{c})$. In addition, it is easy to see that

$$J(\boldsymbol{c})\boldsymbol{c} = \lambda_i(\boldsymbol{c}) - \boldsymbol{x}_i(\boldsymbol{c})^{\mathrm{T}} A_0 \boldsymbol{x}_i(\boldsymbol{c}). \tag{6}$$

Thus, we obtain the Newton method for $\boldsymbol{f}(\boldsymbol{c}) = \boldsymbol{0}$ as follows.

Algorithm 1 The Newton method [13, Method I]

Require: $\lambda_1 < \cdots < \lambda_n$, $A_0, A_1, \ldots, A_n \in \mathbb{R}^{n\times n}$, $\boldsymbol{c}^{(0)} \in \mathbb{R}^n$.

For $A(\boldsymbol{c}^{(0)})$, find the eigenpairs $(\lambda_i^{(0)}, \boldsymbol{x}_i^{(0)})$ for all $1 \le i \le n$

Let $\Lambda^{(0)} := \mathrm{diag}(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})$, $X^{(0)} := [\boldsymbol{x}_1^{(0)}, \ldots, \boldsymbol{x}_n^{(0)}]$

for $k := 0, 1, \ldots$ **do**

  $[J^{(k)}]_{ij} = [\boldsymbol{x}_i^{(k)\mathrm{T}} A_j \boldsymbol{x}_i^{(k)}]_{ij}$

  $[\boldsymbol{d}^{(k)}]_i = \lambda_i^* - \boldsymbol{x}_i^{(k)\mathrm{T}} A_0 \boldsymbol{x}_i^{(k)}$

  $\boldsymbol{c}^{(k+1)} = {J^{(k)}}^{-1} \boldsymbol{d}^{(k)}$

  For $A(\boldsymbol{c}^{(k+1)})$, find the eigenpairs $(\lambda_i^{(k+1)}, \boldsymbol{x}_i^{(k+1)})$ for all $1 \le i \le n$

  Let $\Lambda^{(k+1)} := \mathrm{diag}(\lambda_1^{(k+1)}, \ldots, \lambda_n^{(k+1)})$, $X^{(k+1)} := [\boldsymbol{x}_1^{(k+1)}, \ldots, \boldsymbol{x}_n^{(k+1)}]$

end for

This is straightforward for solving inverse problems. In fact, Algorithm 1 can be extended to inverse generalized eigenvalue problems and inverse quadratic eigenvalue problems [10, 11]. However, we require numerical solutions of the eigenvalue problems for $A(\boldsymbol{c}^{(k+1)})$ in each iteration. There are some quasi-Newton methods without exactly solving such eigenvalue problems. See [13, Method II] and [2, 3, 16] for the details.

In [13, Method III], a different approach is proposed with the use of the matrix exponential and the Cayley transform. Note that the solution of the inverse eigenvalue problem can be described as

$$X^{*\mathrm{T}} A(\boldsymbol{c}^*) X^* = \Lambda^*, \tag{7}$$

where X^* is an orthogonal matrix. Let $\boldsymbol{c}^{(k)}$ and $X^{(k)}$ denote the current approximations of $\boldsymbol{c}^*$ and X^* respectively, where $X^{(k)}$ is assumed to be an orthogonal matrix. Let us write $X^* = X^{(k)} \mathrm{e}^{Y^{(k)}}$, where $Y^{(k)}$ is a skew-symmetric matrix. Then, using (7) and the Taylor series of the exponential function, we have

$$X^{(k)\mathrm{T}} A(\boldsymbol{c}^*) X^{(k)} = \mathrm{e}^{Y^{(k)}} \Lambda^* \mathrm{e}^{-Y^{(k)}} = \Lambda^* + Y^{(k)}\Lambda^* - \Lambda^* Y^{(k)} + \mathcal{O}(\|Y^{(k)}\|^2).$$

Similarly to the standard Newton method, omitting the second order term in $Y^{(k)}$, we obtain the following equation

$$X^{(k)\mathrm{T}} A(\boldsymbol{c}^{(k+1)}) X^{(k)} = \Lambda^* + \widetilde{Y}^{(k)}\Lambda^* - \Lambda^* \widetilde{Y}^{(k)}, \tag{8}$$

where let $\widetilde{Y}^{(k)}$ be the skew-symmetric matrix in the same manner as $Y^{(k)}$. We find $\boldsymbol{c}^{(k+1)}$ by the diagonal elements of the equation above. More specifically, letting

$$[J^{(k)}]_{ij} = \boldsymbol{x}_i^{(k)\mathrm{T}} A_j \boldsymbol{x}_i^{(k)} \quad (1 \le i, j \le n)$$
$$[\boldsymbol{d}^{(k)}]_i = \lambda_i^* - \boldsymbol{x}_i^{(k)\mathrm{T}} A_0 \boldsymbol{x}_i^{(k)} \quad (1 \le i \le n)$$

in a similar way to Algorithm 1, we see

$$\boldsymbol{c}^{(k+1)} = (J^{(k)})^{-1}\boldsymbol{d}^{(k)}.$$

Once $\boldsymbol{c}^{(k+1)}$ is obtained, from the off-diagonal elements of (8), we have

$$[\widetilde{Y}^{(k)}]_{ij} = -[\widetilde{Y}^{(k)}]_{ji} = -\frac{[S^{(k+1)}]_{ij}}{\lambda_j^* - \lambda_i^*} \quad (1 \le i, j \le n, i \ne j). \tag{9}$$

Hence, the skew-symmetric matrix $\widetilde{Y}^{(k)}$ can be obtained. Now construct an orthogonal matrix $Z^{(k)}$ using the Cayley transform

$$Z^{(k)} = (I + \widetilde{Y}^{(k)}/2)(I - \widetilde{Y}^{(k)}/2)^{-1} (\approx \mathrm{e}^{\widetilde{Y}^{(k)}})$$

and compute $X^{(k+1)} = X^{(k)}Z^{(k)}$. This algorithm is the so-called Cayley transform method below.

Algorithm 2 The Cayley transform method [13, Method III].

Require: $\lambda_1 < \cdots < \lambda_n$, $A_0, \ldots, A_n \in \mathbb{R}^{n\times n}$; an orthogonal matrix $X^{(0)} \in \mathbb{R}^{n\times n}$

1: **for** $k = 0, 1, \ldots$ **do**
2: $[J^{(k)}]_{ij} = \boldsymbol{x}_i^{(k)\mathrm{T}} A_j \boldsymbol{x}_i^{(k)} \quad (1 \le i, j \le n)$
3: $[\boldsymbol{d}^{(k)}]_i = \lambda_i^* - \boldsymbol{x}_i^{(k)\mathrm{T}} A_0 \boldsymbol{x}_i^{(k)} \quad (1 \le i \le n)$
4: $\boldsymbol{c}^{(k+1)} = (J^{(k)})^{-1}\boldsymbol{d}^{(k)}$
5: $S^{(k+1)} = X^{(k)\mathrm{T}} A(\boldsymbol{c}^{(k+1)}) X^{(k)}$
6: $[\widetilde{Y}^{(k)}]_{ij} = -[\widetilde{Y}^{(k)}]_{ji} = -[S^{(k+1)}]_{ij}/(\lambda_j^* - \lambda_i^*) \quad (1 \le i, j \le n, i \ne j)$
7: $X^{(k+1)} = X^{(k)}(I - \widetilde{Y}^{(k)}/2)(I + \widetilde{Y}^{(k)}/2)^{-1}$
8: **end for**

This algorithm appears to be important in the research fields of inverse eigenvalue problems. From the mathematical view points, a geometric interpretation is introduced in [4, 5, 7]. To reduce the computational cost, [1, 17] introduce some inexact solvers in line 4.

One may notice that, in general, constructing $J^{(k)}$ requires $\mathcal{O}(n^4)$ operations. However, from the practical view points, there are some examples such as the number of the nonzero elements of A_j is $\mathcal{O}(n)$ for each j. The Toeplitz inverse problem is a typical example, which is used in our numerical tests in Section 6. See Examples 3 and 4 for details. In such situations, Algorithms 1 and 2 can be computed in $\mathcal{O}(n^3)$ operations.

2.2 Strength of the proposed algorithm

Here we note that our new algorithm is based on the matrix equations in (3). Clearly, the second equation in (3) corresponds to (7) in Algorithm 2, where $X^{(0)}$ is assumed to be an orthogonal matrix. In contrast, the first equation in (3) plays a role in the improvement of the orthogonality. Hence, $X^{(k)}$ for $k = 0, 1, \ldots$ are not necessarily be orthogonal. Thus, the Cayley transform $(I - \widetilde{Y}^{(k)}/2)(I + \widetilde{Y}^{(k)}/2)^{-1}$ in the last step in Algorithm 2 is not required in our algorithm. In this sense, our approach is straightforward and convincing. Moreover, on the basis of this feature, this proposed algorithm is extended to inverse generalized eigenvalue problems in Section 5.

In addition, note that this study is relevant to the research of the iterative refinement algorithms for the symmetric eigenvalue problems as follows.

In particular, the Davies–Modi algorithm [12] for symmetric eigenvalue problems is based on the same idea as the Cayley transform method (Algorithm 2). The Davies–Modi algorithm is also based on the matrix equation

$$X^{\mathrm{T}} A X = \Lambda,$$

where A is a symmetric matrix, X is a normalized eigenvector matrix, and Λ is a diagonal matrix whose diagonal elements are the eigenvalues of A. The idea is also seen in Jahn's method [14, 18]. As in Algorithm 2, the Davies–Modi algorithm requires an orthogonal matrix $\widehat{X}$ approximating X. In each iteration, the Cayley transform is indispensable to ensure the orthogonality of the iterative matrices.

Recently, Ogita–Aishima [15] have proposed a new iterative refinement algorithm based on the following matrix equations:

$$\begin{cases} X^{\mathrm{T}} X = I & \text{(orthogonality)} \\ X^{\mathrm{T}} A X = \Lambda & \text{(diagonality)} \end{cases}$$

This corresponds to the proposed algorithm for inverse eigenvalue problems. In other words, this study is interpreted as a unified view on quadratically convergent algorithms for eigenvalue problems and inverse eigenvalue problems based on the matrix equations. To the best of our knowledge, such a unified development of algorithms is provided for the first time.

Finally, we explain the present status of numerical algorithms for multiple eigenvalues in inverse eigenvalue problems. There exists a numerical algorithm using smoothed LU factorization for solving inverse eigenvalue problems, even if there are some multiple eigenvalues [9]. However, this requires n times LU factorization in each iteration, resulting in $\mathcal{O}(n^4)$ operations, even if A_i for $1 \leq i \leq n$ are sparse matrices such as the Toeplitz inverse eigenvalue problems. The smoothed QR factorization method [8] has the same property. Actually, Friedland, Nocedal, and Overton [13] and Dai–Lancaster [10] proposed some modified algorithms for multiple eigenvalues under an assumption on the number of multiple eigenvalues. Recently, detailed convergence analysis based on a relative generalized Jacobian matrix has been shown in [17]. In addition, for the standard eigenvalue problems, Ogita–Aishima algorithm [15] works well for multiple eigenvalues.

Furthermore, [15] shows a reasonable idea to overcome the difficulty for ill-conditioned problems, where the matrix has nearly multiple eigenvalues. Although it appears to be possible to introduce such ideas above in our algorithm for inverse eigenvalue problems, in general some nontrivial analysis is required for handling multiple eigenvalues. In this paper, we assume that all the prescribed eigenvalues are distinct in the same manner as [1, 2, 3]. Handling multiple eigenvalues is regarded as a future work.

3 Proposed algorithm

In this section, we derive a new algorithm based on the relations in (3). For a given $X^{(k)} \in \mathbb{R}^{n\times n}$, define $E^{(k)} \in \mathbb{R}^{n\times n}$ such that

$$X^{(k)} = X^*(I + E^{(k)}), \tag{10}$$

where $X^{(k)}$ is assumed to be sufficiently close to X^*.

First, using $X^{*\mathrm{T}}X^* = I$ in (3), we have

$$I + E^{(k)} + E^{(k)\mathrm{T}} + \Delta_1^{(k)} = X^{(k)\mathrm{T}}X^{(k)}, \quad \Delta_1^{(k)} := E^{(k)\mathrm{T}}E^{(k)}. \tag{11}$$

Since we assume $\|E^{(k)}\|$ is sufficiently small, omitting the quadratic term $\|\Delta_1^{(k)}\| \le \|E^{(k)}\|^2$, we obtain the following matrix equation for $\widetilde{E}^{(k)}$:

$$\widetilde{E}^{(k)} + \widetilde{E}^{(k)\mathrm{T}} = I - X^{(k)\mathrm{T}}X^{(k)}. \tag{12}$$

Next, noting $X^{*\mathrm{T}}A(\boldsymbol{c}^*)X^* = \Lambda^*$ in (3), we have

$$\Lambda^* + \Lambda^*E^{(k)} + E^{(k)\mathrm{T}}\Lambda^* + \Delta_2^{(k)} = X^{(k)\mathrm{T}}A(\boldsymbol{c}^*)X^{(k)}, \quad \Delta_2^{(k)} := E^{(k)\mathrm{T}}\Lambda^*E^{(k)}. \tag{13}$$

As in (12), omitting $\Delta_2^{(k)}$, we have the following equation for $\widetilde{E}^{(k)}$ and $\boldsymbol{c}^{(k+1)}$:

$$\Lambda^* + \Lambda^*\widetilde{E}^{(k)} + \widetilde{E}^{(k)\mathrm{T}}\Lambda^* = X^{(k)\mathrm{T}}A(\boldsymbol{c}^{(k+1)})X^{(k)}. \tag{14}$$

Combining (12) and (14), we obtain the following equations:

$$\begin{cases} I + \widetilde{E}^{(k)} + \widetilde{E}^{(k)\mathrm{T}} = X^{(k)\mathrm{T}}X^{(k)}, \\ \Lambda^* + \Lambda^*\widetilde{E}^{(k)} + \widetilde{E}^{(k)\mathrm{T}}\Lambda^* = X^{(k)\mathrm{T}}A(\boldsymbol{c}^{(k+1)})X^{(k)}, \end{cases} \tag{15}$$

where the elements of $\widetilde{E}^{(k)}$ and $\boldsymbol{c}^{(k+1)}$ are unknown variables.

Since we see $n^2 + n$ unknown variables in (15), at first glance $\mathcal{O}(n^6)$ operations appear to be required for solving the linear system above. However, $\widetilde{E}^{(k)}$ and $\boldsymbol{c}^{(k+1)}$ can be obtained in at most $\mathcal{O}(n^4)$ operations indeed. As in Section 2, if the number of the nonzero elements of A_i for each i is $\mathcal{O}(n)$ such as the Toeplitz inverse problem, $\widetilde{E}^{(k)}$ and $\boldsymbol{c}^{(k+1)}$ can be obtained in $\mathcal{O}(n^3)$ operations as shown below.

First, noting the diagonal parts of $\widetilde{E}$ in the first equation, we see

$$[\widetilde{E}^{(k)}]_{ii} = \frac{\boldsymbol{x}_i^{(k)\mathrm{T}}\boldsymbol{x}_i^{(k)} - 1}{2} \quad (1 \le i \le n). \tag{16}$$

Next, to compute $\boldsymbol{c}^{(k+1)}$, note that the left hand-side of the second equation in (15) satisfies

$$[\Lambda^* + \Lambda^*\widetilde{E}^{(k)} + \widetilde{E}^{(k)\mathrm{T}}\Lambda^*]_{ii} = \lambda_i^*\boldsymbol{x}_i^{(k)\mathrm{T}}\boldsymbol{x}_i^{(k)} \quad (1 \le i \le n) \tag{17}$$

for the diagonal parts. Here, letting

$$[J^{(k)}]_{ij} = \boldsymbol{x}_i^{(k)\mathrm{T}}A_j\boldsymbol{x}_i^{(k)} \quad (1 \le i, j \le n), \tag{18}$$

we have

$$[X^{(k)\mathrm{T}}A(\boldsymbol{c}^{(k+1)})X^{(k)}]_{ii} = [J^{(k)}\boldsymbol{c}^{(k+1)}]_i + \boldsymbol{x}_i^{(k)\mathrm{T}}A_0\boldsymbol{x}_i^{(k)} \quad (1 \le i \le n). \tag{19}$$

Therefore, letting

$$[\boldsymbol{d}^{(k)}]_i = \lambda_i^*\boldsymbol{x}_i^{(k)\mathrm{T}}\boldsymbol{x}_i^{(k)} - \boldsymbol{x}_i^{(k)\mathrm{T}}A_0\boldsymbol{x}_i^{(k)} \quad (1 \le i \le n), \tag{20}$$

we obtain

$$\boldsymbol{c}^{(k+1)} = (J^{(k)})^{-1}\boldsymbol{d}^{(k)}. \tag{21}$$

Finally, we compute the off-diagonal parts of $\widetilde{E}^{(k)}$ as follows. Using (21), in each (i, j) element of (15) we see the following 2×2 linear system

$$\begin{cases} [\widetilde{E}^{(k)}]_{ij} + [\widetilde{E}^{(k)}]_{ji} = \boldsymbol{x}_i^{(k)\mathrm{T}}\boldsymbol{x}_j^{(k)} \\ \lambda_i^*[\widetilde{E}^{(k)}]_{ij} + \lambda_j^*[\widetilde{E}^{(k)}]_{ji} = \boldsymbol{x}_i^{(k)\mathrm{T}}A(\boldsymbol{c}^{(k+1)})\boldsymbol{x}_j^{(k)} \end{cases} \quad (1 \le i, j \le n, i \ne j).$$

Therefore, we obtain

$$[\widetilde{E}^{(k)}]_{ij} = \frac{\lambda_j^*\boldsymbol{x}_i^{(k)\mathrm{T}}\boldsymbol{x}_j^{(k)} - \boldsymbol{x}_i^{(k)\mathrm{T}}A(\boldsymbol{c}^{(k+1)})\boldsymbol{x}_j^{(k)}}{\lambda_j^* - \lambda_i^*} \quad (1 \le i, j \le n, i \ne j), \tag{22}$$

where we now assume all the prescribed eigenvalues are distinct. As a result, we can compute the next step

$$X^{(k+1)} = X^{(k)}(I - \widetilde{E}^{(k)}), \tag{23}$$

where $I - \widetilde{E}^{(k)}$ is the first order approximation of $(I + \widetilde{E}^{(k)})^{-1}$ using the Neumann series. In Algorithm 3, we present the proposed algorithm.

Algorithm 3 The proposed algorithm.

Require: $\lambda_1 < \cdots < \lambda_n,\ A_0, \dots, A_n \in \mathbb{R}^{n\times n};\ \ X^{(0)} \in \mathbb{R}^{n\times n}$

for $k = 0, 1, \dots$ **do**

$\quad R^{(k)} = X^{(k)\mathrm{T}} X^{(k)}$

$\quad [\widetilde{E}^{(k)}]_{ii} = ([R^{(k)}]_{ii} - 1)/2 \quad (1 \le i \le n)$

$\quad [J^{(k)}]_{ij} = \boldsymbol{x}_i^{(k)\mathrm{T}} A_j \boldsymbol{x}_i^{(k)} \quad (1 \le i, j \le n)$

$\quad [\boldsymbol{d}^{(k)}]_i = \lambda_i^* [R^{(k)}]_{ii} - \boldsymbol{x}_i^{(k)\mathrm{T}} A_0 \boldsymbol{x}_i^{(k)} \quad (1 \le i \le n)$

$\quad \boldsymbol{c}^{(k+1)} = (J^{(k)})^{-1} \boldsymbol{d}^{(k)}$

$\quad S^{(k+1)} = X^{(k)\mathrm{T}} A(\boldsymbol{c}^{(k+1)}) X^{(k)}$

$\quad [\widetilde{E}^{(k)}]_{ij} = (\lambda_j^* [R^{(k)}]_{ij} - [S^{(k+1)}]_{ij})/(\lambda_j^* - \lambda_i^*) \quad (1 \le i, j \le n, i \ne j)$

$\quad X^{(k+1)} = X^{(k)}(I - \widetilde{E}^{(k)})$

end for

Compared with Algorithm 2 in terms of the arithmetic operations, the Cayley transform $(I - \widetilde{Y}^{(k)}/2)(I + \widetilde{Y}^{(k)}/2)^{-1}$ is replaced with $X^{(k)\mathrm{T}} X^{(k)}$ corresponding to the refinement of the orthogonality of $X^{(k)}$ in our algorithm. As a result, our approach reduces the arithmetic operations in each iteration.

4 Convergence analysis

In this section, we prove quadratic convergence of the proposed algorithm on the assumption that, for some k, $X^{(k)}$ is sufficiently close to X^*.

Recall that the error of the approximate solution is expressed as $\|X^{(k)} - X^*\| = \|E^{(k)}\|$ in view of $X^{(k)} = X^*(I + E^{(k)})$. The next approximate solution is

$$X^{(k+1)} := X^{(k)}(I - \widetilde{E}^{(k)}) = X^*(I + E^{(k)})(I - \widetilde{E}^{(k)}) = X^*(I + E^{(k)} - \widetilde{E}^{(k)} - E^{(k)}\widetilde{E}^{(k)}).$$

The purpose is to prove

$$\|X^{(k+1)} - X^*\| = \mathcal{O}(\|X^{(k)} - X^*\|^2),$$

which corresponds to

$$\|\widetilde{E}^{(k)} - E^{(k)}\| = \mathcal{O}(\|E^{(k)}\|^2),$$

as $k \to \infty$.

From (11), (13), and (15), we have

$$\begin{cases} (E^{(k)} - \widetilde{E}^{(k)}) + (E^{(k)\mathrm{T}} - \widetilde{E}^{(k)\mathrm{T}}) + \Delta_1^{(k)} = O \\ \Lambda^*(E^{(k)} - \widetilde{E}^{(k)}) + (E^{(k)\mathrm{T}} - \widetilde{E}^{(k)\mathrm{T}})\Lambda^* + \Delta_2^{(k)} = X^{(k)\mathrm{T}} A_\Delta^{(k)} X^{(k)} \end{cases} \tag{24}$$

$$A_\Delta^{(k)} := A(\boldsymbol{c}^*) - A(\boldsymbol{c}^{(k+1)}), \quad \|\Delta_1^{(k)}\| \le \|E^{(k)}\|^2, \quad \|\Delta_2^{(k)}\| \le \|\Lambda^*\|\|E^{(k)}\|^2$$

For the convergence analysis, let

$$[J^*]_{ij} := \boldsymbol{x}_i^{*\mathrm{T}} A_j \boldsymbol{x}_i^*, \tag{25}$$

corresponding to the definition of $J^{(k)}$ in (18). We see the next lemma for J^*.

Lemma 1. *We define J^* as $[J^*]_{ij} = \boldsymbol{x}_i^{*\mathrm{T}} A_j \boldsymbol{x}_i^*$. Suppose that J^* is nonsingular. Then, we have*

$$\|J^{*-1}\|\sqrt{n}\sqrt{\sum_{\ell=1}^{n} \|A_\ell\|^2} \geq 1.$$

Proof. It is easy to see that

$$\|J^{*-1}\|^{-1} \leq \|J^*\| \leq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |\boldsymbol{x}_i^{*\mathrm{T}} A_j \boldsymbol{x}_i^*|^2} \leq \sqrt{n}\sqrt{\sum_{\ell=1}^{n} \|A_\ell\|^2}.$$

This completes the proof. □

Note that, if J^* is nonsingular, X^* is locally unique due to (11) and (13) where the left hand-sides are simple quadratic functions of $E^{(k)}$. The next lemma is essential to prove the quadratic convergence.

Lemma 2. *Let $A_0, \dots, A_n$ be real symmetric $n \times n$ matrices and $\lambda_1^* < \cdots < \lambda_n^*$ be prescribed eigenvalues. Suppose that there exists some X^* such that J^* in (25) is nonsingular, and Algorithm 3 is applied to $X^{(0)} \in \mathbb{R}^{n\times n}$. Moreover, for some k, suppose that*

$$\|E^{(k)}\| \leq \frac{\min_{i\neq j} |\lambda_i^* - \lambda_j^*|}{6n\|\Lambda^*\|(1+\alpha)}, \tag{26}$$

where $X^{(k)} = X^*(I + E^{(k)})$ and

$$\alpha := \|J^{*-1}\|\sqrt{n}\sqrt{\sum_{\ell=1}^{n} \|A_\ell\|^2} \geq 1. \tag{27}$$

Then, we obtain

$$\|\widetilde{E}^{(k)} - E^{(k)}\| \leq \frac{2n\|\Lambda^*\|(1+\alpha)}{C_\alpha^{(k)} \min_{i\neq j} |\lambda_i^* - \lambda_j^*|}\|E^{(k)}\|^2, \tag{28}$$

$$\|\widetilde{E}^{(k)} - E^{(k)}\| \leq \frac{24}{47}\|E^{(k)}\|, \tag{29}$$

where

$$C_\alpha^{(k)} = 1 - \alpha(2 + \|E^{(k)}\|)\|E^{(k)}\|. \tag{30}$$

Proof. First, we estimate the diagonal elements of $\widetilde{E}^{(k)} - E^{(k)}$. We see

$$|[\widetilde{E}^{(k)}]_{ii} - [E^{(k)}]_{ii}| = \frac{|[\Delta_1^{(k)}]_{ii}|}{2} \leq \frac{\|E^{(k)}\|^2}{2} \quad (i = 1, \ldots, n) \tag{31}$$

from the first equation in (24).

Next, we discuss $\boldsymbol{c}^{(k+1)}$ using the second equation in (24). In the left-hand side, we have

$$|[\Lambda^*(E^{(k)} - \widetilde{E}^{(k)}) + (E^{(k)\mathrm{T}} - \widetilde{E}^{(k)\mathrm{T}})\Lambda^* + \Delta_2^{(k)}]_{ii}| \leq \|\Lambda^*\|\|\Delta_1^{(k)}\| + \|\Delta_2^{(k)}\|$$

from (31). It then follows that

$$\begin{aligned}&\sqrt{\sum_{i=1}^{n} |[\Lambda^*(E^{(k)} - \widetilde{E}^{(k)}) + (E^{(k)\mathrm{T}} - \widetilde{E}^{(k)\mathrm{T}})\Lambda^* + \Delta_2^{(k)}]_{ii}|^2} \\ &\leq \sqrt{n}(\|\Lambda^*\|\|\Delta_1^{(k)}\| + \|\Delta_2^{(k)}\|).\end{aligned}$$

Therefore, from the diagonal elements of the second equation in (24), we obtain

$$\begin{aligned}\|\boldsymbol{c}^{(k+1)} - \boldsymbol{c}^*\| &\leq \|J^{(k)-1}\|\sqrt{n}(\|\Lambda^*\|\|\Delta_1^{(k)}\| + \|\Delta_2^{(k)}\|) \\ &\leq 2\|J^{(k)-1}\|\sqrt{n}\|\Lambda^*\|\|E^{(k)}\|^2. \end{aligned} \tag{32}$$

Concerning $\|{J^{(k)}}^{-1}\|$, noting $\boldsymbol{x}_i^{(k)} = \boldsymbol{x}_i^* + \sum_{\ell=1}^{n}[E^{(k)}]_{\ell i}\boldsymbol{x}_\ell^*$ from the definition in (10), we see

$$\begin{aligned}|[J^*]_{ij} - [J^{(k)}]_{ij}| &= |\boldsymbol{x}_i^{*\mathrm{T}} A_j \boldsymbol{x}_i^* - \boldsymbol{x}_i^{(k)\mathrm{T}} A_j \boldsymbol{x}_i^{(k)}| \\ &= |2\boldsymbol{x}_i^{*\mathrm{T}} A_j \sum_{\ell=1}^{n}[E^{(k)}]_{\ell i}\boldsymbol{x}_\ell^* + \sum_{\ell=1}^{n}[E^{(k)}]_{\ell i}\boldsymbol{x}_\ell^{*\mathrm{T}} A_j \sum_{\ell=1}^{n}[E^{(k)}]_{\ell i}\boldsymbol{x}_\ell^*| \\ &\leq (2 + \|E^{(k)}\|)\|E^{(k)}\|\|A_j\|\end{aligned}$$

from $\|\sum_{\ell=1}^{n}[E^{(k)}]_{\ell i}\boldsymbol{x}_\ell^*\| \leq \|E^{(k)}\|$. Using the Weyl's inequality for singular values, we have

$$\|{J^{(k)}}^{-1}\|^{-1} \geq \|J^{*-1}\|^{-1} - \sqrt{n}(2 + \|E^{(k)}\|)\|E^{(k)}\|\sqrt{\sum_{\ell=1}^{n}\|A_\ell\|^2}.$$

Hence, we obtain

$$\|{J^{(k)}}^{-1}\| \leq \|J^{*-1}\| \left(1 - \alpha(2 + \|E^{(k)}\|)\|E^{(k)}\|\right)^{-1} = \|J^{*-1}\|{C_\alpha^{(k)}}^{-1} \tag{33}$$

from (27) and (30). We prove $C_\alpha^{(k)}$ in the right-hand side is positive as follows. Since we assume (26), we have

$$\|E^{(k)}\| \leq \frac{1}{n}\frac{1}{3(1+\alpha)} \leq \min(1/12, 1/6\alpha) \tag{34}$$

for $n \geq 2$ and $\alpha \geq 1$. Hence, we see

$$C_\alpha^{(k)} = 1 - \alpha(2 + \|E^{(k)}\|)\|E^{(k)}\| \geq 1 - 1/3 - 1/72 = \frac{47}{72} > 0. \tag{35}$$

Using the inequalities (32) and (33), we estimate the off-diagonal elements of $\widetilde{E}^{(k)} - E^{(k)}$. Similarly to (22), from (24), we have

$$\begin{aligned} &|[\widetilde{E}^{(k)}]_{ij} - [E^{(k)}]_{ij}| \\ &\leq \frac{|\lambda_j^*||[\Delta_1^{(k)}]_{ij}| + |[\Delta_2^{(k)}]_{ij}| + |\sum_{\ell=1}^n (c_\ell^{(k+1)} - c_\ell^*)\boldsymbol{x}_i^{(k)\mathrm{T}} A_\ell \boldsymbol{x}_j^{(k)}|}{|\lambda_i^* - \lambda_j^*|}. \end{aligned} \tag{36}$$

Noting that

$$\begin{aligned} |\sum_{\ell=1}^n (c_\ell^{(k+1)} - c_\ell^*)\boldsymbol{x}_i^{(k)\mathrm{T}} A_\ell \boldsymbol{x}_j^{(k)}| &\leq \sum_{\ell=1}^n |c_\ell^{(k+1)} - c_\ell^*|\|A_\ell\|(1 + \|E^{(k)}\|)^2 \\ &\leq (1 + \|E^{(k)}\|)^2 \|\boldsymbol{c}^{(k+1)} - \boldsymbol{c}^*\| \sqrt{\sum_{\ell=1}^n \|A_\ell\|^2}, \end{aligned}$$

we have

$$\begin{aligned} &|[\widetilde{E}^{(k)}]_{ij} - [E^{(k)}]_{ij}| \\ &\leq \frac{2\|\Lambda^*\|\|E^{(k)}\|^2 + (1 + \|E^{(k)}\|)^2 \|\boldsymbol{c}^{(k+1)} - \boldsymbol{c}^*\| \sqrt{\sum_{\ell=1}^n \|A_\ell\|^2}}{|\lambda_i^* - \lambda_j^*|} \\ &\leq \frac{2\|\Lambda^*\|\|E^{(k)}\|^2 (1 + \sqrt{n}\|J^{(k)^{-1}}\|(1 + \|E^{(k)}\|)^2 \sqrt{\sum_{\ell=1}^n \|A_\ell\|^2})}{|\lambda_i^* - \lambda_j^*|} \\ &\leq \frac{2\|\Lambda^*\|}{\min_{i \neq j} |\lambda_i^* - \lambda_j^*|} \left(1 + \frac{\alpha(1 + \|E^{(k)}\|)^2}{1 - \alpha(2 + \|E^{(k)}\|)\|E^{(k)}\|}\right) \|E^{(k)}\|^2 \\ &\leq \frac{2\|\Lambda^*\|(1 + \alpha)}{C_\alpha^{(k)} \min_{i \neq j} |\lambda_i^* - \lambda_j^*|} \|E^{(k)}\|^2, \end{aligned}$$

where the second inequality is due to (32), the third inequality is due to (33), and the last inequality is due to the definition of $C_\alpha^{(k)}$ in (30). Using (31) and $\|\widetilde{E}^{(k)} - E^{(k)}\| \leq \sum_{i,j} \sqrt{[\widetilde{E}^{(k)}]_{ij}^2 - [E^{(k)}]_{ij}^2}$, we have (28).

Regarding (29), from (35), we have

$$C_\alpha^{(k)^{-1}} = (1 - \alpha(2 + \|E^{(k)}\|)\|E^{(k)}\|)^{-1} \leq \frac{72}{47}.$$

Therefore, from (26) and (28), we obtain (29). □

On the basis of Lemma 2, we obtain a main theorem that states the quadratic convergence of Algorithm 3.

Theorem 1. *Let $A_0, \dots, A_n$ be real symmetric $n \times n$ matrices and $\lambda_1^* < \cdots < \lambda_n^*$ be prescribed eigenvalues. Suppose that there exists some X^* such that J^* in* (25) *is nonsingular, and Algorithm 3 is applied to $X^{(0)} \in \mathbb{R}^{n\times n}$. Moreover, for some k, suppose that* (26). *Then, we obtain*

$$\|E^{(k+1)}\| \le \frac{359}{564}\|E^{(k)}\|, \tag{37}$$

$$\limsup_{\ell\to\infty} \frac{\|E^{(\ell+1)}\|}{\|E^{(\ell)}\|^2} \le \frac{2n\|\Lambda^*\|(1+\alpha)}{\min_{i\neq j}|\lambda_i^* - \lambda_j^*|} + 1. \tag{38}$$

Proof. Using

$$X^{(k+1)} = X(I + E^{(k+1)}) = X^{(k)}(I - \widetilde{E}^{(k)}) = X(I + E^{(k)})(I - \widetilde{E}^{(k)}),$$

we have

$$E^{(k+1)} = (E^{(k)} - \widetilde{E}^{(k)}) - E^{(k)}\widetilde{E}^{(k)} = (E^{(k)} - \widetilde{E}^{(k)}) - E^{(k)}(\widetilde{E}^{(k)} - E^{(k)} + E^{(k)}).$$

It then follows that

$$\begin{aligned} \|E^{(k+1)}\| &\le \|E^{(k)} - \widetilde{E}^{(k)}\| + \|E^{(k)}\|\|\widetilde{E}^{(k)} - E^{(k)}\| + \|E^{(k)}\|^2 \\ &\le \frac{24}{47}\|E^{(k)}\| + \frac{24}{47}\cdot\frac{1}{12}\|E^{(k)}\| + \frac{1}{12}\|E^{(k)}\| \\ &= \frac{359}{564}\|E^{(k)}\| \end{aligned}$$

due to (29) and (34). The first inequality above and (28) indicate (38). This completes the proof. □

5 Toward inverse generalized eigenvalue problems

In this section, we extend Algorithm 3 and its convergence analysis to inverse generalized eigenvalue problems in [8, 9, 10].

5.1 Problem setting

Let $A_0, A_1, \dots, A_n, B_0, B_1, \dots, B_n \in \mathbb{R}^{n\times n}$ be symmetric matrices. In addition, let

$$A(\boldsymbol{c}) = A_0 + c_1 A_1 + \cdots + c_n A_n, \tag{39}$$

$$B(\boldsymbol{c}) = B_0 + c_1 B_1 + \cdots + c_n B_n. \tag{40}$$

The purpose is to obtain $\boldsymbol{c}^*$ such that $A(\boldsymbol{c}^*)\boldsymbol{x}_i^* = \lambda_i^* B(\boldsymbol{c}^*)\boldsymbol{x}_i^*$ for the prescribed eigenvalues $\lambda_1 < \cdots < \lambda_n$. Now we assume $\boldsymbol{c}^*$ is locally unique and $B(\boldsymbol{c}^*)$ is positive definite. In addition, an approximate matrix $X^{(0)}$ of $X^* := [\boldsymbol{x}_1^*, \dots, \boldsymbol{x}_n^*]$ is given, where X^* is normalized as $X^{*\mathrm{T}} B(\boldsymbol{c}^*) X^* = I$.

For this problem, Newton method is constructed in [10], which is an extension of Algorithm 1.

Algorithm 4 The Newton method [10]

Require: $\lambda_1 < \cdots < \lambda_n$, $A_0, \ldots, A_n, B_0, \ldots, B_n \in \mathbb{R}^{n\times n}$, $\boldsymbol{c}^{(0)} \in \mathbb{R}^n$.

Solve the generalized eigenvalue problem $A(\boldsymbol{c}^{(0)})\boldsymbol{x}_i^{(0)} = \lambda_i^{(0)} B(\boldsymbol{c}^{(0)})\boldsymbol{x}_i^{(0)}$, i.e., find the eigenpairs $(\lambda_i^{(0)}, \boldsymbol{x}_i^{(0)})$ for all $1 \le i \le n$

Let $\Lambda^{(0)} := \mathrm{diag}(\lambda_1^{(0)}, \ldots, \lambda_n^{(0)})$, $X^{(0)} := [\boldsymbol{x}_1^{(0)}, \ldots, \boldsymbol{x}_n^{(0)}]$

for $k := 0, 1, \ldots$ **do**

  $[J^{(k)}]_{ij} = [\boldsymbol{x}_i^{(k)\mathrm{T}}(A_j - \lambda_i^{(k)} B_j)\boldsymbol{x}_i^{(k)}]_{ij}$

  $[\boldsymbol{d}^{(k)}]_i = \lambda_i^* - \lambda_i^{(k)} - [\boldsymbol{x}_i^{(k)\mathrm{T}}(A_0 - \lambda_i^{(k)} B_0)\boldsymbol{x}_i^{(k)}]_i$

  $\boldsymbol{c}^{(k+1)} = {J^{(k)}}^{-1}\boldsymbol{d}^{(k)}$

  Solve $A(\boldsymbol{c}^{(k+1)})\boldsymbol{x}_i^{(k+1)} = \lambda_i^{(k+1)} B(\boldsymbol{c}^{(k+1)})\boldsymbol{x}_i^{(k+1)}$ for all $1 \le i \le n$

  Let $\Lambda^{(k+1)} := \mathrm{diag}(\lambda_1^{(k+1)}, \ldots, \lambda_n^{(k+1)})$, $X^{(k+1)} := [\boldsymbol{x}_1^{(k+1)}, \ldots, \boldsymbol{x}_n^{(k+1)}]$

end for

Remark 1. *The Cayley transform method (Algorithm 2) cannot be extended to the inverse generalized eigenvalue problems unless* $B_i = O$ *for all* $1 \le i \le n$*. If* $B_i = O$ *for all* $1 \le i \le n$ *and* $X^{(0)}$ *satisfies* $X^{(0)\mathrm{T}} B_0 X^{(0)} = I$*, Algorithm 2 can be applied to the inverse generalized eigenvalue problems.*

5.2 Proposed algorithm

Similarly to Section 3, we focus on the matrix equation

$$\begin{cases} X^{\mathrm{T}} B(\boldsymbol{c}) X = I \\ X^{\mathrm{T}} A(\boldsymbol{c}) X = \Lambda^* \end{cases}$$

Letting

$$X^{(k)} = X^*(I + E^{(k)}), \tag{41}$$

we have the following relations

$$\begin{cases} I + E^{(k)} + E^{(k)\mathrm{T}} + \Delta_1^{(k)} = X^{(k)\mathrm{T}} B(\boldsymbol{c}^*) X^{(k)}, \\ \Lambda^* + \Lambda^* E^{(k)} + E^{(k)\mathrm{T}}\Lambda^* + \Delta_2^{(k)} = X^{(k)\mathrm{T}} A(\boldsymbol{c}^*) X^{(k)}, \end{cases} \tag{42}$$
$$\|\Delta_1^{(k)}\| \le \|E^{(k)}\|^2, \quad \|\Delta_2^{(k)}\| \le \|\Lambda^*\|\|E^{(k)}\|^2.$$

Omitting the quadratic terms, we obtain the key matrix equation

$$\begin{cases} I + \widetilde{E}^{(k)} + \widetilde{E}^{(k)\mathrm{T}} = X^{(k)\mathrm{T}} B(\boldsymbol{c}^{(k+1)}) X^{(k)} \\ \Lambda^* + \Lambda^* \widetilde{E}^{(k)} + \widetilde{E}^{(k)\mathrm{T}}\Lambda^* = X^{(k)\mathrm{T}} A(\boldsymbol{c}^{(k+1)}) X^{(k)} \end{cases} \tag{43}$$

to derive a new algorithm.

Because of the unknown matrix $X^{(k)\mathrm{T}}B(\boldsymbol{c}^{(k+1)})X^{(k)}$ in the right-hand side, the diagonal elements $[\widetilde{E}^{(k)}]_{ii}$ cannot directly obtained different from (16). However, we can obtain $\boldsymbol{c}^{(k+1)}$ as follows. From the diagonal elements of (43), we have

$$\begin{cases} 1 + 2[\widetilde{E}^{(k)}]_{ii} = \boldsymbol{x_i}^{(k)\mathrm{T}} B(\boldsymbol{c}^{(k+1)}) \boldsymbol{x}_i{}^{(k)} \\ \lambda_i^* + 2\lambda_i^*[\widetilde{E}^{(k)}]_{ii} = \boldsymbol{x_i}^{(k)\mathrm{T}} A(\boldsymbol{c}^{(k+1)}) \boldsymbol{x}_i{}^{(k)} \end{cases}$$

which imply

$$\begin{aligned} &\lambda_i^* + 2\lambda_i^*[\widetilde{E}^{(k)}]_{ii} - \lambda_j^*(1 + 2[\widetilde{E}^{(k)}]_{ii}) \\ &= \boldsymbol{x_i}^{(k)\mathrm{T}}(A_0 - \lambda_i^* B_0)\boldsymbol{x}_i{}^{(k)} + \sum_{j=1}^{n} \boldsymbol{x_i}^{(k)\mathrm{T}}(A_j - \lambda_i^* B_j)\boldsymbol{x}_i{}^{(k)} c_i^{(k+1)} \\ &= 0. \end{aligned}$$

Hence, for $i, j = 1, \ldots, n$, letting

$$[J^{(k)}]_{ij} = [\boldsymbol{x}_i^{(k)\mathrm{T}}(A_j - \lambda_i^* B_j)\boldsymbol{x}_i^{(k)}]_{ij}, \tag{44}$$

$$[\boldsymbol{d}^{(k)}]_i = -[\boldsymbol{x}_i^{(k)\mathrm{T}}(A_0 - \lambda_i^* B_0)\boldsymbol{x}_i^{(k)}]_i, \tag{45}$$

we obtain

$$\boldsymbol{c}^{(k+1)} = J^{(k)^{-1}} \boldsymbol{d}^{(k)}. \tag{46}$$

Therefore, we obtain $\widetilde{E}^{(k)}$ in the same manner as Section 3. In Algorithm 5, we present the proposed algorithm.

Algorithm 5 The Proposed algorithm

Require: $\lambda_1 < \cdots < \lambda_n$, $A_0, \ldots, A_n, B_0, \ldots, B_n \in \mathbb{R}^{n \times n}$, $X^{(0)} \in \mathbb{R}^{n \times n}$.

for $k := 0, 1, \ldots$ **do**
- $[J^{(k)}]_{ij} = [\boldsymbol{x}_i^{(k)\mathrm{T}}(A_j - \lambda_i^* B_j)\boldsymbol{x}_i^{(k)}]_{ij}$
- $[\boldsymbol{d}^{(k)}]_i = -[\boldsymbol{x}_i^{(k)\mathrm{T}}(A_0 - \lambda_i^* B_0)\boldsymbol{x}_i^{(k)}]_i$
- $\boldsymbol{c}^{(k+1)} = J^{(k)^{-1}} \boldsymbol{d}^{(k)}$
- $R^{(k)} = X^{(k)\mathrm{T}} B(\boldsymbol{c}^{(k+1)}) X^{(k)}$
- $[\widetilde{E}^{(k)}]_{ii} = ([R^{(k)}]_{ii} - 1)/2 \quad (1 \le i \le n)$
- $S^{(k+1)} = X^{(k)\mathrm{T}} A(\boldsymbol{c}^{(k+1)}) X^{(k)}$
- $[\widetilde{E}^{(k)}]_{ij} = (\lambda_j^*[R^{(k)}]_{ij} - [S^{(k+1)}]_{ij})/(\lambda_j^* - \lambda_i^*) \quad (1 \le i, j \le n, i \ne j)$
- $X^{(k+1)} = X^{(k)}(I - \widetilde{E}^{(k)})$

end for

5.3 Convergence analysis

As in Section 4, we use (42) and (43). In other words, we focus on

$$\begin{cases} (E^{(k)} - \widetilde{E}^{(k)}) + (E^{(k)\mathrm{T}} - \widetilde{E}^{(k)\mathrm{T}}) + \Delta_1^{(k)} = X^{(k)\mathrm{T}} B_\Delta^{(k)} X^{(k)} \\ \Lambda^*(E^{(k)} - \widetilde{E}^{(k)}) + (E^{(k)\mathrm{T}} - \widetilde{E}^{(k)\mathrm{T}})\Lambda^* + \Delta_2^{(k)} = X^{(k)\mathrm{T}} A_\Delta^{(k)} X^{(k)} \end{cases} \tag{47}$$

$$B_\Delta^{(k)} := B(\boldsymbol{c}^*) - B(\boldsymbol{c}^{(k+1)}), \quad A_\Delta^{(k)} := A(\boldsymbol{c}^*) - A(\boldsymbol{c}^{(k+1)})$$

Note that

$$\|\Delta_1^{(k)}\| \leq \|E^{(k)}\|^2, \quad \|\Delta_2^{(k)}\| \leq \|\Lambda^*\|\|E^{(k)}\|^2. \tag{48}$$

Similarly to (25), let

$$[J^*]_{ij} := \boldsymbol{x}_i^{*\mathrm{T}}(A_j - \lambda_i^* B_j)\boldsymbol{x}_i^*, \tag{49}$$

corresponding to the definition of $J^{(k)}$. Then, we can obtain the following convergence theorem.

Theorem 2. *Let $A_0, \dots, A_n$ be real symmetric $n \times n$ matrices and $\lambda_1^* < \cdots < \lambda_n^*$ be prescribed eigenvalues. Suppose that there exists some X^* such that J^* in (49) is nonsingular, and Algorithm 3 is applied to $X^{(0)} \in \mathbb{R}^{n\times n}$. Moreover, for some k, suppose that*

$$\|E^{(k)}\| \leq \frac{\min_{i\neq j}|\lambda_i - \lambda_j|}{6n\|\Lambda^*\|(1+\alpha)}, \tag{50}$$

where $X^{(k)} = X^*(I + E^{(k)})$ and

$$\alpha := \|J^{*-1}\|\beta\sqrt{n}\sqrt{\max_{1\leq m\leq n}\sum_{\ell=1}^{n}\|A_\ell - \lambda_m^* B_\ell\|^2} \geq 1, \quad \beta := \|B(\boldsymbol{c}^*)^{-1}\|. \tag{51}$$

Then, we obtain

$$\|E^{(k+1)}\| \leq \frac{359}{564}\|E^{(k)}\|, \tag{52}$$

$$\limsup_{k\to\infty}\frac{\|E^{(k+1)}\|}{\|E^{(k)}\|^2} \leq \frac{2n\|\Lambda^*\|(1+\alpha)}{\min_{i\neq j}|\lambda_i^* - \lambda_j^*|} + 1. \tag{53}$$

Proof. First, we estimate $\|\boldsymbol{c}^{(k+1)} - \boldsymbol{c}^*\|$. Similarly to (32), we have

$$\begin{aligned} \|\boldsymbol{c}^{(k+1)} - \boldsymbol{c}^*\| &\leq \|J^{(k)-1}\|\sqrt{n}(\|\Lambda^*\|\|\Delta_1^{(k)}\| + \|\Delta_2^{(k)}\|) \\ &\leq 2\|J^{(k)-1}\|\sqrt{n}\|\Lambda^*\|\|E^{(k)}\|^2. \end{aligned} \tag{54}$$

Concerning $\|J^{(k)^{-1}}\|$, noting $\boldsymbol{x}_i^{(k)} = \boldsymbol{x}_i^* + \sum_{\ell=1}^n [E^{(k)}]_{\ell i}\boldsymbol{x}_\ell^*$ from the definition, we see

$$\begin{aligned}
&|[J^*]_{ij} - [J^{(k)}]_{ij}| \\
&= |\boldsymbol{x}_i^{*\mathrm{T}}(A_j - \lambda_i^* B_j)\boldsymbol{x}_i^* - \boldsymbol{x}_i^{(k)\mathrm{T}}(A_j - \lambda_i^* B_j)\boldsymbol{x}_i^{(k)}| \\
&= |2\boldsymbol{x}_i^{*\mathrm{T}}(A_j - \lambda_i^* B_j)\sum_{\ell=1}^n [E^{(k)}]_{\ell i}\boldsymbol{x}_\ell^* + \sum_{\ell=1}^n [E^{(k)}]_{\ell i}\boldsymbol{x}_\ell^{*\mathrm{T}}(A_j - \lambda_i^* B_j)\sum_{\ell=1}^n [E^{(k)}]_{\ell i}\boldsymbol{x}_\ell^*| \\
&\leq (2 + \|E^{(k)}\|)\|E^{(k)}\|\|A_j - \lambda_i^* B_j\|\beta
\end{aligned}$$

from $\|\boldsymbol{x}_i^*\| \leq \sqrt{\beta}$ and $\|\sum_{\ell=1}^n [E^{(k)}]_{\ell i}\boldsymbol{x}_\ell^*\| \leq \|E^{(k)}\|\sqrt{\beta}$. Noting the Weyl's inequality for singular values, we have

$$\|J^{(k)^{-1}}\|^{-1} \geq \|J^{*-1}\|^{-1} - (2 + \|E^{(k)}\|)\|E^{(k)}\|\sqrt{n}\sqrt{\max_{1\leq m\leq n}\sum_{\ell=1}^n \|A_\ell - \lambda_m^* B_\ell\|^2\beta}.$$

Hence, we obtain

$$\|J^{(k)^{-1}}\| \leq \|J^{*-1}\|\left(1 - \alpha(2 + \|E^{(k)}\|)\|E^{(k)}\|\right)^{-1} \tag{55}$$

in the same manner as (33). Thus, for $i \neq j$, we have

$$\begin{aligned}
&|[\widetilde{E}^{(k)}]_{ij} - [E^{(k)}]_{ij}| \\
&\leq \frac{|\lambda_j^*||[\Delta_1^{(k)}]_{ij}| + |[\Delta_2^{(k)}]_{ij}| + |\sum_{\ell=1}^n (c_\ell^{(k+1)} - c_\ell^*)\boldsymbol{x}_i^{(k)\mathrm{T}}(A_\ell - \lambda_j^* B_\ell)\boldsymbol{x}_j^{(k)}|}{|\lambda_i^* - \lambda_j^*|}.
\end{aligned}$$

Noting that

$$\begin{aligned}
&|\sum_{\ell=1}^n (c_\ell^{(k+1)} - c_\ell^*)\boldsymbol{x}_i^{(k)\mathrm{T}}(A_\ell - \lambda_j^* B_\ell)\boldsymbol{x}_j^{(k)}| \\
&\leq \sum_{\ell=1}^n |c_\ell^{(k+1)} - c_\ell^*|\|A_\ell - \lambda_j^* B_\ell\|(1 + \|E^{(k)}\|)^2\beta \\
&\leq (1 + \|E^{(k)}\|)^2\|\boldsymbol{c}^{(k+1)} - \boldsymbol{c}^*\|\sqrt{\sum_{\ell=1}^n \|A_\ell - \lambda_j^* B_\ell\|^2\beta},
\end{aligned}$$

we obtain

$$
\begin{aligned}
&|[\widetilde{E}^{(k)}]_{ij} - [E^{(k)}]_{ij}| \\
&\le \frac{2\|\Lambda^*\|\|E^{(k)}\|^2 + (1+\|E^{(k)}\|)^2\|\boldsymbol{c}^{(k+1)} - \boldsymbol{c}^*\|\sqrt{\sum_{\ell=1}^{n}\|A_\ell - \lambda_j^* B_\ell\|^2}\beta}{|\lambda_i^* - \lambda_j^*|} \\
&\le \frac{2\|\Lambda^*\|\|E^{(k)}\|^2(1+\sqrt{n}\|J^{(k)^{-1}}\|(1+\|E^{(k)}\|)^2\sqrt{\sum_{\ell=1}^{n}\|A_\ell - \lambda_j^* B_\ell\|^2}\beta}{|\lambda_i^* - \lambda_j^*|} \\
&\le \frac{2\|\Lambda^*\|}{\min_{i\neq j}|\lambda_i^* - \lambda_j^*|}\left(1 + \frac{\alpha(1+\|E^{(k)}\|)^2}{1-\alpha(2+\|E^{(k)}\|)\|E^{(k)}\|}\right)\|E^{(k)}\|^2 \\
&\le \frac{2\|\Lambda^*\|(1+\alpha)}{C_\alpha^{(k)}\min_{i\neq j}|\lambda_i^* - \lambda_j^*|}\|E^{(k)}\|^2,
\end{aligned}
$$

where

$$
C_\alpha^{(k)} := 1 - \alpha(2+\|E^{(k)}\|)\|E^{(k)}\|.
$$

Regarding the diagonal elements of $\widetilde{E}^{(k)} - E^{(k)}$, for any $\lambda_j^* \neq \lambda_i^*$, note

$$
\begin{aligned}
&|[\widetilde{E}^{(k)}]_{ii} - [E^{(k)}]_{ii}| \\
&\le \frac{|\lambda_j^*||[\Delta_1^{(k)}]_{ii}| + |[\Delta_2^{(k)}]_{ii}| + |\sum_{\ell=1}^{n}(c_\ell^{(k+1)} - c_\ell^*)\boldsymbol{x}_i^{(k)\mathrm{T}}(A_\ell - \lambda_j^* B_\ell)\boldsymbol{x}_i^{(k)}|}{|\lambda_i^* - \lambda_j^*|} \\
&\le \frac{2\|\Lambda^*\|(1+\alpha)}{C_\alpha^{(k)}\min_{i\neq j}|\lambda_i^* - \lambda_j^*|}\|E^{(k)}\|^2.
\end{aligned}
$$

Therefore, we obtain

$$
\|\widetilde{E}^{(k)} - E^{(k)}\| \le \frac{2n\|\Lambda^*\|(1+\alpha)}{C_\alpha^{(k)}\min_{i\neq j}|\lambda_i^* - \lambda_j^*|}\|E^{(k)}\|^2
$$

in the same manner as Lemma 2. Moreover, since we see

$$
\|E^{(k+1)}\| \quad \le \quad \|E^{(k)} - \widetilde{E}^{(k)}\| + \|E^{(k)}\|\|\widetilde{E}^{(k)} - E^{(k)}\| + \|E^{(k)}\|^2
$$

in the same manner as the proof in Theorem 1, we obtain (52) and (53). □

6 Numerical tests

In this section, we report some numerical results to illustrate the convergence theory of the proposed algorithms and to compare their basic convergence behavior with the existing algorithms. All our tests were performed in MATLAB. For each numerical example, let the initial matrix $X^{(0)}$ be the eigenvector matrix of $A(\boldsymbol{c}^{(0)})$. Recall that Algorithm 1 is the standard Newton method, Algorithm 2 is the Cayley transform method, and Algorithm 3 is the proposed algorithm.

Example 1. This is an example in [13, Example 1]. Let $n = 8$, and

$$A_0 = \begin{bmatrix} 0 & 4 & -1 & 1 & 1 & 5 & -1 & 1 \\ 4 & 0 & -1 & 2 & 1 & 4 & -1 & 2 \\ -1 & -1 & 0 & 3 & 1 & 3 & -1 & 3 \\ 1 & 2 & 3 & 0 & 1 & 2 & -1 & 4 \\ 1 & 1 & 1 & 1 & 0 & 1 & -1 & 5 \\ 5 & 4 & 3 & 2 & 1 & 0 & -1 & 6 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \end{bmatrix}.$$

For $1 \leq i \leq n$, let $[A_i]_{ii} = 1$, and $[A_i]_{\ell m} = 0$ for all $\ell, m \neq i$. In addition, let $\lambda_i^* = c_i^{(0)} = 10i$ for $1 \leq i \leq n$. Table 1 displays the values of the errors $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$, for each algorithm.

Table 1: Numerical results for $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$ in Example 1

	Algorithm 1		Algorithm 2		Algorithm 3	
k	$\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$	$\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$	$\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$
0	3.29E-01	1.02E+01	3.29E-01	1.02E+01	3.29E-01	1.02E+01
1	1.11E-01	2.06E-00	1.49E-01	2.06E-00	1.67E-01	2.06E-00
2	1.97E-02	3.06E-01	2.24E-02	3.56E-01	1.99E-02	3.56E-01
3	5.17E-04	8.19E-03	4.33E-04	8.33E-03	4.67E-04	7.09E-03
4	4.64E-07	7.16E-06	4.55E-07	6.48E-06	5.52E-07	5.68E-06
5	3.39E-13	5.28E-12	2.09E-13	3.90E-12	2.96E-13	4.55E-12

Example 2. In this example, we use a random matrix. For each A_i, we generate a gaussian matrix Ω, and let $A_i := \Omega + \Omega^{\mathrm{T}}$. Then, we generate $\boldsymbol{c}^*$ with entries randomly chosen. Then we compute the eigenvalues of $A(\boldsymbol{c}^*)$. The initial guess $\boldsymbol{c}^{(0)}$ is formed by $\boldsymbol{c}^{(0)} = \boldsymbol{c}^* + \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is also chosen randomly. Table 2 displays the values of the errors $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$ for $n = 50$, for each algorithm.

Example 3. In this example, we use Toeplitz matrices as our A_i in (1):

$$A_0 = O,\ A_1 = I,\ A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \ldots, A_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \ddots & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \quad (56)$$

In this example, we consider $n = 50$. We first generate $\boldsymbol{c}^*$ with entries randomly chosen. Then we compute the eigenvalues of $A(\boldsymbol{c}^*)$. The initial

Table 2: Numerical results for $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$ in Example 2

k	Algorithm 1 $\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$	Algorithm 2 $\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$	Algorithm 3 $\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$
0	1.18E-02	3.64E-03	1.18E-02	3.64E-03	1.18E-02	3.64E-03
1	1.85E-03	6.84E-04	1.84E-03	6.84E-04	1.84E-03	6.84E-04
2	2.77E-05	1.10E-05	2.58E-05	1.05E-05	2.66E-05	1.07E-05
3	1.13E-08	4.41E-09	1.39E-08	5.43E-09	1.62E-08	6.35E-09
4	2.04E-13	7.66E-13	5.31E-13	2.02E-13	6.13E-14	2.47E-14

guess $\boldsymbol{c}^{(0)}$ is formed by $\boldsymbol{c}^{(0)} = \boldsymbol{c}^* + \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is also chosen randomly. Table 3 displays the values of the errors $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$ for $n = 50$, for each algorithm.

Table 3: Numerical results for $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$ in Example 3

k	Algorithm 1 $\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$	Algorithm 2 $\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$	Algorithm 3 $\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$
0	1.58E-01	3.64E-02	1.58E-01	3.64E-02	1.58E-01	3.64E-02
1	1.18E-02	8.67E-03	1.11E-02	8.67E-03	1.75E-02	8.67E-03
2	3.80E-04	3.15E-04	2.98E-04	3.23E-04	4.08E-04	2.43E-04
3	5.44E-07	5.33E-07	4.78E-07	5.91E-07	2.89E-07	3.24E-07
4	2.56E-12	2.62E-12	2.43E-12	3.01E-12	4.82E-13	5.34E-13

Example 4. For $n = 100$, we consider the inverse Toeplitz eigenvalue problem in the same manner as Example 3. In this example, $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$ are shown in Table 4.

Next, we consider the inverse generalized eigenvalue problems. Recall that Algorithm 4 is the standard Newton method, and Algorithm 5 is the proposed algorithm.

Example 5. This is an example in [10, Example 1]. Let $n = 5$, $A_0 =$

Table 4: Numerical results for $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$ in Example 4

	Algorithm 1		Algorithm 2		Algorithm 3	
k	$\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$	$\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$	$\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$
0	1.59E-01	5.31E-03	1.59E-01	5.31E-03	1.59E-01	5.31E-03
1	9.18E-03	9.18E-04	9.15E-03	9.18E-04	1.29E-02	9.18E-04
2	2.64E-04	8.63E-06	2.44E-04	1.16E-05	2.91E-04	1.43E-05
3	2.75E-08	1.08E-09	2.04E-08	2.21E-09	1.50E-07	5.76E-09
4	7.32E-14	5.94E-14	3.55E-13	2.12E-13	2.16E-13	7.62E-14

$\mathrm{diag}(9, 11, 10, 8, 14)$, $B_0 = \mathrm{diag}(11, 13, 15, 11, 10)$, $A_1 = B_1 = I$,

$$A_2 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = B_3, \quad A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = B_5.$$

In addition, let $\boldsymbol{c}^* = [1, 1, 1, 1, 1]^{\mathrm{T}}$ and $\boldsymbol{c}^{(0)} = [1.1, 1.2, 1.3, 1.4, 1.5]^{\mathrm{T}}$. Table 5 displays the values of the errors $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$, for each algorithm.

Example 6. In this example, we use (56) as our A_i to construct a Toeplitz matrix $A(\boldsymbol{c})$. In addition, let $B_0 = I$, and B_i for $1 \le i \le n$ are defined as follows: $[B_i]_{ii} = 1$, and $[B_i]_{\ell m} = 0$ for all $\ell, m \ne i$. In this example, we consider $n = 50$. We first generate $\boldsymbol{c}^*$ with entries randomly chosen. Then we solve the generalized eigenvalue problem $A(\boldsymbol{c}^*)\boldsymbol{x} = \lambda^* B(\boldsymbol{c}^*)$. The initial guess $\boldsymbol{c}^{(0)}$ is formed by $\boldsymbol{c}^{(0)} = \boldsymbol{c}^* + \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is also chosen randomly. Table 6 displays the values of the errors $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$ for $n = 50$, for each algorithm.

In these examples, the quadratic convergence can be observed in some neighborhood of the exact solution, and the proposed algorithms have very similar local behavior to the existing methods.

Table 5: Numerical results for $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$ in Example 5

	Algorithm 4		Algorithm 5	
k	$\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$	$\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$
0	1.51E-01	7.41E-01	1.51E-01	7.41E-01
1	1.02E-01	1.26E-00	7.23E-02	9.53E-01
2	1.32E-02	1.95E-01	1.23E-02	6.07E-02
3	2.84E-04	4.15E-03	2.82E-04	1.26E-03
4	3.31E-07	5.47E-06	9.47E-08	2.26E-07
5	3.53E-13	5.39E-12	2.63E-14	1.78E-13

Table 6: Numerical results for $\|E^{(k)}\|$ and $\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$ in Example 6

	Algorithm 4		Algorithm 5	
k	$\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$	$\|E^{(k)}\|$	$\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^*\|$
0	1.58E-01	3.64E-02	1.58E-01	3.64E-02
1	1.18E-02	8.67E-03	1.75E-02	8.67E-03
2	3.80E-04	3.15E-04	4.08E-04	2.43E-04
3	5.44E-07	5.33E-07	2.89E-07	3.24E-07
4	2.56E-12	2.62E-12	6.51E-13	6.79E-13

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