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Abstract

Some discrete inequalities such as the Sobolev inequality gives useful a priori estimates for numerical schemes. Although they had been known for the simplest forward difference operator, those for central difference type opereators had been left open until quite recently in Kojima–Matsuo–Furihata (2016) a unified way to discuss them was found. Still, due to some technical reasons, the result was limited to a narrow range of central difference operators. In this letter, we provide a new proof that gives a complete answer regarding the discrete Sobolev inequality and the discrete Gagliardo–Nirenberg inequality with the nonlinear Schrödinger equation index.

1 Introduction

The aim of this letter is to give a new proof of certain discrete inequalities, which solves an open problem left in the recent study Kojima–Matsuo–Furihata [1].

The background of this goal is as follows. We consider discrete versions of some inequalities involving the Sobolev norms, such as the Sobolev inequality

$$\|u\|_\infty \le c_1 \|u\|_{W^{1,2}(\mathbb{S})}, \tag{1}$$

or the Gagliardo–Nirenberg inequality

$$\|u\|_{L^p(\mathbb{S})} \le c_2 \|u\|^{\sigma}_{W^{1,r}(\mathbb{S})} \|u\|^{1-\sigma}_{L^q(\mathbb{S})}, \tag{2}$$

where $1 \leq p, q, r \leq \infty$ and $0 \leq \sigma \leq 1$ are constants satisfying

$$\frac{1}{p} = \sigma\left(\frac{1}{r} - 1\right) + (1 - \sigma)\frac{1}{q},$$

and constants c_1, c_2 are independent of u. The space $W^{1,r}(\mathbb{S})$ is the standard Sobolev space, $L^p(\mathbb{S})$ is the Lebesgue space, and $\|\cdot\|_\infty$ and $\|\cdot\|_{W^{1,r}(\mathbb{S})}$ are the norms of $L^\infty(\mathbb{S})$ and $W^{1,r}(\mathbb{S})$ respectively. We also use similar standard notation. In this letter, we consider only one dimensional, circle setting $\mathbb{S}$ to avoid cumbersome discussions around boundaries.

Such inequalities are useful to establish some a priori estimates regarding solutions of partial differential equations. For example, the cubic nonlinear Schrödinger equation (NLS) $\mathrm{i}u_t = u_{xx} + |u|^2 u$ has the invariants $\|u\|_2^2 =$ const. and $\|u_x\|_2^2 - \|u\|_4^4/2 =$ const., which then yield an a priori estimate $\|u(t,\cdot)\|_\infty < +\infty$ from the Sobolev inequality and the Gagliardo–Nirenberg inequality with $p = 4, q = 2, r = 2, \sigma = 1/4$ (we call it the "NLS-index" below.)

This is also the case for some numerical schemes that are carefully constructed so that such important invariants are (in some sense) preserved. Akrivis *et al.* [2] considered such a Galerkin scheme for NLS and proved that the numerical solution enjoys the same sup-norm stability following the continuous discussion above. In this case, we use the continuous version of the above inequalities.

When we consider finite difference schemes, the situation turns a bit sour, since there the continuous inequalities no longer work and we have to construct their discrete versions, which are not clear from the continuous versions. Matsuo *et al.* [3] (see also [4]) considered an invariants preserving finite difference scheme for NLS (which is essentially the same scheme as those in [2, 5]), and by establishing a discrete version of Sobolev inequality on the circle:

$$\|\boldsymbol{U}\|_\infty \leq c\|\boldsymbol{U}\|_{W_{\mathrm{d}}^{1,2}(\delta^+)}, \tag{3}$$

and similarly a discrete Gagliardo–Nirenberg inequality (with the NLS-index; we omit the concrete form here), they proved the finite difference solutions keep the sup-norm stability. $\mathbb{S}_N$ is the discretized circle

$$\mathbb{S}_N = \{(U_k \in \mathbb{C}) \mid U_k = U_{k \bmod N}\},$$

and $\boldsymbol{U} \in \mathbb{S}_N$ is an approximate solution on it (it actually depends on time, but since it is not important in the present letter, we drop the time index.) The discrete Sobolev norm employed in [3] is given by

$$\|\boldsymbol{U}\|_{W_{\mathrm{d}}^{1,2}(\delta^+)} = \left(\sum_{k=0}^{N-1} (|U_k|^2 + |\delta^+ U_k|^2)\Delta x\right)^{1/2}.$$

Notice that *it depends on the definition of the finite difference operator* δ^+, which is the forward difference operator here.

Later on, the finite difference scheme was extended to arbitrary spatial order in [6]. The main idea there was to replace the simplest forward difference δ^+ with the $2s$-order central finite differences $\delta^{\langle 1\rangle,2s}$ (the precise definition will be given below.) Although this study itself was a success in that the resulting schemes work very well, its theoretical analysis was left open since the associated discrete inequalities (i.e., those where δ^+ is replaced by $\delta^{\langle 1\rangle,2s}$) remained open. Even for the simplest case in this class, i.e., $s = 1$ (which corresponds to the standard central difference operator $\delta^{\langle 1\rangle,2}$), this is a tough task, despite the apparent simpleness of the problem. This can be, for example, understood in the following way. Since $\delta^{\langle 1\rangle,2} = (\delta^+ + \delta^-)/2$, and the sums of the forward and backward finite differences coincide on the discrete circle, we have

$$\|D_{\delta^{\langle 1\rangle,2}}\boldsymbol{U}\|_2^2 \leq \|D_{\delta^+}\boldsymbol{U}\|_2^2.$$

Thus, for the Sobolev inequality, the desired inequality

$$\|\boldsymbol{U}\|_\infty^2 \leq c\|\boldsymbol{U}\|_{W_{\mathrm{d}}^{1,2}(\delta^{\langle 1\rangle,2})} = c(\|D_{\delta^{\langle 1\rangle,2}}\boldsymbol{U}\|_2^2 + \|\boldsymbol{U}\|_2^2)$$

is purely stronger than the known version (3), if it holds. The same applies to the Gagliardo–Nirenberg inequality.

This difficulty has been solved only quite recently in Kojima–Matsuo–Furihata [1] (after more than a decade since [6].) They succeeded in settling the problem in the case of $\delta^{\langle 1\rangle,2}$. Furthermore, they introduced a clever trick to "reduce" the general case $\delta^{\langle 1\rangle,2s}$ $(s \geq 2)$ to $\delta^{\langle 1\rangle,2}$, so that a unified proof can be simultaneously given for them avoiding cumbersome discussions for each (complicated) difference operator. Still, there remained a limitation that their proof was valid only for $s \leq 7$ and $s = \infty$ (the spectral difference operator). This limitation essentially came from their technical strategy based on linear algebra. The inequality is, however, expected to hold for every s, since we already have its lowest and highest limits ($s = 1$ and ∞); actually the authors of [1] raised this as a conjecture in its last part and said that "Preliminary numerical tests by the present authors support this view."

The present letter is to prove this conjecture. Below, in Section 2, we briefly review [1]. Then in Section 3 we give the new proof. Section 4 is devoted to other remarks. Throughout this letter we mainly focus on the Sobolev inequality, which is sufficient to illustrate how the new proof works.

2 Original ideas of analyzing central-difference type operators

In this section, we explain the outline of [1]. We define the standard central-difference type operators in the following form.

Definition 1. *An operator* $\delta^{\langle 1 \rangle,2s}$ *is a standard central-difference type operator if it is in the form*

$$\delta^{\langle 1 \rangle,2s} U_k = \sum_{j=1}^{s} \beta_j^{(s)} \frac{U_{k+j} - U_{k-j}}{2j\Delta x}, \tag{4}$$

and $\delta^{\langle 1 \rangle,2s}$ *is an approximation of* ∂_x *of* $O(\Delta x^{2s})$.

The statement on the accuracy can be explicitly written for small Δx as

$$\frac{\mathrm{d}}{\mathrm{d}x} f(a) = \sum_{j=1}^{s} \beta_j^{(s)} \frac{f(a + j\Delta x) - f(a - j\Delta x)}{2j\Delta x} + O(\Delta x^{2s}), \tag{5}$$

which is useful in the argument in the next section. The coefficients $\beta_j^{(s)}$ are uniquely determined to gain the accuracy (see, for example, [7]). The local expression (4) can be represented in matrix form

$$D_{\delta^{\langle 1 \rangle,2s}} = \sum_{j=1}^{s} \beta_j^{(s)} \frac{R^j - L^j}{2j\Delta x} \tag{6}$$

where

$$L = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}, \quad R = L^{-1}.$$

Kojima *et al.* [1] tried to prove (3) for $\delta^{\langle 1 \rangle,2s}$ ($s = 1, 2, \ldots, \infty$). Their strategy is as follows.

1. Prove (3) for $\delta^{\langle 1 \rangle,2}$, i.e., the lowest order version.
2. Then try to "reduce" other operators $\delta^{\langle 1 \rangle,2s}$ ($s \geq 2$) to $\delta^{\langle 1 \rangle,2}$ so that (3) is established also for them.

The step 1 has been successfully shown.

Proposition 1 ([1, Lemma 2.8])**.** *The discrete Sobolev inequality* (3) *holds for* $\delta^{\langle 1 \rangle,2}$ *(i.e.,* δ^+ *replaced with* $\delta^{\langle 1 \rangle,2}$*.)*

The key in the proof is to find a useful continuous function that connects *discrete* and *continuous*. Then the authors has succeeded in translating the discrete problem to the continuous inequality, which is readily known. Although similar techniques can be found in the literature for simpler difference operators (for example, [8]), finding such a nice function is not an obvious task in the case of central difference type operators.

Next, let us consider the step 2. Let us define a matrix S_N associated with $\delta^{\langle 1 \rangle, 2s}$ by

$$S_N = \sum_{j=1}^{s} \frac{\beta_j^{(s)}}{j} (L^{j-1} + L^{j-3} + \cdots + R^{j-1}).$$

This matrix relates $D_{\delta^{\langle 1 \rangle, 2s}}$ to $D_{\delta^{\langle 1 \rangle, 2}}$:

$$D_{\delta^{\langle 1 \rangle, 2s}} = S_N D_{\delta^{\langle 1 \rangle, 2}}.$$

(See [1, Lemma 2.3].) The next concept describes how “safe” this relation is.

Definition 2 (p-reducibility)**.** *An standard central-difference type operator $\delta^{\langle 1 \rangle, 2s}$ is p-reducible to $\delta^{\langle 1 \rangle, 2}$ if there exists a constant C independent of N such that $\|{S_N}^{-1}\|_p < C$ holds.*

If $\delta^{\langle 1 \rangle, 2s}$ is 2-reducible to $\delta^{\langle 1 \rangle, 2}$,

$$\begin{aligned}
\|\boldsymbol{U}\|_\infty^2 &\leq c \|\boldsymbol{U}\|^2_{W_{\mathrm{d}}^{1,2}(\delta^{\langle 1 \rangle, 2})} \\
&= c\left(\|\boldsymbol{U}\|_2^2 + \|D_{\delta^{\langle 1 \rangle, 2}} \boldsymbol{U}\|_2^2 \right) \\
&= c\left(\|\boldsymbol{U}\|_2^2 + \|{S_N}^{-1} D_{\delta^{\langle 1 \rangle, 2s}} \boldsymbol{U}\|_2^2 \right) \\
&\leq c\left(\|\boldsymbol{U}\|_2^2 + (\|{S_N}^{-1}\|_2 \|D_{\delta^{\langle 1 \rangle, 2s}} \boldsymbol{U}\|_2)^2 \right) \\
&\leq \max\{c, cC\} \|\boldsymbol{U}\|^2_{W_{\mathrm{d}}^{1,2}(\delta^{\langle 1 \rangle, 2s})}
\end{aligned}$$

holds, and this proves the discrete Sobolev inequality for $\delta^{\langle 1 \rangle, 2s}$. Note that p-reducibility demands $\|{S_N}^{-1}\|_2$ is bounded from above for all N; otherwise, the right hand side of the above will blows up in the limit of $N \to \infty$ and (although it is mathematically correct for every fixed N) the discrete inequality loses practical meaning.

The discussion above reveals the fact that the 2-reducibility of $\delta^{\langle 1 \rangle, 2s}$ is essential in establishing the discrete Sobolev inequality. The authors of [1] employed an linear algebra approach to prove this; S_N is diagonally dominant if $s \leq 7$.

Proposition 2 ([1, Lemma 2.7])**.** *If S_N is diagonally dominant, $\delta^{\langle 1 \rangle, 2s}$ is 1-reducible to $\delta^{\langle 1 \rangle, 2}$.*

Proposition 3 ([1, Lemma 2.6])**.** *If $\delta^{\langle 1\rangle,2s}$ is 1-reducible to $\delta^{\langle 1\rangle,2}$, then it is p-reducible to $\delta^{\langle 1\rangle,2}$ for all $p \geq 1$.*

Combining these propositions we see for $s \leq 7$ the central difference type operators are 2-reducible to $\delta^{\langle 1\rangle,2}$, which completes the desired proof for the discrete Sobolev inequality.

Note that in this approach we have a stronger property than necessary, the 1-reducibility. An advantage of this is that it is also possible to establish a discrete version of the Gagliardo–Nirenberg inequality for general index (see [1, Theorem 3.2]). The approach is, however, not applicable for $s \geq 8$, since there S_N is no longer diagonally dominant. The only result obtained in [1] for $s \geq 8$ is that for the case $s = \infty$, which is summarized as follows.

Proposition 4 ([1, Theorem 3.6])**.** *The discrete Sobolev inequality and the discrete Gagliardo–Nirenberg inequality with the NLS-index hold for $s = \infty$.*

In the next section, we seek a completely different, an analytic approach to cover $8 \leq s < \infty$.

3 New proof

In this section, we give a new proof that holds for every s. The key is to directly evaluate the eigenvalues of S_N, which gives a sharp estimate $\|{S_N}^{-1}\|_2 = 1$.

With $\omega_N = \exp(2\pi \mathrm{i}/N)$, the eigenvalues of R can be represented as ${\omega_N}^k$ $(k = 0, \cdots, N-1)$. Therefore, the eigenvalues of S_N are, for $k = 0, \ldots, N-1$,

$$\sum_{j=1}^{s} \frac{\beta_j^{(s)}}{j} \left({\omega_N}^{(j-1)k} + {\omega_N}^{(j-3)k} + \cdots + {\omega_N}^{-(j-1)k} \right)$$
$$= \begin{cases} 1 & (k = 0), \\ \sum_{j=1}^{s} \frac{\beta_j^{(s)}}{j} \frac{\sin(2\pi kj/N)}{\sin(2\pi k/N)} & (k = 1, \cdots, N-1). \end{cases}$$

Note that S_N always has an eigenvalue equal to 1.

If we introduce an interpolating function

$$f_s(x) = \sum_{j=1}^{s} \frac{\beta_j^{(s)}}{j} \frac{\sin jx}{\sin x} \qquad (0 \leq x < 2\pi), \tag{7}$$

it suffices to show $f_s(x) \geq 1$. We first note an interesting fact that $\sin jx / \sin x$ can be expanded by $(1 - \cos x)^k$ $(0 \leq k \leq j-1)$, which can be proved by induction. Thus, let us write

$$f_s(x) = \sum_{k=0}^{s-1} c_k^{(s)} (1 - \cos x)^k \qquad (0 \leq x < 2\pi). \tag{8}$$

The situation is, however, much more favorable; if we compute the concrete forms of the first three, we find

$$\begin{aligned} f_1(x) &= 1, \\ f_2(x) &= 1 + \frac{1}{3}(1-\cos x), \\ f_3(x) &= 1 + \frac{1}{3}(1-\cos x) + \frac{2}{15}(1-\cos x)^2. \end{aligned}$$

This gives rise to a stronger conjecture that (i) $c_k^{(s)}$ does not depend on s, i.e., there is only one series $\{c_k\}_{k=0}^{\infty}$, and (ii) all c_k's are positive. Obviously this proves $f_s(x) \geq 1$. Below we prove this conjecture.

We first prove (i). To this end, let us first show the following lemma that describes the behavior of $f_s(x)$ around $x = 0$.

Lemma 1. *For any small positive x we have*

$$f_s(x) = \frac{x}{\sin x} + O(x^{2s}).$$

Proof. By using (5) with $f(x) = \sin x$ and $a = 0$, we see

$$\begin{aligned} 1 &= \sum_{j=1}^{s} \beta_j^{(s)} \frac{\sin j\Delta x}{j\Delta x} + O(\Delta x^{2s}) \\ &= \frac{\sin \Delta x}{\Delta x} f_s(\Delta x) + O(\Delta x^{2s}). \end{aligned}$$

Since (5) is valid for any Δx satisfying $0 < \Delta x < 1$, we have the claim. □

Around $x = 0$, the term $x/\sin x$ can be expanded as follows.

$$\frac{x}{\sin x} = \sum_{k=0}^{\infty} c_k (1-\cos x)^k \quad (|x| < \pi). \tag{9}$$

Introducing $z = \sin(x/2)$, we can rewrite it as

$$\frac{\sin^{-1} z}{z\sqrt{1-z^2}} = \sum_{k=0}^{\infty} 2^k c_k z^{2k} \quad (|z| < 1). \tag{10}$$

This is valid for $|z| < 1$ and justifies the expansion (9). Using this, we have the claim (i).

Lemma 2. *For any $s \geq 1$, we have $c_k^{(s)} = c_k$ $(k = 0, \ldots, s-1)$.*

Proof. Using the expansion (9) in Lemma 1, together with (8), we see

$$\sum_{k=0}^{s-1} c_k^{(s)} (1-\cos x)^k = \sum_{k=0}^{\infty} c_k (1-\cos x)^k + O(x^{2s})$$

for all $0 \leq x < 1$. Again setting $z = \sin(x/2)$, this equation can be represented by z as

$$\sum_{k=0}^{s-1} 2^k c_k^{(s)} z^{2k} = \sum_{k=0}^{\infty} 2^k c_k z^{2k} + O(z^{2s}) \tag{11}$$

for small positive z. This equation proves the lemma. □

Note that the expansion (9) is only valid for $|x| < \pi$, while $f_s(x)$ is defined on $[0, 2\pi)$ in (8). But this is enough to identify the unknown coefficients $c_k^{(s)}$.

Now let us turn our attension to the claim (ii).

Lemma 3. *The coefficients c_k's in* (10) (*and accordingly* (9)) *are all positive.*

Proof. By combining the expansions

$$\sin^{-1} z = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} z^{2n+1},$$

$$\frac{1}{z\sqrt{1-z^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} z^{2n-1}$$

for $|z| < 1$, we see

$$2^k c_k = \sum_{l=0}^{k} \frac{(2l)!}{4^l (l!)^2 (2l+1)} \frac{(2(k-l))!}{4^{k-l}((k-l)!)^2} > 0.$$

□

Collecting all the above lemmas, we finally have the desired theorem.

Theorem 1. *For any positive integer s, we have $\|S_N{}^{-1}\|_2 = 1$. That is, $\delta^{\langle 1 \rangle, 2s}$ is 2-reducible to $\delta^{\langle 1 \rangle, 2}$ for every $s < \infty$.*

Proof. Taking the limit $x \to 0$ in (9), we see $c_0 = 1$ (which can be also observed in the concrete examples above.) Thus all the eigenvalues of S_N is equal to or greater than 1, while S_N actually has an eigenvalue equal to 1. □

Since the case $s = \infty$ has been already settled in [1], we can summarize the results as follows.

Corollary 1 (Discrete Sobolev inequality for $\delta^{\langle 1 \rangle, 2s}$)**.** *For any $s = 1, 2, \ldots, \infty$, we have*

$$\|\boldsymbol{U}\|_\infty^2 \leq c \|\boldsymbol{U}\|_{W_{\mathrm{d}}^{1,2}(\delta^{\langle 1 \rangle, 2s})},$$

where c is a constant independent of $\boldsymbol{U}$ and N.

By a similar discussion, we can also establish the discrete Gagliardo–Nirenberg inequality of the NLS-index (we omit the proof.)

Corollary 2 (Discrete Gagliardo–Nirenberg inequality for $\delta^{\langle 1\rangle,2s}$)**.** *For any* $s = 1, 2, \ldots, \infty$*, we have*

$$\|\boldsymbol{U}\|_{L^4_{\mathrm{d}}}^4 \leq c\|\boldsymbol{U}\|_{W^{1,2}_{\mathrm{d}}(\delta^{\langle 1\rangle,2s})}\|\boldsymbol{U}\|_{L^2_{\mathrm{d}}}^3,$$

where c *is a constant independent of* $\boldsymbol{U}$ *and* N*.*

4 Concluding remarks

In this letter, we gave a new proof of some discrete inequalities regarding central difference type operators. This gives a positive answer to the conjecture raised in the previous study Kojima–Matsuo–Furihata [1]. The key there is to directly evaluate the eigenvalues of the reduction matrices (S_N) in an analytic manner. This also supports the view proposed in [1] that the idea of "reducing" operators is promising in handling complicated difference operators. Note that in the present approach, we *never* uses the concrete values of the coefficients $\beta_j^{(s)}$, which is extremely advantageous when we have to deal with higher-order operators.

Some concluding comments are in order. First, there still remains an unsolved problem that whether the discrete Gagliardo–Nirenberg inequality with general index holds or not for $s \geq 8$ (including $s = \infty$). It is not even clear if it is likely to hold. Possibly some careful numerical tests are needed to identify this point. Unfortunately the techniques employed in this paper do not seem to extend to this general case (at least naturally), since the evaluation of the eigenvalues essentially fully utilizes the property of L^2 norm.

Second, two or three dimensional cases should be considered. It seems the discussion in the present paper (and in [1]) basically carries to such cases as far as the domain is rectangular and we consider only uniform grids (in each spatial direction.)

Finally, in order to adapt to more practical situations, we need to generalize the result to more general boundary conditions and (non-uniform) grids. Under such general situations, however, the definition of finite difference method itself becomes a challenge, and it seems some breakthrough is necessary to obtain a unified perspective there.

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