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An Algorithm for the Problem of Minimum Weight Packing of Arborescences with Matroid Constraints

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Abstract

As a common generalization of previous arborescence packings, Cs. Király and Szigeti [14] introduced the reachability-based matroid-restricted packing of arborescences. That paper gave a characterization when such a packing exists and a polynomial algorithm for the unweighted case. Here we provide a polynomial algorithm for the weighted case. We reduce the problem to the weighted matroid intersection problem by exploiting the underlying intersecting submodular bi-set function.

1 Introduction

Let $D = (V + s, A)$ be a *rooted digraph*, that is a digraph with a designated root vertex s. The arcs leaving s are called *root arcs*. An *s-arborescence* is an acyclic subgraph in which s has in-degree zero and every other vertex has in-degree one. For an s-arborescence T and a vertex v of T, $T[s, v]$ denotes the unique path from s to v. A set of arc-disjoint s-arborescences is called a *packing* of s-arborescences. For $X \subseteq V$, $P(X)$ denotes the set of vertices in V from which a vertex in X is reachable by a directed path in D. For $X, Y \subseteq V + s$ and an arc set $B \subseteq A$, $\partial^B_Y(X)$ denotes the set of arcs in B from $Y \setminus X$ to X. If $Y = V + s$, then Y is often omitted. If $B = A$, that is, the arcs set of D, then B is also omitted.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be matroids on the set $\partial_s(V)$ of root arcs and the arc set A of D, respectively. A packing $T_1, \dots, T_k$ of s-arborescences in D is said to be

- $\mathcal{M}_2$*-restricted* if the union of the arc sets of the arborescences $T_1, \dots, T_k$ forms an independent set of $\mathcal{M}_2$;
- $\mathcal{M}_1$*-based* if, for each $v \in V$, the set of root arcs used in the paths from s to v in the arborescence packing forms a base of $\mathcal{M}_1$, i.e., $\{\partial(V) \cap A(T_i[s, v]) : T_i \text{ contains } v\}$ is a base of $\mathcal{M}_1$;
- $\mathcal{M}_1$*-reachability-based* if, for each $v \in V$, the set of root arcs used in the paths from s to v in the arborescence packing forms an independent set in $\mathcal{M}_1$ of size $r_1(\partial_s(P(v)))$.

The following theorem due to Cs. Király and Szigeti [14] characterizes when an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences exists.

Theorem 1.1. *Let $D = (V + s, A)$ be a rooted digraph, $\mathcal{M}_1 = (\partial(V), r_1)$ and $\mathcal{M}_2 = (A, r_2)$ two matroids such that $\mathcal{M}_2$ is the direct sum of matroids $\mathcal{M}_v = (\partial(v), r_v)$ for $v \in V$. There exists an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences in D if and only if*

$$r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial_s(P(X))) \text{ for all } X \subseteq V \text{ and } F \subseteq \partial_s(X). \quad (1)$$

For $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packings, (1) can be simplified as follows.

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Corollary 1.2. *Let $D = (V + s, A)$ be a rooted digraph, $\mathcal{M}_1 = (\partial(V), r_1)$ and $\mathcal{M}_2 = (A, r_2)$ two matroids such that $\mathcal{M}_2$ is the direct sum of matroids $\mathcal{M}_v = (\partial(v), r_v)$ for $v \in V$. There exists an $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-arborescences in D if and only if*

$$r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial(V)) \text{ for all } \emptyset \neq X \subseteq V \text{ and } F \subseteq \partial_s(X). \tag{2}$$

The paper [14] also provided a polynomial algorithm to find an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences in D if there exists one. That algorithm used a submodular function minimization algorithm for verifying (1). Here we will show that (1) can be verified by repeated applications of matroid intersection.

The main contribution of this paper is to provide a polynomial algorithm for the weighted case. Our approach is the following. In Phase 1, we find a minimum weight arc set that can be decomposed into a reachability-based packing of arborescences and then, in Phase 2, we find the required decomposition. The second phase doesn't depend on the weighting, so we can use the algorithm developed in [14]. Thus our focus in this paper is to find a minimum weight arc set for a packing. The idea for finding such an arc set of minimum weight is to show that it is a common base of two matroids, one of them being $\mathcal{M}_2$. The construction of the other matroid was done by exploiting the underlying submodular *bi-set* function. The application of bi-sets for arborescence packings was introduced by Bérczi and Frank [1] and then later developed by Bérczi, T. Király and Kobayashi [2]. We continue this development by showing how an intersecting submodular bi-set function induces a matroid.

We should also remark that Frank [8] used the same approach to reduce a rooted k-connection problem to matroid intersection, where the matroid induced by a modular bi-set function is considered.

Since the construction is rather involved, in Section 3, we first consider the matroid-based packing problem, a special case of the reachability-based packing. In this special case the problem is reduced to the matroid intersection problem between $\mathcal{M}_2$ and a matroid induced by an intersecting submodular *set* function.

For more details on the historical background and recent development of arborescence packings, see [14]. The matroid terminology used here will follow [16].

2 Constructing Matroids from Submodular Functions

In this section we review a construction of matroids from intersecting submodular set functions and then extend it to bi-set functions.

2.1 Set functions

Let S be a finite set. Two sets $X, Y \subseteq V$ are *intersecting* if $X \cap Y \neq \emptyset$. The family $\mathcal{Q}$ of subsets of S is said to be *intersecting* if $X \cup Y, X \cap Y \in \mathcal{Q}$ for every intersecting $X, Y \in \mathcal{Q}$. A function $f : \mathcal{Q} \to \mathbb{R}$ on an intersecting family $\mathcal{Q}$ is called *intersecting submodular* if $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for any intersecting $X, Y \in \mathcal{Q}$, and it is called *monotone* if $f(X) \leq f(Y)$ for every $X, Y \in \mathcal{Q}$ with $X \subseteq Y$. Initiated by Edmonds and Rota [7] or Edmonds [3], several authors gave constructions of matroids from (intersecting) submodular functions. We use the following form (see Section 13.4.1 of [9], or Section 3.4(c) in [10]).

Theorem 2.1. *Let $\mathcal{Q}$ be an intersecting family of subsets of a finite set S and $f : \mathcal{Q} \to \mathbb{Z}_{\geq 0}$ a monotone intersecting submodular set function. Then*

$$\mathcal{I}_f = \{Y \subseteq S : |X| \leq f(X) \ \forall X \in \mathcal{Q}, X \subseteq Y\}$$

forms the independent set family of a matroid $\boldsymbol{\mathcal{M}_f}$ and

$$\boldsymbol{P_f} := \{x \in \mathbb{R}^S : x(X) \leq f(X) \ \forall X \in \mathcal{Q}, \ 0 \leq x(v) \leq 1 \ \forall v \in S\}$$

is the convex hull of the incidence vectors of the independent sets of $\mathcal{M}_f$.

2.2 Bi-set functions

We use the following terminologies for *bi-sets*. Let $D = (V, A)$ be a digraph. For $B \subseteq A$, let $V(B)$ be the set of the endvertices of the arcs in B while let $H(B)$ be the set of heads of the arcs in B. The set of all bi-sets $\{\mathsf{X} = (X_O, X_I) : X_I \subseteq X_O \subseteq V\}$ is denoted by $\mathcal{P}_2(V)$ or simply by $\mathcal{P}_2$. For $\mathsf{X} = (X_O, X_I) \in \mathcal{P}_2$ and $B \subseteq A$, X_O and X_I denote the outer-set X_O and the inner-set X_I of X, respectively, $B(\mathsf{X}) := \{uv \in B : u \in X_O, v \in X_I\}$ and $i_B(\mathsf{X}) := |B(\mathsf{X})|$. Note that, for $X \subseteq V$, $i_B(X) = i_B((X, X))$. For $\mathsf{X} \in \mathcal{P}_2$, $Y \subseteq V + s$ and $B \subseteq A$, $\partial^B_Y(\mathsf{X})$ denotes the set of arcs in B from $Y \setminus \mathsf{X}_O$ to X_I. For $\mathsf{X}, \mathsf{Y} \in \mathcal{P}_2$, we denote $\mathsf{X} \subseteq \mathsf{Y}$ if $\mathsf{X}_I \subseteq \mathsf{Y}_I$ and $\mathsf{X}_O \subseteq \mathsf{Y}_O$. The intersection $\cap$ and the union $\cup$ of bi-sets $\mathsf{X}, \mathsf{Y} \in \mathcal{P}_2$ are defined by $\mathsf{X} \cap \mathsf{Y} := (X_O \cap Y_O, X_I \cap Y_I)$ and $\mathsf{X} \cup \mathsf{Y} := (X_O \cup Y_O, X_I \cup Y_I)$. Bi-sets X and Y are said to be *intersecting* if $X_I \cap Y_I \neq \emptyset$.

Note that for $\mathsf{X}, \mathsf{Y} \in \mathcal{P}_2$,

$$A(\mathsf{X}) \cap A(\mathsf{Y}) = A(\mathsf{X} \cap \mathsf{Y}), \tag{3}$$

$$A(\mathsf{X}) \cup A(\mathsf{Y}) \subseteq A(\mathsf{X} \cup \mathsf{Y}). \tag{4}$$

A family $\mathcal{F}$ of bi-sets is *intersecting* if $\mathsf{X} \cap \mathsf{Y} \in \mathcal{F}$ and $\mathsf{X} \cup \mathsf{Y} \in \mathcal{F}$ for any intersecting bi-sets $\mathsf{X}, \mathsf{Y} \in \mathcal{F}$. A function $f : \mathcal{F} \to \mathbb{R}$ is *intersecting submodular* if $f(\mathsf{X}) + f(\mathsf{Y}) \geq f(\mathsf{X} \cup \mathsf{Y}) + f(\mathsf{X} \cap \mathsf{Y})$ for all intersecting $\mathsf{X}, \mathsf{Y} \in \mathcal{F}$.

Frank [8, Theorem 3.3] proved the following statement for modular bi-set functions. We remark that the same argument works for intersecting submodular bi-set functions.

Theorem 2.2. *Let $D = (V, A)$ be a digraph, $\mathcal{F}$ an intersecting bi-set family on V, and $f : \mathcal{F} \to \mathbb{Z}_{\geq 0}$ an intersecting submodular bi-set function. Then*

$$\mathcal{I} := \{B \subseteq A : i_B(\mathsf{X}) \leq f(\mathsf{X})\ \forall \mathsf{X} \in \mathcal{F}\}$$

forms the family of independent sets of a matroid on A.

Proof. Let $\mathcal{A} = \{F \subseteq A : \exists \mathsf{X} \in \mathcal{F}, (V(F), H(F)) \subseteq \mathsf{X}\}$, and define $h : \mathcal{A} \to \mathbb{Z}$ by $h(F) = \min\{f(\mathsf{X}) : (V(F), H(F)) \subseteq \mathsf{X} \in \mathcal{F}\}$ for $F \in \mathcal{A}$.

Take any $F_1, F_2 \in \mathcal{A}$, and let $X_i \in \mathcal{H}$ be a minimizer in the definition of $h(F_i)$ for $i = 1, 2$. Note that

$$\begin{aligned} (V(F_1 \cap F_2), H(F_1 \cap F_2)) &\subseteq (V(F_1) \cap V(F_2), H(F_1) \cap H(F_2)) \subseteq \mathsf{X}_1 \cap \mathsf{X}_2 \\ (V(F_1 \cup F_2), H(F_1 \cup F_2)) &= (V(F_1) \cup V(F_2), H(F_1) \cup H(F_2)) \subseteq \mathsf{X}_1 \cup \mathsf{X}_2. \end{aligned} \tag{5}$$

To see that $\mathcal{A}$ is an intersecting family, suppose that $F_1 \cap F_2 \neq \emptyset$. Then $H(F_1) \cap H(F_2) \neq \emptyset$, and hence $(\mathsf{X}_1)_I \cap (\mathsf{X}_2)_I \neq \emptyset$. As $\mathcal{F}$ is an intersecting family, $\mathsf{X}_1 \cap \mathsf{X}_2 \in \mathcal{F}$ and $\mathsf{X}_1 \cup \mathsf{X}_2 \in \mathcal{F}$. Therefore (5) implies that $F_1 \cap F_2 \in \mathcal{A}$ and $F_1 \cup F_2 \in \mathcal{A}$, and $\mathcal{A}$ is indeed an intersecting family.

Also (5) and the intersecting submodularity of f implies the intersecting submodularity of h as follows:

$$h(F_1) + h(F_2) = f(\mathsf{X}_1) + f(\mathsf{X}_2) \geq f(\mathsf{X}_1 \cap \mathsf{X}_2) + f(\mathsf{X}_1 \cup \mathsf{X}_2) \geq h(F_1 \cap F_2) + h(F_1 \cup F_2).$$

We show now that $B \in \mathcal{I}$ if and only if $|F| \leq h(F)$ for every $F \subseteq B$ with $F \in \mathcal{A}$. Indeed, if $B \in \mathcal{I}$, then for any $F \subseteq B$ with $F \in \mathcal{A}$ and for any X with $(V(F), H(F)) \subseteq \mathsf{X} \in \mathcal{F}$, $|F| \leq i_B(\mathsf{X}) \leq f(\mathsf{X})$ and hence $|F| \leq h(F)$. On the other hand, if $|F| \leq h(F)$ for every $F \subseteq B$ with $F \in \mathcal{A}$, then for any $\mathsf{X} \in \mathcal{F}$, $B(\mathsf{X}) \in \mathcal{A}$ and hence $i_B(\mathsf{X}) = |B(\mathsf{X})| \leq h(B(\mathsf{X})) \leq f(\mathsf{X})$.

The statement now follows from Theorem 2.1. □

3 Minimum Weight Packing: Matroid-based Case

As a warm up for the next section, in this section we consider the minimum weight matroid-based packing problem, a special case of the reachability based packing problem.

3.1 Algorithm

Let $D = (V + s, A)$ be a rooted digraph, $\mathcal{M}_1 = (\partial_s(V), r_1)$ and $\mathcal{M}_2 = (A, r_2) = \oplus_{v\in V}\mathcal{M}_v$ two matroids, where $\mathcal{M}_1$ is a matroid of rank k. As we explained in the introduction, our goal is to find a minimum weight arc set that can be decomposed into a matroid-based matroid-restricted packing of s-arborescences. We show how to reduce the problem to the weighted matroid intersection problem.

The following set function b on A, introduced in [13], will play an important role:

$$b(H) := k|V(H) - s| - k + r_1(H \cap \partial_s(V)) \quad \forall\emptyset \neq H \subseteq A.$$

Observe that b is non-negative integer valued, monotone and intersecting submodular on $2^A \setminus \{\emptyset\}$, and hence by Theorem 2.1,

$$\mathcal{I}_b := \{B \subseteq A : |H| \leq b(H)\ \forall\emptyset \neq H \subseteq B\}$$

forms the independent set family of a matroid $\mathcal{M}_b$ on A. Section 4 of [13] provides a polynomial algorithm to decide whether a set B belongs to $\mathcal{I}_b$ or not.

Lemma 3.1. *Suppose that $\mathcal{M}_2 = \oplus_{v\in V}\mathcal{M}_v$ and each $\mathcal{M}_v$ has rank k. Then $B \subseteq A$ is the arc set of an $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-arborescences if and only if B is a common independent set of $\mathcal{M}_2$ and $\mathcal{M}_b$ of size $k|V|$.*

Proof. We first prove that, in both directions,

$$|\partial^B(v)| = k \qquad \forall v \in V. \tag{6}$$

Indeed, if B is the arc set of a packing, then (6) follows from the definition of $\mathcal{M}_1$-based packings. If B is a common independent set of size $k|V|$, then $k|V| = \sum_{v\in V} k \geq \sum_{v\in V} r_2(\partial^B(v)) = \sum_{v\in V} |\partial^B(v)| = |B| = k|V|$. Thus (6) follows.

By (6), for any $X \subseteq V$ and $F \subseteq \partial^B_s(X)$,

$$k|X| = \sum_{v\in X} |\partial^B(v)| = |B(X) \cup \partial^B(X)|. \tag{7}$$

Also, in both directions, B is independent in $\mathcal{M}_2$, and hence

$$|\partial^B(X) - F| = r_2(\partial^B(X) - F). \tag{8}$$

First suppose that $B \subseteq A$ is the arc set of an $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-arborescences. By Corollary 1.2,

$$r_1(F) + r_2(\partial^B(X) - F) \geq k \qquad (\forall\emptyset \neq X \subseteq V, F \subseteq \partial^B_s(X)). \tag{9}$$

To show that $B \in \mathcal{I}_b$, we prove that $|H| \leq b(H)\ \forall\emptyset \neq H \subseteq B$. Take any $H \subseteq B$ with $H \neq \emptyset$ and let $X := V(H) - s$ and $F := H \cap \partial_s(V)$. Then

$$|B(X) \cup \partial^B(X)| \geq |H| + |\partial^B(X)| - |F|. \tag{10}$$

By adding (7), (8), (10) and (9), we get the inequality $|H| \leq b(H)$, implying that $B \in \mathcal{I}_b$.

Now suppose that B is a common independent set of $\mathcal{M}_2$ and $\mathcal{M}_b$ of size $k|V|$. To verify that B satisfies (9), take any $X \subseteq V$ with $X \neq \emptyset$ and $F \subseteq \partial^B_s(X)$ and let $H := B(X) \cup F$. Then we again have (10). Also since B is independent in $\mathcal{M}_b$ and $H \subseteq B$,

$$|H| \leq b(H) = k|X| - k + r_1(F). \tag{11}$$

By adding (7), (8), (10) and (11), we get (9). By Corollary 1.2, digraph $(V+s, B)$ contains a $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-arborescences and, since its size is exactly $k|V| = |B|$, its arc set coincides with B. $\square$

Theorem 3.2. *Let $D = (V+s, A)$ be a rooted digraph, $c : A \to \mathbb{R}$, $\mathcal{M}_1 = (\partial(V), r_1)$ and $\mathcal{M}_2 = (A, r_2)$ two matroids such that $\mathcal{M}_2$ is the direct sum of matroids $\mathcal{M}_v = (\partial(v), r_v)$ for $v \in V$. There exists a polynomial algorithm to decide whether D has an $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-arborescences and to find one of minimum weight if D has at least one such packing.*

Proof. Let k be the rank of $\mathcal{M}_1$. If $\mathcal{M}_v$ has rank less than k for some $v \in V$, then we can immediately conclude that there is no $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing. If $\mathcal{M}_v$ has rank at least k, then we may suppose that each $\mathcal{M}_v$ has rank exactly k by truncating it at k. Hence by Lemma 3.1 and Edmonds' weighted matroid intersection algorithm [6], we can find the arc set of minimum weight that can be decomposed into an $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-arborescences in D. The required decomposition can be then obtained by the algorithm of [14, Section 6]. $\square$

3.2 Polyhedral aspects

An immediate corollary of Lemma 3.1 is a polyhedral description of the characteristic vectors of the arc sets of the matroid-based matroid-restricted packings of arborescences as the intersection of two base polyhedra due to Edmonds [3]. In this subsection we provide a slightly different description which is more natural and fits better to Corollary 1.2.

Theorem 3.3. *Let $D = (V + s, A)$, $\mathcal{M}_1 = (\partial_s(V), r_1)$, $\mathcal{M}_2 = (A, r_2) = \oplus_{v\in V}\mathcal{M}_v$ where $\mathcal{M}_1$ and each $\mathcal{M}_v$ is a matroid of rank k. Let $\boldsymbol{P_{D,\mathcal{M}_1,\mathcal{M}_2}}$ be defined by the following linear system*

$$\begin{aligned} x(\partial(X) - F) &\geq k - r_1(F) \quad \forall\, \emptyset \neq X \subseteq V,\ \forall F \subseteq \partial_s(X), && (12)\\ r_2(J) &\geq x(J) \qquad\quad \forall\, J \subseteq \partial(v),\ \forall v \in V, && (13)\\ x(a) &\geq 0 \qquad\qquad\ \ \forall\, a \in A, && (14)\\ x(A) &= k|V|. && (15)\end{aligned}$$

Then $P_{D,\mathcal{M}_1,\mathcal{M}_2}$ is an integer polyhedron and its vertices are the characteristic vectors of the arc sets of the $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packings of s-arborescences in $(D, \mathcal{M}_1, \mathcal{M}_2)$.

Proof. First, we replace (12) by another inequality which is more convenient to apply the results of the previous section.

Claim 3.4. (12) *is equivalent to*

$$b(H) \geq x(H) \quad \forall\, \emptyset \neq H \subseteq A \qquad (16)$$

provided that (13) – (15) *are satisfied.*

Proof. Since $\mathcal{M}_2 = \oplus_{v\in V}\mathcal{M}_v$ and each matroid $\mathcal{M}_v$ is of rank k, (13) implies that $k|V| \geq \sum_{v\in V} r_2(\partial(v)) \geq \sum_{v\in V} x(\partial(v)) \geq x(A)$. Then, by (15), $x(v) = k$ for every $v \in V$. Hence (12) holds if and only if the following inequality holds for any nonempty $X \subseteq V$ and $F \subseteq \partial_s(X)$:

$$k|V(F \cup A(X)) - s| = k|X| = \sum_{v\in X} x(\partial(v)) = x(\partial(X) \cup A(X)) \geq k - r_1(F) + x(F \cup A(X)).$$

The latter condition is equivalent to (16) by (14). $\square$

By Theorem 2.1, the polyhedron P_b, defined by the inequalities (14), (16) and $x(a) \leq 1\ \forall a \in A$, is the convex hull of the incidence vectors of the independent sets of the matroid $\mathcal{M}_b$ (defined in the previous section). By Edmonds [4], the polyhedron P_2, which is the convex hull of the incidence vectors of the independent sets of the matroid $\mathcal{M}_2$, is defined by the inequalities (13) and (14). Then, by Edmonds [3], $P_3 := P_b \cap P_2$ is an integer polyhedron and is defined by (13), (14) and (16). (Note that the condition $x(a) \leq 1\ \forall a \in A$ is implied by (13) applied to $J = \{a\}$.) As we have seen above, (15) is a valid inequality for P_3. Then, by (15), $P_{D,\mathcal{M}_1,\mathcal{M}_2}$ (whose defining inequalities are, by Claim 3.4, (13)–(16)), is a face of the integer polyhedron P_3 and hence $P_{D,\mathcal{M}_1,\mathcal{M}_2}$ is also integer. Then, by Lemma 3.1 and Edmonds [3], the theorem follows. $\square$

4 Minimum Weight Packing: Reachability-based Case

4.1 Reducing to the weighted matroid intersection problem

Let $D = (V + s, A)$ be a rooted digraph, $\mathcal{M}_1 = (\partial_s^A(V), r_1)$ and $\mathcal{M}_2 = (A, r_2) = \oplus_{v \in V} \mathcal{M}_v$ two matroids. Let $m(v) = r_1(\partial_s^A(P(v))$ for each $v \in V$. Suppose also that a weight $c : A \to \mathbb{R}$ is given. In this section we prove that the problem of computing a minimum weight $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences can be reduced to the matroid intersection problem. To this end we first consider the case when the instance satisfies the following three conditions:

$$r_2(\partial^A(v)) = m(v) \quad (\forall v \in V), \tag{17}$$

$$\text{each root arc belongs to every base of } \mathcal{M}_2, \tag{18}$$

$$\partial_s^A(v) \text{ is independent in } \mathcal{M}_1 \text{ for every } v \in V. \tag{19}$$

These assumptions can be achieved by truncation of $\mathcal{M}_2$ and by subdivision of the root arcs; the detailed expositions are postponed to the end of this subsection.

When (18) holds, the cut condition (1) holds if and only if the inequality holds for every $X \subseteq V$ and $F = \partial_s^A(X)$. Hence in view of Theorem 1.1 our goal is to find a minimum weight arc set $B \subseteq A$ satisfying the following two conditions:

$$r_1(\partial_s^B(X)) + r_2(\partial_V^B(X)) \geq r_1(\partial_s^A(P(X)) \quad (\forall X \subseteq V). \tag{20}$$

$$|B| = \sum_{v \in V} m(v). \tag{21}$$

Theorem 4.1. *Let $D = (V + s, A)$ be a rooted digraph with $c : A \to \mathbb{R}$, $\mathcal{M}_1 = (\partial_s^A(V), r_1)$ and $\mathcal{M}_2 = (A, r_2) = \oplus_{v \in V} \mathcal{M}_v$ two matroids. Suppose that (17) and (18) are satisfied. Then a minimum weight arc set $B \subseteq A$ of an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing can be computed by solving a minimum weight matroid intersection problem.*

Our algorithm makes use of the following clever setting of a bi-set family and a bi-set function introduced by Bérczi and Frank [1] to understand the theorem by Kamiyama et al [12], and further developed by Bérczi et al [2].

Let us define $\sim$ as follows: for $u, v \in V$, $u \sim v$ if and only if $\partial_s^A(P(u)) = \partial_s^A(P(v))$. It is easy to see that $\sim$ is an equivalence relation. We call the equivalence classes $\boldsymbol{A_1, \ldots, A_\ell}$ as *atoms* of D. For every root arc e_i, let $\boldsymbol{U_i}$ be the set of vertices in V which can be reached from s via the arc e_i in D. Let

$$\begin{aligned}
\boldsymbol{\mathcal{F}} &:= \{\mathsf{X} \in \mathcal{P}_2 : \exists 1 \leq j \leq \ell,\ \emptyset \neq \mathsf{X}_I \subseteq A_j, (\mathsf{X}_O \setminus \mathsf{X}_I) \cap A_j = \emptyset\}, \\
\boldsymbol{I_{\mathsf{X}}} &:= \{e_i \in \partial_s^A(V) : \mathsf{X}_I \subseteq U_i, e_i \notin \partial_s^A(\mathsf{X}_I), (\mathsf{X}_O \setminus \mathsf{X}_I) \cap U_i = \emptyset\} && (\forall \mathsf{X} \in \mathcal{F}), \\
\boldsymbol{J_{\mathsf{X}}} &:= \{e_i \in \partial_s^A(V) : \mathsf{X}_I \subseteq U_i, \text{ and either } e_i \in \partial_s^A(\mathsf{X}_I) \text{ or } (\mathsf{X}_O \setminus \mathsf{X}_I) \cap U_i \neq \emptyset\} && (\forall \mathsf{X} \in \mathcal{F}), \\
\boldsymbol{p(\mathsf{X})} &:= r_1(I_{\mathsf{X}} \cup J_{\mathsf{X}}) - r_1(J_{\mathsf{X}}) && (\forall \mathsf{X} \in \mathcal{F}).
\end{aligned}$$

Note that for any $\mathsf{X} \in \mathcal{F}$,

$$I_{\mathsf{X}} \cup J_{\mathsf{X}} = \partial_s^A(P(\mathsf{X}_I)). \tag{22}$$

The following lemma motivates us to look at the function p. Although the lemma follows implicitly from Bérczi et al [2], we give a simpler (specialized) proof for completeness.

Lemma 4.2. *For any $B \subseteq A$, the following two conditions are equivalent:*

$$|\partial_V^B(X)| \geq r_1(\partial_s^A(P(X))) - r_1(\partial_s^A(X)) \qquad (\forall X \subseteq V) \tag{23}$$

$$|\partial_V^B(\mathsf{X})| \geq p(\mathsf{X}) \qquad (\forall \mathsf{X} \in \mathcal{F}) \tag{24}$$

Proof. (23)⇒ (24): This direction was explicitly discussed in [2] and the proof goes as follows. Suppose that (23) holds. For (24), consider any $\mathsf{X} = (X_O, X_I) \in \mathcal{F}$. Let $\mathsf{Y} := (Y_O, Y_I) := (X_I \cup (V \setminus \bigcup_{i \notin J_{\mathsf{X}}} U_i), X_I)$. By the definition of Y_O,

$$J_{\mathsf{X}} = J_{\mathsf{Y}} = \partial_s^A(Y_O). \tag{25}$$

Moreover, since no arc leaves $\bigcup_{i \notin J_{\mathsf{X}}} U_i$,

$$\partial_V^B(Y_O) \subseteq \partial_V^B(\mathsf{X}). \tag{26}$$

By $Y_I \subseteq Y_O \subseteq P(X_I) = P(Y_I)$, we have $P(Y_I) = P(Y_O)$, so that

$$I_{\mathsf{X}} \cup J_{\mathsf{X}} = \partial_s^A(P(X_I)) = \partial_s^A(P(Y_I)) = \partial_s^A(P(Y_O)), \tag{27}$$

where the first equality follows from (22). Theses arguments provide (24) as follows:

$$\begin{aligned} p(\mathsf{X}) = r_1(I_{\mathsf{X}} \cup J_{\mathsf{X}}) - r_1(J_{\mathsf{X}}) = r_1(\partial_s^A(P(Y_O))) - r_1(\partial_s^A(Y_O)) & \qquad \text{(by (25) and (27))} \\ \leq |\partial_V^B(Y_O)| & \qquad \text{(by (23))} \\ \leq |\partial_V^B(\mathsf{X})| & \qquad \text{(by (26))}. \end{aligned}$$

(24)⇒ (23): Suppose that (24) holds. To verify that (23) holds, take any $X \subseteq V$. We construct a directed graph D_{atom} on the set of all atoms obtained from D by contracting the set of vertices of each atom to a vertex. Then D_{atom} is acyclic. Let $\boldsymbol{v_0 = s, v_1, \ldots, v_\ell}$ be a topological order of this graph. We denote the atoms so that atom A_i corresponds to vertex v_i. Suppose that the atoms that intersect X are $A_{h_1}, \ldots A_{h_k}$, and let

$$\begin{aligned} \boldsymbol{\mathsf{X}_j} &:= (P(A_{h_j}) \cap X, A_{h_j} \cap X) \text{ for } 1 \leq j \leq k, \\ \boldsymbol{K_j} &:= \partial_s^A(\bigcup_{i=1}^{j} P(A_{h_i})), \\ \boldsymbol{L_j} &:= \partial_s^A(X \cap \bigcup_{i=1}^{j} A_{h_i}). \end{aligned}$$

With this setting of X_j, we have $\mathsf{X}_j \in \mathcal{F}$ and

$$|\partial_V^B(X)| = \sum_{j=1}^{k} |\partial_V^B(\mathsf{X}_j)|. \tag{28}$$

We prove by induction on i that

$$\sum_{j=1}^{i} p(\mathsf{X}_j) \geq r_1(K_i) - r_1(L_i) \qquad (\forall 1 \leq i \leq k). \tag{29}$$

If $i = 1$, then v_{h_i} is a source in D_{atom}, and hence by (22),

$$p(\mathsf{X}_1) = r_1(I_{\mathsf{X}_1} \cup J_{\mathsf{X}_1}) - r_1(J_{\mathsf{X}_1}) = r_1(\partial_s^A(P(A_{h_1}))) - r_1(\partial_s^A(X \cap A_{h_1})) = r_1(K_1) - r_1(L_1).$$

Suppose that (29) is satisfied for i' with $1 \leq i' < i$. The submodularity and the monotonicity of r_1 give

$$r_1(I_{\mathsf{X}_i} \cup J_{\mathsf{X}_i}) + r_1(J_{\mathsf{X}_i} \cup K_{i-1}) \geq r_1(J_{\mathsf{X}_i}) + r_1(I_{\mathsf{X}_i} \cup J_{\mathsf{X}_i} \cup K_{i-1}). \tag{30}$$

Also, by $L_{i-1} \subseteq K_{i-1}$, we have $K_{i-1} \cup (L_{i-1} \cup (J_{\mathsf{X}_i} \setminus K_{i-1})) = J_{\mathsf{X}_i} \cup K_{i-1}$ and $K_{i-1} \cap (L_{i-1} \cup (J_{\mathsf{X}_i} \setminus K_{i-1})) = L_{i-1}$. Hence, by the submodularity of r_1, we have

$$r_1(K_{i-1}) + r_1(L_{i-1} \cup (J_{\mathsf{X}_i} \setminus K_{i-1})) \geq r_1(L_{i-1}) + r_1(J_{X_i} \cup K_{i-1}). \tag{31}$$

Combining those inequalities we get

$$\begin{aligned}
\sum_{j=1}^{i} p(\mathsf{X}_j) &= r_1(I_{\mathsf{X}_i} \cup J_{\mathsf{X}_i}) - r_1(J_{\mathsf{X}_i}) + \sum_{j=1}^{i-1} p(\mathsf{X}_j) \\
&\geq r_1(I_{\mathsf{X}_i} \cup J_{\mathsf{X}_i}) - r_1(J_{\mathsf{X}_i}) + r_1(K_{i-1}) - r_1(L_{i-1}) && \text{(by induction)} \\
&\geq r_1(I_{\mathsf{X}_i} \cup J_{\mathsf{X}_i} \cup K_{i-1}) - r_1(L_{i-1} \cup (J_{\mathsf{X}_i} \setminus K_{i-1})) && \text{(by (30) and (31))} \\
&= r_1(K_i) - r_1(L_i) && \text{(by definition).}
\end{aligned}$$

Thus (29) holds. In particular, by setting $i = k$, we get

$$\begin{aligned}
|\partial_V^B(X)| &= \sum_{j=1}^{k} |\partial_V^B(\mathsf{X}_j)| && \text{(by (28))} \\
&\geq \sum_{j=1}^{k} p(\mathsf{X}_j) && \text{(by (23))} \\
&\geq r_1(K_k) - r_1(L_k) && \text{(by (29))} \\
&= r_1(\partial^A(P(X))) - r_1(\partial_s^A(X)) && \text{(by definition)}
\end{aligned}$$

as we stated. □

We define a bi-set function $b : \mathcal{F} \to \mathbb{Z}$ as follows:

$$\boldsymbol{b(\mathsf{X})} := m(\mathsf{X}_I) - |\partial_s^A(\mathsf{X}_I)| - p(\mathsf{X}) \qquad (\forall \mathsf{X} \in \mathcal{F}).$$

Lemma 4.3. *$\mathcal{F}$ is an intersecting family and b is a non-negative intersecting submodular bi-set function on $\mathcal{F}$.*

Proof. Let $\mathsf{X} = (X_O, X_I)$ and $\mathsf{Y} = (Y_O, Y_I)$ be two intersecting bi-sets in $\mathcal{F}$. By the definition of $\mathcal{F}$ and $X_I \cap Y_I \neq \emptyset$, there exists a unique atom A_k containing both X_I and Y_I, and consequently $X_I \cap Y_I$ and $X_I \cup Y_I$. Since $(X_O \setminus X_I) \cap A_k = \emptyset = (Y_O \setminus Y_I) \cap A_k$, we have $((X_O \cap Y_O) \setminus (X_I \cap Y_I)) \cap A_k = \emptyset = ((X_O \cup Y_O) \setminus (X_I \cup Y_I)) \cap A_k$, and hence $\mathsf{X} \cap \mathsf{Y}, \mathsf{X} \cup \mathsf{Y} \in \mathcal{F}$, that is, $\mathcal{F}$ is intersecting.

To see the intersecting submodularity of b, it suffices to show, by the modularity of $m(\mathsf{X}_I) - |\partial_s^A(\mathsf{X}_I)|$, that p is intersecting supermodular. By the definition of an atom and (22), we have

$$\partial_s^A(P(A_k)) = I_{\mathsf{X}} \cup J_{\mathsf{X}} = I_{\mathsf{Y}} \cup J_{\mathsf{Y}} = I_{\mathsf{X}\cap\mathsf{Y}} \cup J_{\mathsf{X}\cap\mathsf{Y}} = I_{\mathsf{X}\cup\mathsf{Y}} \cup J_{\mathsf{X}\cup\mathsf{Y}}, \tag{32}$$

Also by the definition of J,

$$J_{\mathsf{X}} \cup J_{\mathsf{Y}} = J_{\mathsf{X}\cup\mathsf{Y}} \quad \text{and} \quad J_{\mathsf{X}} \cap J_{\mathsf{Y}} \supseteq J_{\mathsf{X}\cap\mathsf{Y}}. \tag{33}$$

Thus

$$\begin{aligned}
p(X) + p(Y) &= r_1(I_{\mathsf{X}} \cup J_{\mathsf{X}}) - r_1(J_{\mathsf{X}}) + r_1(I_{\mathsf{Y}} \cup J_{\mathsf{Y}}) - r_1(J_{\mathsf{Y}}) \\
&\leq r_1(I_{\mathsf{X}} \cup J_{\mathsf{X}}) + r_1(I_{\mathsf{Y}} \cup J_{\mathsf{Y}}) - r_1(J_{\mathsf{X}} \cap J_{\mathsf{Y}}) - r_1(J_{\mathsf{X}} \cup J_{\mathsf{Y}}) && \text{(by submodularity)} \\
&\leq r_1(I_{\mathsf{X}\cap\mathsf{Y}} \cup J_{\mathsf{X}\cap\mathsf{Y}}) + r_1(I_{\mathsf{X}\cup\mathsf{Y}} \cup J_{\mathsf{X}\cup\mathsf{Y}}) - r_1(J_{\mathsf{X}\cap\mathsf{Y}}) - r_1(J_{\mathsf{X}\cup\mathsf{Y}}) && \text{(by (32) and (33))} \\
&= p(\mathsf{X} \cap \mathsf{Y}) + p(\mathsf{X} \cup \mathsf{Y}).
\end{aligned}$$

Finally, to see that b is non-negative, take any $\mathsf{X} \in \mathcal{F}$. Let $d = r_1(\partial_s^A(P(\mathsf{X}_I)))$. By (22), $r_1(I_\mathsf{X} \cup J_\mathsf{X}) = d$. Also, since X_I is contained in an atom, the definition of atoms implies $r_1(\partial_s^A(P(v))) = d$ for every $v \in \mathsf{X}_I$. Hence, picking any $v \in \mathsf{X}_I$, we have

$$\begin{aligned}
b(\mathsf{X}) &= \sum_{u \in \mathsf{X}_I} (r_1(\partial_s^A(P(u)) - |\partial_s^A(u)|) - p(\mathsf{X}) \\
&\geq r_1(\partial_s^A(P(v))) - |\partial_s^A(v)| - r_1(I_\mathsf{X} \cup J_\mathsf{X}) + r_1(J_\mathsf{X}) && (\text{by } |\partial_s^A(u)| \leq r_1(\partial_s^A(P(u))) \text{ from (19)}) \\
&= r_1(J_\mathsf{X}) - |\partial_s^A(v)| && (\text{by } r_1(\partial_s^A(P(v))) = d = r_1(I_\mathsf{X} \cup J_\mathsf{X})) \\
&\geq r_1(\partial_s^A(v)) - |\partial_s^A(v)| && (\text{by } \partial_s^A(v) \subseteq J_\mathsf{X}) \\
&\geq 0 && (\text{by (19)}).
\end{aligned}$$

Thus b is non-negative. □

Let $\boldsymbol{\mathcal{I}} := \{B \subseteq A : i_B(\mathsf{X}) \leq b(\mathsf{X}) \ \forall \mathsf{X} \in \mathcal{F}\}$. Then, by Lemma 4.3 and Theorem 2.2, $\mathcal{I}$ forms the independent set family of a matroid on A. This matroid is denoted by $\boldsymbol{\mathcal{M}}$.

Theorem 4.1 now follows from the following lemma.

Lemma 4.4. *Suppose that (17) and (18) are satisfied. Then $B \subseteq A$ is the arc set of an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences if and only if B is a common independent set of $\mathcal{M}_2$ and $\mathcal{M}$ of size $m(V)$.*

Proof. First let us mention that the rank of $\mathcal{M}_2$ is $m(V)$. Indeed, by (17),

$$r_2(A) = \sum_{v \in V} r_2(\partial^A(v)) = \sum_{v \in V} r_1(\partial_s^A(P(v))) = \sum_{v \in V} m(v) = m(V). \tag{34}$$

Note that in both directions B is independent in $\mathcal{M}_2$. Moreover, we have

$$\begin{aligned}
&|\partial^B(v)| = m(v) \quad (\forall v \in V), && (35)\\
&\partial_s^A(V) \subseteq B, && (36)\\
&|B| = m(V), && (37)
\end{aligned}$$

in both directions. Indeed, if a packing exists then (35) and (36) follow from the definition of reachability-based packing and (18). Thus $|B| = \sum_{v \in V} |\partial^B(v)| = \sum_{v \in V} r_1(\partial_s^A(P(v)) = \sum_{v \in V} m(v) = m(V)$, where the second equation also follows from the definition of reachability-packing.

On the other hand, if B is a common independent set with $|B| = m(V)$, then, by (17) and the independence of B in $\mathcal{M}_2$,

$$m(V) = \sum_{v \in V} m(v) = \sum_{v \in V} r_2(\partial^A(v)) \geq \sum_{v \in V} r_2(\partial^B(v)) = \sum_{v \in V} |\partial^B(v)| = |B| = m(V),$$

implying (35). It follows, by (34), that B is a base of $\mathcal{M}_2$ and hence, by (18), that $\partial_s^A(V) \subseteq B$, i.e., (36) holds.

By (36), the independence condition for $\mathcal{M}$, that is, $i_B(\mathsf{X}) \leq b(\mathsf{X})$ $(\forall \mathsf{X} \in \mathcal{F})$, is written as

$$i_B(\mathsf{X}) \leq m(X_I) - |\partial_s^B(X_I)| - p(\mathsf{X}) \qquad (\mathsf{X} \in \mathcal{F}). \tag{38}$$

On the other hand, since

$$m(X_I) \;=\; \sum_{v \in X_I} |\partial^B(v)| = i_B(\mathsf{X}) + |\partial^B(\mathsf{X})| \tag{39}$$

holds by (35), (38) is equivalent to

$$p(\mathsf{X}) \leq |\partial_V^B(\mathsf{X})| \qquad (\forall \mathsf{X} \in \mathcal{F}). \tag{40}$$

On the other hand, Lemma 4.2 and (36) imply that (40) is equivalent to

$$|\partial^B(X)| \geq r_1(\partial_s^A(P(X))) - r_1(\partial_s^B(X)) \qquad (\forall X \subseteq V). \tag{41}$$

The last condition is equivalent to (20), that is, the condition for B to be the arc set of an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing (under the condition that $|B| = m(V)$). □

We now show how to solve the general case of the minimum weight packing problem (without assuming (17), (18) and (19)). Let $(D = (V + s, A), c, \mathcal{M}_1, \mathcal{M}_2)$ be an instance of the problem. Suppose that there exists a feasible ($\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted) packing. Let B_0 be the union of the arc sets of the arborescences in this packing. Then, for each vertex $v \in V$, $\partial^{B_0}(v)$ is independent in $\mathcal{M}_v$ (and hence in $\mathcal{M}_2$) of size $r_1(\partial_s^A(P(v))$. By truncating each matroid $\mathcal{M}_v$ at $r_1(\partial_s^A(P(v))$, B_0 certifies that the problem still has a solution. (Conversely, every feasible packing of the latter problem is a feasible packing of the former one.) Hence we may suppose (17) by truncating each matroid $\mathcal{M}_v$ at the preprocessing.

Next, to achieve (18) and (19) we construct a new instance $(D', c', \mathcal{M}_1', \mathcal{M}_2')$ from $(D, c, \mathcal{M}_1, \mathcal{M}_2)$. We first remove from D all the root-arcs that are loops in $\mathcal{M}_1$. D' is obtained from the remaining digraph by subdividing each root arc sv to sv' and $v'v$ by inserting a new vertex v', and we set $c'(v'v) = c(sv), c'(sv') = 0$ and $c'(a) = c(a)$ for all non root arcs. M_1' is obtained from M_1 by replacing its ground set by $\partial_s^{A'}(V')$. Each new vertex v' in D' has in-degree one, and we assign a free matroid $\mathcal{M}_{v'}$ to each such v'. For the original vertices u of D, $\mathcal{M}_u'$ is obtained from $\mathcal{M}_u$ by replacing each root arc su by $u'u$. Then (18) and (19) are satisfied for the new instance.

The two instances are equivalent in the sense that from a feasible packing for $(D', c', \mathcal{M}_1', \mathcal{M}_2')$ one can easily construct a feasible packing for $(D, c, \mathcal{M}_1, \mathcal{M}_2)$ of the same weight and vice versa.

4.2 Algorithmic aspects

In Theorem 4.1 and a discussion at the end of the last subsection, we have seen that computing the minimum weight arc set of a feasible packing can be done by solving the weighted matroid intersection problem between $\mathcal{M}_2$ and $\mathcal{M} = (A, \mathcal{I})$. It remains to provide a polynomial-time independence oracle for $\mathcal{M}$. We will show that the independence of each arc set B can be checked by solving matroid intersection problems repeatedly.

Lemma 4.5. *We can decide in polynomial time whether a set B of arcs belongs to $\mathcal{I}$ or not.*

Proof. We will use the definitions from the last subsection. Using any searching algorithm, the sets U_i, and hence the partition of V into atoms, I_{X} and J_{X} for any $\mathsf{X} \in \mathcal{F}$ can be computed in polynomial time. Also it can be decided in polynomial time whether a bi-set X belongs to $\mathcal{F}$ or not.

Recall that B is independent in $\mathcal{I}$ if and only if $i_B(\mathsf{X}) \leq b(\mathsf{X})$ for every $\mathsf{X} \in \mathcal{F}$. Since there are $O(n)$ atoms, we may focus on checking the inequality for every $\mathsf{X} \in \mathcal{F}$ with $X_I \subseteq A_i$ for a fixed atom A_i. In other words, our goal is to check

$$i_B(\mathsf{X}) \leq m(\mathsf{X}_I) - |\partial_s^A(\mathsf{X}_I)| - r_1(I_{\mathsf{X}} \cup J_{\mathsf{X}}) + r_1(J_{\mathsf{X}}) \quad (\forall \mathsf{X} \in \mathcal{F}_i := \{\mathsf{Y} \in \mathcal{F} : \mathsf{Y}_I \subseteq A_i\}).$$

By (22), $r_1(I_{\mathsf{X}} \cup J_{\mathsf{X}})$ is constant over $\mathcal{F}_i$. Therefore, it suffices to design an algorithm for checking the following condition for a given $k : A_i \to \mathbb{Z}_+$ and $\ell \in \mathbb{Z}_+$:

$$i_B(\mathsf{X}) \leq k(X_I) - \ell + r_1(J_{\mathsf{X}}) \qquad (\forall \mathsf{X} \in \mathcal{F}_i). \tag{42}$$

We first solve the case when $\ell = 0$, that is, checking

$$i_B(\mathsf{X}) \leq k(X_I) + r_1(J_{\mathsf{X}}) \qquad (\forall \mathsf{X} \in \mathcal{F}_i), \tag{43}$$

and then show how to deal with the general case. The special case when $\ell = 0$ will be done by reducing the problem to the *independence matching problem*, which is known to be equivalent to the matroid intersection problem (see, e.g., [15]). In the independence matching problem, we are given a bipartite graph $G = (U, W; E)$ and a matroid $\mathcal{M}_W$ on W. A matching M in G is said to be *independent* if $W(M)$ is independent in $\mathcal{M}_W$, where $W(M)$ denotes the endvertices of M in W.

In order to define G and $\mathcal{M}_W$ appropriately we need the following definitions. For an arc a, let $t(a)$ and $h(a)$ be the tail and the head of a, respectively. Let $B_1 = B(A_i)$ and $B_2 = \partial_V^B(A_i)$. For each vertex $v \in A_i$, we prepare $k(v)$ copies $v_1, \ldots, v_{k(v)}$ of v, and let $kA_i = \{v_1, \ldots, v_{k(v)} : v \in A_i\}$ be the set of all those copies. Then we define an auxiliary bipartite graph $G = (U, W; E)$ as follows:

$$U = B_1 \cup B_2 \ (\subseteq B),$$
$$W = kA_i \cup \partial_s^A(V),$$
$$E = E_1 \cup E_2 \cup E_3 \cup E_4,$$

where

$$E_1 = \{av_i : a \in B_1, v \in A_i, h(a) = v \text{ or } t(a) = v\}$$
$$E_2 = \{av_i : a \in B_2, v \in A_i, h(a) = v\}$$
$$E_3 = \{ae_j : a \in B_1 \cup B_2, e_j \in \partial_s^A(A_i), h(a) = h(e_i)\}$$
$$E_4 = \{ae_j : a \in B_2, e_j \in \partial_s^A(V) \setminus \partial_s^A(A_i), t(a) \in U_j\}.$$

A matroid $\mathcal{M}_W$ is defined to be the sum of the free matroid on kA_i and $\mathcal{M}_1$. The rank function of $\mathcal{M}_W$ is denoted by r_W.

Claim 4.6. *(43) holds if and only if G has an independent matching that covers U.*

Proof. The Rado-Perfect theorem [18, 17] (see [15, (2.75)]) says that G has an independent matching of size d if and only if $|U \setminus C| + r_W(\Gamma(C)) \geq d$ for every $C \subseteq U$, where $\Gamma(C)$ denotes the set of neighbors of C in G. Hence G has an independent matching that covers U if and only if

$$|C| \leq r_W(\Gamma(C)) \qquad (C \subseteq U). \tag{44}$$

We show (43) is equivalent to (44).

Suppose that (43) holds. To see (44) take any $C \subseteq U$. We may suppose $C \neq \emptyset$. Take $\mathsf{X} = (X_O, X_I)$ such that $X_I = H(C) \cup T(C \cap B_1)$ and $X_O = V(C)$, where H and T denote the set of heads and tails of arcs of C in D, respectively. Then $X_I \neq \emptyset$, and we have $\mathsf{X} \in \mathcal{F}_i$. Also from the construction we have

$$C \subseteq B(\mathsf{X}), \tag{45}$$
$$J_{\mathsf{X}} = \Gamma(C) \cap \partial_s^A(V). \tag{46}$$

Hence we have

$$\begin{aligned} |C| &\leq i_B(\mathsf{X}) && \text{(by (45))} \\ &\leq k(X_I) + r_1(J_{\mathsf{X}}) && \text{(by (43))} \\ &= |\Gamma(C) \cap kA_i| + r_1(\Gamma(C) \cap \partial_s^A(V)) && \text{(by (46))} \\ &= r_W(\Gamma(C)). \end{aligned}$$

Thus (44) holds.

Conversely, suppose (44) holds. Take any $\mathsf{X} \in \mathcal{F}_i$. Observe that $B(\mathsf{X}) \subseteq U$ and each element in $\Gamma(B(\mathsf{X})) \cap kA_i$ is a copy of a vertex in X_I. Hence

$$|\Gamma(B(\mathsf{X})) \cap kA_i| \leq k(X_I). \tag{47}$$

By the construction we also have

$$\Gamma(B(\mathsf{X})) \cap \partial_s^A(V) = J_{\mathsf{X}}. \tag{48}$$

Hence we have

$$\begin{aligned} i_B(\mathsf{X}) = |B(\mathsf{X})| &\leq r_W(\Gamma(B(\mathsf{X}))) && \text{(by (44))} \\ &= |\Gamma(B(\mathsf{X})) \cap kA_i| + r_1(\Gamma(B(\mathsf{X})) \cap \partial_s^A(V)) \\ &\leq k(X_I) + r_1(J_{\mathsf{X}}) && \text{(by (47) and (48)).} \end{aligned}$$

Thus (43) holds. □

By Claim 4.6, we can check whether B satisfies (43) in polynomial time by a matroid intersection algorithm. It remains to extend the approach to the case when $\ell > 0$. We do this by using a standard technique for checking the independence in a so-called count matroid observed by Imai [11].

Let us use the auxiliary graph $G = (U, W; E)$ and $\mathcal{M}_U, \mathcal{M}_W$ defined above, and assume that B satisfies (43) for $\ell = 0$. We consider checking

$$i_B(X) \leq k(X_I) - \ell + r_1(\mathsf{X}) \qquad (\forall \mathsf{X} \in \mathcal{F}_i^a := \{\mathsf{X} \in \mathcal{F}_i : a \in B(\mathsf{X})\}) \tag{49}$$

for a fixed $a \in U$. We prepare a new auxiliary bipartite graph $G^a = (U^a, W; E^a)$ obtained from G by replacing a with $\ell + 1$ copies $a_0, \ldots, a_\ell$ (and then replacing each edge $ax \in E$ in G incident to $a \in U$ with $\ell + 1$ copies $a_0x, \ldots, a_\ell x$). Applying the same proof as that in Claim 4.6 we have the following:

Claim 4.7. *Suppose that B satisfies (43). Then (49) holds if and only if G^a has an independent matching that covers U^a.*

In view of this claim one can check (42) in polynomial time first by checking (43) by computing a maximum independent matching in G and then checking (49) by computing a maximum independent matching in G^a for every $a \in U$. This completes the proof. □

We have polynomial independence oracles for $\mathcal{M}_2$ and, by Lemma 4.5, also for $\mathcal{M}$. Then we have the following result.

Theorem 4.8. *Let $D = (V + s, A)$ be a rooted digraph, $\mathcal{M}_1 = (\partial(V), r_1)$ and $\mathcal{M}_2 = (A, r_2)$ two matroids such that $\mathcal{M}_2$ is the direct sum of the matroids $\mathcal{M}_v = (\partial(v), r_v)$ for $v \in V$. Let c be a weighting on the arc set A. There exists a polynomial algorithm to find an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences in D of minimum weight.*

Proof. By Lemma 4.4, Lemma 4.5 and Edmonds' weighted matroid intersection algorithm [6], we can find the arc set of minimum weight that can be decomposed into an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences in D. The required decomposition can be then obtained by the algorithm of [14]. □

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