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Abstract

This paper deals with hierarchically decentralized control structure for large-scaled dynamical networked systems by aggregation. Our main idea to clarify the trade-off and the role-sharing of the global and the local controllers is to introduce a model set named ”Global/Local Shared Model Set,” which should be taken in both the global and local cites. We set up a fairly general framework and derive the global and local control problems. We then clarify the trade-off through the size of the model set and demonstrate it by a simple example.

1 Introduction

In recent years, systems to be treated in various fields of engineering including control have become large and complex. Typical examples include meteorological phenomena, energy network systems, traffic flow networks and biological systems. They can be regarded as hierarchical networked dynamical systems, and several new frameworks to treat such systems from the view point control have been proposed (See e.g. [1–3]). Hara et. al. proposed a new research area so called “Glocal Control” meaning that both desired global and local behaviors are achieved by local actions of measurement and control [3]. The key framework is based on hierarchical networked systems with multiple-resolution in time and space, and each layer has its own objective which might be conflict with other layers’ objectives. Hence, one of the big issues to realize the glocal control is to establish a unified way of handling global/local objectives properly as a hierarchically decentralized control.

We consider the following situation. There are a bunch of subsystems which are slightly different each other, and we assume that each of them is equipped with a local controller which can be designed independently so that it stabilizes its own subsystem and optimizes a certain local objective. There also exists a so called global controller which uses the average or sum of a certain quantity of the locally controlled subsystems to coordinate all the subsystems properly for optimizing a certain global objective.

There are two main reasons to consider such a situation. The first reason is from the practical view point. A typical example is electric power network systems, where the global control objective is to make the balance of demand and supply of the total power of multiple generators, and each generator is locally controlled to achieve the local performance better. The second reason is from the theoretical view point. As seen in [4], averaging or low-rank inter-layer interactions in general is quite effective to achieve the rapid consensus, and the property is fit to the glocal control concept based on hierarchical networked systems with multiple time/space resolutions [3].

There are two theoretical key issues to be investigated for hierarchically decentralized control by aggregation, namely (i) how to guarantees the stability of whole system?, and (ii) how to derive the global/local trade-off relation and how to compromise it?

To this end, we introduce a model set named "*Global/Local Shared Model Set*," which is defined by a standard LFT form consisting of the nominal model and norm-bounded perturbations. The nominal model is set for both the aggregated system and each locally controlled subsystem to be followed within a certain error bound. Then, each local controller is designed to make the resultant feedback loop system of the local subsystem to be in the class and simultaneously attain a local control performance. On the other hand, the global controller is designed to attain control performance for this nominal model under consideration of the errors between the nominal model and the local feedback loop systems. Thus, through changing the size of the model set we will clarify the trade-off in the hierarchically decentralized control systems and provide a simple illustrative example to show the effectiveness of the approach.

Notation: RH_∞: stable rational ring
$\mathcal{S}_c(P)$: a set of all stabilizing controllers for P

2 General problem setting

We here propose a fairly general setting, which is represented by a block diagram depicted in Fig. 1. The system consists of two layers, the upper layer for global control and the lower layer for local control. They are connected each other by aggregation $\frac{1}{N}\mathbf{1}^\top$ from bottom to up and distribution $\mathbf{1}$ from up to bottom, where

$\mathbf{1} := [1, 1, \dots, 1]^\top$ with size N.

$$\boldsymbol{G}(s) = \begin{bmatrix} G_{yu}(s) & G_{yw}(s) & G_{yv}(s) \\ G_{zu}(s) & G_{zw}(s) & G_{zv}(s) \\ G_{ru}(s) & G_{rw}(s) & G_{rv}(s) \end{bmatrix},$$

$$\boldsymbol{L}(s) = \begin{bmatrix} L_{vr}(s) & L_{vw}(s) & L_{vu}(s) \\ L_{zr}(s) & L_{zw}(s) & L_{zu}(s) \\ L_{yr}(s) & L_{yw}(s) & L_{yu}(s) \end{bmatrix},$$

$$L_{*\sharp}(s) = \text{diag}\left\{ L_{*\sharp}^{(i)}(s) \right\},$$

$$\boldsymbol{C}_\ell(s) = \text{diag}\left\{ C_\ell^{(i)}(s) \right\}.$$

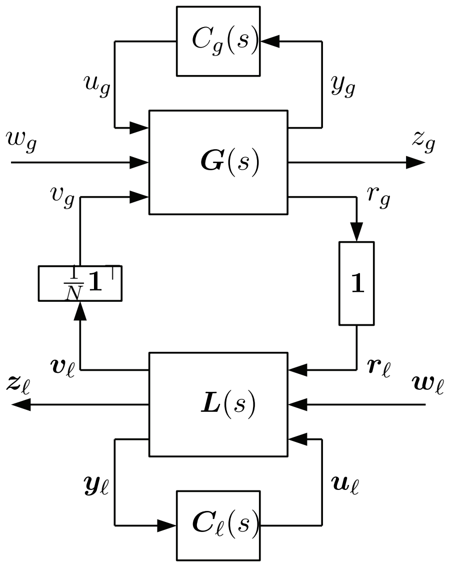

Fig. 1: Structure of total system

The upper and lower layer generalized plants are represented by $\boldsymbol{G}(s)$ and $\boldsymbol{L}(s)$, respectively. The block diagonal property of $\boldsymbol{L}(s)$ means that the lower layer is a collection of independent subsystems. Each subsystem thus has an independent local controller $C_\ell^{(i)}(s)$, and hence we have the block diagonal property of $\boldsymbol{C}_\ell(s)$.

The collections of inputs and outputs of the local controllers are denoted by $\boldsymbol{y}_\ell$ and $\boldsymbol{u}_\ell$, respectively. Signals $\boldsymbol{v}_\ell$ and $\boldsymbol{r}_\ell$ are N-dimensional vectors which correspond to signals to link the upper layer, i.e., $v_g = \frac{1}{N}\mathbf{1}^\top \boldsymbol{v}_\ell$ and $\boldsymbol{r}_\ell = \mathbf{1} r_g$. Signals $\boldsymbol{w}_\ell$ and $\boldsymbol{z}_\ell$ are the collections of input and output variables for representing the local objective, respectively. The element-wise representation of $\boldsymbol{u}_\ell$ is given by $\boldsymbol{u}_\ell = [u_1, u_2, \dots, u_N]^\top$, and the same notation is used for $\boldsymbol{y}_\ell$, $\boldsymbol{v}_\ell$, $\boldsymbol{r}_\ell$, $\boldsymbol{w}_\ell$, and $\boldsymbol{z}_\ell$. The upper layer is controlled by the global controller $C_g(s)$ with input u_g and output y_g. Signals v_g and r_g are the aggregated signal from the lower layer and the reference signal to be sent out to the lower layer, respectively. Signals w_g and z_g are the input and output variables for representing the global objective, respectively.

Our main idea to achieve the two requirements (i) and (ii) mentioned above, or (i) stability of total system and (ii) global/local performance trade-off, is to introduce a set of model set named "*Global/Local Shared Model Set*," which is defined by a standard LFT form as

$$\mathcal{M}_\delta := \left\{ \tilde{\boldsymbol{M}} = \mathcal{F}_\ell\left(M_o(s), \boldsymbol{\Delta}(s)\right) : \ \|\boldsymbol{\Delta}\|_\infty \le \delta \right\}, \tag{1}$$

where

$$M_o(s) = \begin{bmatrix} M_0(s) & M_1(s) \\ M_2(s) & 0 \end{bmatrix}. \tag{2}$$

The set is expected to be shared by both upper and lower layers in the following sense. The upper layer expects that each local agent is controlled such that the closed-loop transfer function from r_i to v_i denoted by $\Phi_{Lvr}^{(i)}(s)$ belongs to $\mathcal{M}_\delta$, and hence the lower layer tries to optimize the local objective related to the closed-loop transfer function from w_i to z_i denoted by $\Phi_{Lzw}^{(i)}(s)$ under the requirement from the upper layer, where

$$\Phi_{Lvr}^{(i)} := \mathcal{F}_\ell\left(\begin{bmatrix} L_{vr}^{(i)} & L_{vu}^{(i)} \\ L_{yr}^{(i)} & L_{yu}^{(i)} \end{bmatrix}, C_\ell^{(i)}\right), \quad \Phi_{Lzw}^{(i)} := \mathcal{F}_\ell\left(\begin{bmatrix} L_{zw}^{(i)} & L_{zu}^{(i)} \\ L_{yw}^{(i)} & L_{yu}^{(i)} \end{bmatrix}, C_\ell^{(i)}\right).$$

Then, the upper layer designs the upper layer controller $C_g(s)$ so that the global control performance represented by the transfer function from w_g to z_g is optimized under uncertainty channel $\mathcal{M}_\delta$ connected in between r_g and v_g. Note that $\mathcal{M}_\delta$ includes the classes of additive and multiplicative perturbations and that the averaging does not change the size of uncertainty δ as will be shown in the next section.

Consequently, we can split the global/local controller design into two independent designs, *Global Controller Design* and *Local Controller Design*.

Global Controller Design:

$$\min_{C_g \in \mathcal{S}_c(G_{yu})} \left\{ \max_{\tilde{M} \in \mathcal{M}_\delta} \|\Phi_{Gzw}\|_\infty \right\}, \tag{3}$$

where

$$\Phi_{Gzw} := \mathcal{F}_u\left\{\mathcal{F}_\ell\left\{\begin{bmatrix} \begin{pmatrix} G_{yu} & G_{yw} \\ G_{zu} & G_{zw} \\ G_{ru} & G_{rw} \end{pmatrix} & \begin{pmatrix} G_{yv} \\ G_{zv} \\ G_{rv} \end{pmatrix} \end{bmatrix}, \tilde{M}\right\}, C_g\right\}$$

Local Controller Design:

$$\min_{C_\ell^{(i)} \in \mathcal{S}_c(L_{yu}^{(i)})} \left\|\Phi_{Lzw}^{(i)}\right\|_\infty \quad \text{s.t.} \quad \Phi_{Lvr}^{(i)}(s) \in \mathcal{M}_\delta \tag{4}$$

The global and local problems are a robust performance problem and a 2-disk problem, respectively, and hence they are not so easy to derive the optimal controllers. However, we can investigate the global/local performance trade-off by the uncertainty level δ. We can readily see that the smaller δ leads to the better global performance and that the larger δ yields the better local performance. Note that we have a freedom of the selection of $M_o(s)$, although we assume in this paper that $M_o(s)$ is a priori selected for simplicity.

3 A typical situation with multiplicative perturbations

3.1 Aggregation of local systems

In this section, we give detailed formulations for a case of aggregation of local subsystems, where we replace notations $C_\ell^{(i)}(s)$ and $\Phi_{Lvr}^{(i)}(s)$ defined in the previous section by $C_\ell^{(i)}(s) = C_i(s)$ and $\Phi_{Lvr}^{(i)}(s) = \Phi_i(s)$ for simplicity.

We assume that the whole system consists of the following N local and SISO subsystems;

$$v_i = P_i(s)u_{oi}, \quad P_i(s) = \frac{n_i(s)}{d_i(s)}, \quad n_i(s),\, d_i(s) \in RH_\infty,$$

where $i\ (= 1,\, 2,\, \ldots,\, N)$ represents the index of a local subsystem, u_{oi} is the input, v_i is the output, $P_i(s)$ is the transfer function represented by a coprime factorization over the stable rational ring, and n_i and d_i are the numerator and the denominator. Suppose that the input u_{oi} consists of a local input u_i and a global input $r_i = r_g$ such as $u_{oi} = u_i + r_i$ and the local input u_i is generated by a local controller $C_i(s)$ as

$$u_i = C_i(s)y_i, \;\; y_i = v_i + w_i \tag{5}$$

Suppose that $C_i(s)$ is designed to stabilize $P_i(s)$ and then it is represented by Youla parametrization such as

$$C_i(s) = -\frac{\alpha_i(s) + d_i(s)q_i(s)}{\beta_i(s) - n_i(s)q_i(s)}, \quad n_i(s)\alpha_i(s) + d_i(s)\beta_i(s) = 1, \tag{6}$$

where $\alpha_i(s), \beta_i(s), q_i(s) \in RH_\infty$. The local controller $C_i(s)$ is also designed to attain given local control performances.

We consider a case that the output of the group of these local subsystems is the average of their local outputs such as $v_g = \frac{1}{N}\sum_{i=1}^N v_i$ where we call v_g as *the aggregated output* of the group of the local subsystems. Define a transfer function $\Phi_i(s)$ from the global input $r_i = r_g$ to the local output v_i as

$$v_i = \frac{P_i(s)}{1 - P_i(s)C_i(s)} r_i =: \Phi_i(s) r_g. \tag{7}$$

Thus $\Phi_i(s)\ (= 1,\, 2,\, \ldots,\, N)$ can be regarded as the transfer function of the locally controlled subsystem composed of the local system $P_i(s)$ and the local controller $C_i(s)$.

Consequently, the local feedback system can be shown in Fig. 2, where the i-th component of $\boldsymbol{L}(s)$ is given by

$$\boldsymbol{L}(s) = \begin{bmatrix} P_i & 0 & P_i \\ W_{SL}P_i & W_{SL} & W_{SL}P_i \\ P_i & 1 & P_i \end{bmatrix}. \tag{8}$$

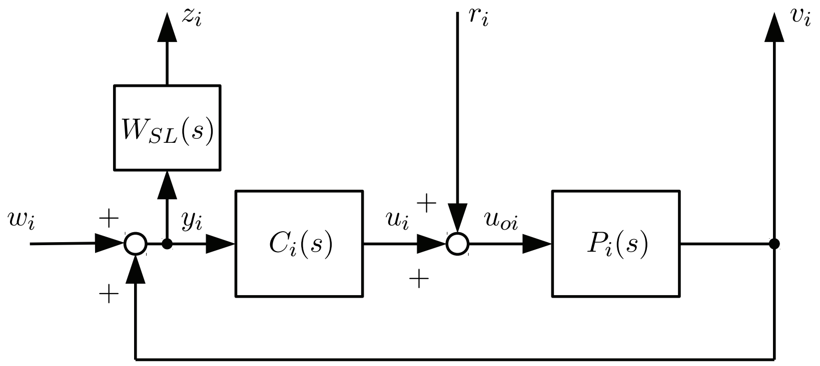

Fig. 2: Feedback loop of local subsystem

Also define $\boldsymbol{M}$ as the aggregated subsystems such as

$$\boldsymbol{M}(s) := \sum_{i=1}^{N} \frac{1}{N} \Phi_i(s), \tag{9}$$

then, the aggregated output v_g can be represented as $v_g = \frac{1}{N}\sum_{i=1}^{N} \Phi_i(s) r_i = \boldsymbol{M}(s) r_g$. We call $\boldsymbol{M}(s)$ as *the aggregated transfer function* from r_g to v_g.

Next, the global controller $C_g(s)$ is designed to generate the global input u_g from the global output y_g such as

$$u_g = C_g(s) y_g. \tag{10}$$

The purpose of the global controller $C_g(s)$ is basically to attain the stability of the whole system and global control performances.

The strategy of the hierarchically decentralized control system considered in this paper is divided into three layers; lower layer, upper layer, and middle layer as follows: (lower layer) design of the local controllers $C_i(s)$ in the lower layer for given local control objectives, (upper layer) design of the global controller $C_g(s)$ in the upper layer for a given global control objective, (middle layer) design of a nominal model $M_o(s)$ to which the locally controlled subsystems are designed to be close.

The actual design procedure of the whole system is as follows: At first, set a stable nominal model $M_o(s)$ appropriately in the middle layer and its information is broadcasted to each local subsystem. The control objective for a local controller $C_i(s)$ are both of to make its closed loop system composed of $P_i(s)$ and $C_i(s)$ to be close to the nominal model $M_o(s)$ and to attain a given local control performance. On the other hand, the control objective for the global controller $C_g(s)$ in the upper layer is to attain a given global control performance with consideration of the error between $\boldsymbol{M}(s)$ and $M_o(s)$, that is, a robust control performance. When the selected nominal model $M_o(s)$ is not appropriate for both of the local control performance and the global control performance, we reset $M_o(s)$ and repeat the above procedure.

In the following subsections, we give actual formulae for the above mentioned control strategy and clarify the relationship between controller designs in the upper layer and the lower layer, and the setting of the model set $\mathcal{M}_\delta$. Then we discuss trade-offs between the attained global control performance and the local control performance.

3.2 Hierarchically decentralized control design

At first, in the lower layer, we consider to design a local controller $C_i(s)$ which simultaneously satisfies a given local control performance and a multiplicative perturbed model matching problem or an additive perturbed model matching problem between the local closed loop system $\Phi_i(s)$ and the nominal model $M_o(s)$ as follows:

$$\text{find } C_i(s) \text{ for each } i \text{ s.t. } \left\| W_{SL} \frac{1}{1 - P_i C_i} \right\|_\infty < 1 \tag{11}$$

$$\left\| \frac{\Phi_i - M_o}{M_o} \right\|_\infty = \left\| \frac{1}{M_o} \left(\frac{P_i}{1 - P_i C_i} - M_o \right) \right\|_\infty < \delta \tag{12}$$

$$\text{or } \|\Phi_i - M_o\|_\infty = \left\| \frac{P_i}{1 - P_i C_i} - M_o \right\|_\infty < \delta \tag{13}$$

where $W_{SL}(s)$ is a given weight function for the local performance of sensitivity reduction and δ represents the size of the model set $\mathcal{M}_\delta$. Note that $\Phi_i(s)$ can be represented as

$$\Phi_i(s) = \frac{P_i(s)}{1 - P_i(s)C_i(s)} = \frac{n_i}{d_i}(d_i\beta_i - d_i n_i q_i) = n_i(\beta_i - n_i q_i) \tag{14}$$

and also similarly

$$\frac{1}{1 - P_i(s)C_i(s)} \tag{15}$$

then, the above problem can be represented as follows:

$$\text{find } q_i(s) \in RH_\infty \text{ for each } i \text{ s.t. } \|W_{SL} d_i(\beta_i - n_i q_i)\|_\infty < 1 \tag{16}$$

$$\left\| M_o^{-1}(n_i(\beta_i - n_i q_i) - M_o) \right\|_\infty < \delta \tag{17}$$

$$\text{or } \|n_i(\beta_i - n_i q_i) - M_o\|_\infty < \delta \tag{18}$$

Next, denote $\Delta_i(s)$ as the multiplicative error or the additive error between $\Phi_i(s)$ and $M_o(s)$;

$$\Delta_i(s) := \frac{\Phi_i(s) - M_o(s)}{M_o(s)} = \frac{1}{M_o} \left(\frac{P_i}{1 - P_i C_i} - M_o \right) \tag{19}$$

$$\text{or } \Delta_i(s) := \Phi_i(s) - M_o(s) = \frac{P_i}{1 - P_i C_i} - M_o, \tag{20}$$

and the multiplicative error between the aggregated system $\boldsymbol{M}(s)$ and the nominal model $M_o(s)$ can be represented as

$$\frac{M - M_o}{M_o} = \frac{1}{M_o}\left(\sum_{i=1}^{N}\frac{1}{N}\Phi_i - M_o\right) = \frac{1}{N}\sum_{i=1}^{N}\frac{\Phi_i - M_o}{M_o} = \frac{1}{N}\sum_{i=1}^{N}\Delta_i =: \Delta \tag{21}$$

and the additive error can also be represented as

$$M - M_o = \sum_{i=1}^{N}\frac{1}{N}\Phi_i - M_o = \frac{1}{N}\sum_{i=1}^{N}(\Phi_i - M_o) = \frac{1}{N}\sum_{i=1}^{N}\Delta_i =: \Delta. \tag{22}$$

Consequently, when the error condition (12) on subsystem i, that is, $\|\Delta_i\|_\infty < \delta$ is satisfied, we get

$$\|\Delta\|_\infty = \left\|\frac{1}{N}\sum_{i=1}^{N}\Delta_i\right\|_\infty < \delta \tag{23}$$

in both cases of the multiplicative error and the additive error.

On the other hand, in the upper layer, the global controller $C_g(s)$ is designed for satisfying the following global control objective under consideration of the perturbation (23), that is, the robust performance problem, on the aggregated system:

$$\text{find } C_g(s) \in \mathcal{S}_c(M_o) \ \text{ s.t. } \ \left\|W_S\frac{1}{1-(1+\tilde{\Delta})M_oC_g}\right\|_\infty < 1 \ ; \ \forall\tilde{\Delta} \text{ s.t. } \left\|\tilde{\Delta}\right\|_\infty < \delta \tag{24}$$

in the multiplicative error case and

$$\text{find } C_g(s) \in \mathcal{S}_c(M_o) \ \text{ s.t. } \ \left\|W_S\frac{1}{1-(M_o+\tilde{\Delta})C_g}\right\|_\infty < 1 \ ; \ \forall\tilde{\Delta} \text{ s.t. } \left\|\tilde{\Delta}\right\|_\infty < \delta \tag{25}$$

in the additive error case, where $W_S(s)$ is an appropriate weight function.

The proble (24) can be also reduced to [5]

$$\text{find } C_g(s) \in \mathcal{S}_c(M_o) \ \text{ s.t. } \ |W_SS| + |\delta T| \le 1, \forall\omega, \tag{26}$$

where $S(s)$ and $T(s)$ are

$$S(s) := \frac{1}{1 - M_o(s)C_g(s)}, \quad T(s) := \frac{M_o(s)C_g(s)}{1 - M_o(s)C_g(s)} \tag{27}$$

and the probem (25) can be reduced to [5]

$$\text{find } C_g(s) \in \mathcal{S}_c(M_o) \ \text{ s.t. } \ |W_SS| + |\delta U| \le 1, \forall\omega, \tag{28}$$

where

$$U(s) := \frac{C_g(s)}{1 - M_o(s)C_g(s)}. \tag{29}$$

Remark 3.1 *In these robust control performance problems, the detailed information on each subsystem $\Phi_i(s)$ or each $\Delta_i(s)$ is not necessary and only δ is necessary. This implies the detailed information is aggregated and the computation complexity can be restrained in the whole control system design. Such control design strategy is necessary for controlling large-scaled systems.*

Note that $M_o(s)$ is assumed to be stable, then its stabilizing controller $C_g(s)$ is given as

$$C_g(s) = -\frac{q(s)}{1 - M_o(s)q(s)}, \quad q(s) \in RH_\infty, \tag{30}$$

and then $S(s), T(s)$ and $U(s)$ can be represented as

$$S(s) = 1 - M_o(s)q(s), \quad T(s) = M_o(s)q(s), \quad U = \frac{C_g}{1 - M_oC_g} = q. \tag{31}$$

Therefore, the robust performance problems in the upper layer can be reduced into the following:

$$\text{find } q(s) \in RH_\infty \quad \text{s.t.} \quad |W_S(1 - M_oq)| + |\delta M_oq| \leq 1, \ \forall s = j\omega \tag{32}$$

in the multiplicative error case and

$$\text{find } q(s) \in RH_\infty \quad \text{s.t.} \quad |W_S(1 - M_oq)| + |\delta q| \leq 1, \ \forall s = j\omega \tag{33}$$

in the additive error case.

Although the nominal model $M_o(s) \in RH_\infty$ is also a design parameter, we here assume for simplicity that it is set appropriately in advance. Then, we have the following methods of hierarchically decentralized control design.

[Hierarchically Decentralized Control Design (Multiplicative Error Case)]

Upper layer:

$$\text{find } q(s) \in RH_\infty \ \text{ s.t. } \| \, |W_S(1 - M_o q)| + |\delta M_o q| \, \|_\infty \leq 1 \tag{34}$$

Lower layer:

$$\text{find } q_i(s) \in RH_\infty \ \text{ for each } \ i \ \text{ s.t. } \|W_{SL} d_i(\beta_i - n_i q_i)\|_\infty < 1 \tag{35}$$

$$\left\| M_o^{-1}(n_i(\beta_i - n_i q_i) - M_o) \right\|_\infty < \delta \tag{36}$$

[Hierarchically Decentralized Control Design (Additive Error Case)]

Upper layer:

$$\text{find } q(s) \in RH_\infty \ \text{ s.t. } \| \, |W_S(1 - M_o q)| + |\delta q| \, \|_\infty \leq 1 \tag{37}$$

Lower layer:

$$\text{find } q_i(s) \in RH_\infty \ \text{ for each } \ i \ \text{ s.t. } \|W_{SL} d_i(\beta_i - n_i q_i)\|_\infty < 1 \tag{38}$$

$$\|n_i(\beta_i - n_i q_i) - M_o\|_\infty < \delta \tag{39}$$

Remark 3.2 (multiplicative error case) Note first that $M_o(s)$ is fixed in this paper and then $q(s)$ and $q_i(s)$ are the design parameters. In (34), even if we set $q(s)$ in the upper layer appropriately, it is known that there exists an unavoidable trade-off between δ and the magnitude of W_{SL}. On the other hand, in order to improve the local control performance (35) in the lower layer, an arbitrary high control performance is attained by a setting $q_i \to \frac{\beta_i}{n_i}$ when the zeros of n_i is stable. Then, when we set $q_i = (1 + \epsilon_i)\frac{\beta_i}{n_i}$ where ϵ_i has an enough small gain, (36) is represented by

$$\left\| M_o^{-1}(n_i(\beta_i - n_i q_i) - M_o) \right\|_\infty = \left\| -n_i \beta_i \epsilon_i M_o^{-1} - 1 \right\|_\infty < \delta. \tag{40}$$

Therefore, from (40), it is known that the possible δ which satisfies (40) becomes large. In summary, there exists a trade-off between the global control performance and the local control performance and it is represented by means of the size δ of the model set $\mathcal{M}_\delta$.

Remark 3.3 (additive error case) From (37), it is known that we can make the global control performance improved arbitrarily by setting q of enough low gain and $M_o \simeq q^{-1}$. As a result, M_o becomes a high gain system. On the other hand, in order to improve the control performance in the lower layer, it is attained by $q_i \simeq \frac{\beta_i}{n_i}$ when $\frac{\beta_i}{n_i}$ is stable (in order to make q proper, multiply an appropriate factor to $\frac{\beta_i}{n_i}$ if necessary [5]). Thus, (36) becomes approximately $\|M_o\|_\infty < \delta$ and it contradicts to the design policy of M_o for the upper layer, that is, it becomes a high gain system as mentioned above. From above consideration, it is known that there exists a trade-off between the design policies in the upper layer and the lower layer by using the setting of the nominal model M_o.

We demonstrate the performance trade-off discussed above in Section 4 with a concrete problem in which classes of plants and controllers are specified.

3.3 Sensitivity trade-off condition

In the previous subsections, we introduce the hierarchically decentralized control design with the multiplicative/additive perturbation representation and discuss the trade-off between the global control performance in the upper layer and the local control performance in the lower layer. In this subsection, we give an explicit condition which represents the similar trade-off as follows:

Theorem 3.1 *In both of the multiplicative/additive perturbation representations, the global sensitivity function S in the upper layer and the local sensitivity function S_i in the lower layer satisfy the following:*

$$|1-S| \cdot \left|1-\frac{P_i}{M_o}S_i\right| < 1,\ \forall\omega \tag{41}$$

Proof. At first, we consider the case of the multiplicative perturbation representation. In the upper layer, the robust stability condition

$$\|\delta M_o q\|_\infty = \|\delta(1-S)\|_\infty \leq 1, \tag{42}$$

that is,

$$|\delta(1-S)| \leq 1,\ \forall\omega, \tag{43}$$

should be satisfied. On the other hand, the model matching condition in the lower layer

$$\begin{aligned} &\left\|\delta^{-1}M_o^{-1}(M_o-\Phi_i)\right\|_\infty \\ &= \left\|\delta^{-1}M_o^{-1}(M_o-P_iS_i)\right\|_\infty, \end{aligned} \tag{44}$$

that is,

$$\left|\delta^{-1}M_o^{-1}(M_o-P_iS_i)\right| < 1,\ \forall\omega \tag{45}$$

should be simultaneously satisfied. Conditions (43) and (45) are also represented as

$$|1-S| \leq \left|\frac{1}{\delta}\right|,\ \forall\omega, \tag{46}$$

$$\left|1-\frac{P_i}{M_o}S_i\right| < |\delta|,\ \forall\omega, \tag{47}$$

respectively. By multiplying (46) and (47), we get (41).

Next, we consider the case of the additive perturbation representation. In the upper layer, the robust stability condition

$$\|\delta q\|_\infty = \left\|\delta M_o^{-1}(1-S)\right\|_\infty \leq 1, \tag{48}$$

that is,

$$|\delta M_o^{-1}(1-S)| \leq 1,\ \forall\omega, \tag{49}$$

should be satisfied. On the other hand, the model matching condition in the lower layer

$$\|M_o - \Phi_i\|_\infty = \|M_o - P_i S_i\|_\infty < \delta, \tag{50}$$

that is,

$$|M_o - P_i S_i| < \delta,\ \forall\omega, \tag{51}$$

should be simultaneously satisfied. Conditions (49) and (51) are also represented as

$$|1-S| \leq \left|\frac{M_o}{\delta}\right|,\ \forall\omega, \tag{52}$$

$$\left|1 - \frac{P_i}{M_o}S_i\right| < \left|\frac{\delta}{M_o}\right|,\ \forall\omega, \tag{53}$$

respectively. By multiplying (52) and (53), we also get (41).

Remark 3.4 The inequality (41) is a necessary condition for the robust stability of the whole system, however it is observed that simultaneously restraining the gains of the sensitivity S in the upper layer and the sensitivity S_i in the lower layer small, may violate (41). From this, condition (41) represents a trade-off between the control performances in the upper layer and the lower layer.

4 An Illustrative Example

This section demonstrates the global/local performance trade-off by a very simple example in a case of the multiplicative perturbation representation defined in the following subsections.

4.1 Problem Description

Local Subsystems: The plant of each local system is represented by

$$P_i(s) = \frac{k_i}{s+h_i},\ h_i,\ k_i > 0, \tag{54}$$
$$k_i \in [\underline{k}_p, \overline{k}_p],\ \underline{k}_p,\ \overline{k}_p > 0,\ h_i \in [\underline{h}_p, \overline{h}_p],\ \underline{h}_p > 0,\ \overline{h}_p < 1,$$

and the objective is to satisfy

$$\text{(ia)} \quad \left\| W_{SL} \frac{1}{1 - P_i C_i} \right\|_\infty < 1 \ ; \ W_{SL}(s) := \frac{\eta_\ell}{s+1}, \ \eta_\ell > 0 \tag{55}$$

The requirement (ia) is to reduce the sensitivity and η_ℓ represents the local performance to be maximized.

Global System: The original global control objective is to satisfy

$$\text{(g)} \quad \left\| W_S \frac{1}{1 - \boldsymbol{M} C_g} \right\|_\infty < 1 \ ; \ W_S(s) = \frac{\eta_g}{\tau s + 1}, \ \eta_g > 0, \ \tau > 0, \tag{56}$$

where $\boldsymbol{M}$ is defined by (9) and it represents the average of $\Phi_i(s)$. The global objective is to reduce the sensitivity, i.e., to maximize η_g.

Shared Model Set $\mathcal{M}_\delta$: The shared model set $\mathcal{M}_\delta$ is given by

$$\mathcal{M}_\delta := \left\{ \tilde{M} \mid \tilde{M} = M_o(1 + \Delta), \ \|\Delta\|_\infty < \delta \right\} \ ; \ M_o(s) = \frac{b}{s+a}, \ a, \ b > 0. \tag{57}$$

4.2 Synthesis of Global and Local Controllers

Global Controller Design:

First note that (34) is a function of $\hat{q}(s) := M_o(s) q(s)$, which is strictly proper and stable, (34) can be rewritten as

$$\hat{q}(s) \in RH_\infty \ \& \ \text{strictly proper s.t.} \ \| \, |W_S(1 - \hat{q})| + |\delta \hat{q}| \, \|_\infty \leq 1. \tag{58}$$

The problem is normally difficult to solve, and hence we will consider one of standard sufficient condition of (58) which is represented by [5]

$$\| \, |W_S(1 - \hat{q})|^2 + |\delta \hat{q}|^2 \, \|_\infty \leq 1/2, \tag{59}$$

or equivalently

$$\left\| \begin{bmatrix} \frac{\eta_g}{1+\tau s} \\ \delta \end{bmatrix} \hat{q}(s) - \begin{bmatrix} \frac{\eta_g}{1+\tau s} \\ 0 \end{bmatrix} \right\|_\infty^2 \leq \frac{1}{2}. \tag{60}$$

Note that we may assume that $\tau = 1$ in $W_S(s)$ without loss of generality. Then, multiplying an inner matrix function defined by

$$\begin{bmatrix} \eta_g & (1-s)\delta \\ -(1+s)\delta & \eta_g \end{bmatrix} \left(\sqrt{\eta_g^2 + \delta^2} + \delta s \right)^{-1} \tag{61}$$

from left yields

$$\left\| \begin{bmatrix} A_1(s)\hat{q}(s) - B_1(s) \\ B_2(s) \end{bmatrix} \right\|_\infty^2 \leq \frac{1}{2}, \tag{62}$$

where

$$A_1(s) := \frac{\sqrt{\eta_g^2+\delta^2} - \delta s}{1+s}, \tag{63}$$

$$B_1(s) := \frac{\eta_g^2}{(1+s)\left(\sqrt{\eta_g^2+\delta^2}+\delta s\right)},\ \ B_2(s) := \frac{\eta_g \delta}{\sqrt{\eta_g^2+\delta^2}+\delta s}. \tag{64}$$

It is clear that $\|B_2\|_\infty^2 \le \frac{1}{2}$ is a necessary condition for the feasibility. Under this condition the norm condition (59) is rewritten as

$$|A_1(j\omega)\hat{q}(j\omega) - B_1(j\omega)|^2 \ < \ \frac{1}{2} - |B_2(j\omega)|^2;\ \ \forall\omega. \tag{65}$$

Hence, focusing on

$$\begin{aligned} &\frac{1}{2} - B_2(s)B_2(-s) \\ =&\frac{1}{2}\left[1 - \frac{2\eta_g^2\delta^2}{\left(\sqrt{\eta_g^2+\delta^2}+\delta s\right)\left(\sqrt{\eta_g^2+\delta^2}-\delta s\right)}\right] \\ =&\frac{\eta_g^2+\delta^2-2\eta_g^2\delta^2-\delta^2 s^2}{2\left(\sqrt{\eta_g^2+\delta^2}+\delta s\right)\left(\sqrt{\eta_g^2+\delta^2}-\delta s\right)} =: \hat{B}_2(s)\hat{B}_2(-s), \end{aligned} \tag{66}$$

we get its unimodular factor $\hat{B}_2(s)$ as

$$\hat{B}_2(s) := \frac{s + \sqrt{1+\frac{\eta_g^2(1-2\delta^2)}{\delta^2}}}{\sqrt{2}\left(s + \frac{\sqrt{\eta_g^2+\delta^2}}{\delta}\right)} \tag{67}$$

This leads to an equivalent norm condition expressed as

$$\left\|\frac{A_1(s)}{\hat{B}_2(s)}\hat{q}(s) - \frac{B_1(s)}{\hat{B}_2(s)}\right\|_\infty \ \le \ 1\ . \tag{68}$$

This is a fairly standard 1-block H_∞ optimal control problem except the restriction on the class of $\hat{q}(s)$, which should be strictly proper and stable. Fortunately, the value of $\frac{B_1(s)}{\hat{B}_2(s)}$ at $s=\infty$ is zero. Hence, we only need to take care the finite non-minimum phase zero of $\frac{A_1(s)}{\hat{B}_2(s)}$, or $A_1(s)$, which is given by

$$s_z = \sqrt{1+\left(\frac{\eta_g}{\delta}\right)^2}. \tag{69}$$

Finally, thanks to a famous result of Nevanlinna-Pick interpolation problem, we can see that the necessary and sufficient condition for (59) is given by

$$\text{(i)} \quad \left| \frac{B_1(s_z)}{\hat{B}_2(s_z)} \right| \leq 1, \quad \text{(ii)} \quad \|B_2\|_\infty^2 \leq \frac{1}{2}. \tag{70}$$

We can first observe that he condition (ii) $\|B_2\|_\infty^2 \leq \frac{1}{2}$ gives the following constraints on achievable η_g :

$$\text{any positive } \eta_g : \ 0 < \delta \leq \frac{1}{\sqrt{2}} \tag{71}$$

$$\eta_g \leq \frac{\delta}{\sqrt{2\delta^2 - 1}} : \ \frac{1}{\sqrt{2}} < \delta < 1, \tag{72}$$

because $B_2(s)$ is a strictly proper 1-st order stable system so that

$$\|B_2\|_\infty^2 = |B_2(0)|^2 = \frac{\eta_g^2 \delta^2}{\eta_g^2 + \delta^2}.$$

In the following, we will shows that the condition (i) $\left| \frac{B_1(s_z)}{\hat{B}_2(s_z)} \right| \leq 1$ holds true if (ii) is satisfied.

Substitute $s = s_z = \sqrt{1 + \left(\frac{\eta_g}{\delta}\right)^2}$ into

$$\frac{B_1(s)}{\hat{B}_2(s)} = \frac{\sqrt{2}\eta_g^2}{\delta} \frac{1}{(s+1)\left(s + \sqrt{1 + \frac{\eta_g^2(1-2\delta^2)}{\delta^2}}\right)},$$

we have

$$\left| \frac{B_1(s_z)}{\hat{B}_2(s_z)} \right| = \frac{\sqrt{2}\eta_g^2}{\delta} \frac{1}{\left(\sqrt{1 + \frac{\eta_g^2}{\delta^2}} + 1\right)\left[\sqrt{1 + \frac{\eta_g^2}{\delta^2}} + \sqrt{1 + (1 - 2\delta^2)\frac{\eta_g^2}{\delta^2}}\right]}$$

Let

$$t := \frac{\eta_g^2}{\delta^2} \ (> 0),$$

the condition (i) becomes

$$\frac{\sqrt{2}\delta t}{\left(\sqrt{t+1} + 1\right)\left[\sqrt{t+1} + \sqrt{1 + (1 - 2\delta^2)\, t}\right]} \leq 1.$$

Case 1: If $0 < \delta \leq \frac{1}{\sqrt{2}}$, it is clear that

$$\sqrt{1 + (1 - 2\delta^2)\, t} \geq 1,$$

and hence we get

$$\frac{\sqrt{2}\delta t}{\left(\sqrt{t+1}+1\right)\left[\sqrt{t+1}+\sqrt{1+\left(1-2\delta^2\right)t}\right]} \le \frac{\sqrt{2}\delta t}{\left(\sqrt{t+1}+1\right)^2}$$

We also see that

$$\frac{\sqrt{2}\delta t}{\left(\sqrt{t+1}+1\right)^2} - 1 = \frac{\left(\sqrt{2}\delta - 1\right)t - 2\sqrt{t+1} - 2}{\left(\sqrt{t+1}+1\right)^2} < 0, \ \ \forall t > 0$$

holds. It is clear from above two inequalities that that (i) hold trues.

Case 2: If $\frac{1}{\sqrt{2}} < \delta < 1$, we define $U = \sqrt{t+1}$ and $V = \sqrt{\left(1-2\delta^2\right)t+1}$. It is clear that:

$$U^2 - V^2 = (t+1) - \left[\left(1-2\delta^2\right)t+1\right] = 2\delta^2 t$$

Substitute U and V into (2), the condition (i) becomes:

$$\frac{U^2-V^2}{\sqrt{2}\delta\left(U+1\right)\left(U+V\right)} \le 1$$

$$\Leftrightarrow \frac{U^2-V^2}{\sqrt{2}\delta\left(U+1\right)\left(U+V\right)} - 1 \le 0$$

$$\Leftrightarrow \frac{\left(1-\sqrt{2}\delta\right)U^2 - V^2 - \sqrt{2}\delta\left(U+V+UV\right)}{\sqrt{2}\delta\left(U+1\right)\left(U+V\right)} \le 0$$

It is clear that (8) hold trues for every $U > 0, V > 0$ since $1-\sqrt{2}\delta < 0$.

In conclusion, the condition (i) hold trues if (ii) is satisfied for any $\delta \in (0,1)$, and the optimal global performance $\eta_g^*(\delta)$ is simply expressed as follows:

[Global Control Performance Limitation] $\eta_g^*(\delta)$

$$\eta_g^*(\delta) = \begin{cases} \frac{\delta}{\sqrt{2\delta^2-1}}, & \delta > \frac{1}{\sqrt{2}} \\ +\infty, & \text{otherwise} \end{cases} \tag{73}$$

Fig. 3 shows the plot (dotted line) of $\eta_g^*(\delta)$.

Local Controller Design:

The objective is to satisfy both of (12) and (55) for a class of $\Phi_i(s)$ given by

$$\text{(ib)} \quad \Phi_i(s) := \frac{P_i(s)}{1-P_i(s)C_i(s)} = \frac{k_i}{s+a_i}, \ a_i > 0. \tag{74}$$

The constraint (ib) is for specifying the class of desirable responses. By using the Youla parametrization of $C_i(s)$, any $a_i > 0$ can be attained and it is a design parameter. In order to describe the optimal $\eta^*_{\ell i}(\delta)$ of $P_i(s)$ for a given δ, we define the following functions and notations:

$$
\begin{aligned}
a^*(\delta) &:= \frac{k_i}{b(1-\delta)}a \\
X^*(a^*) &:= -h_i^2 + \sqrt{h_i^4 + (a^*)^2(1-h_i^2) - h_i^2} \\
R_\alpha(a^*) &:= \left(\frac{(X^*(a^*)+1)\,(X^*(a^*)+(a^*)^2)}{X^*(a^*)+h_i^2}\right)^{\frac{1}{2}}, \ R_\beta(a^*) := \frac{a^*}{h_i} \\
\underline{\delta} &:= \frac{\overline{k}_p - \underline{k}_p}{\overline{k}_p + \underline{k}_p}, \ h_o := \sqrt{\frac{h_i^2}{1-h_i^2}}
\end{aligned}
$$

The following is the summary of the optimal $\eta^*_{\ell i}(\delta)$ (See the Appendix for the detailed derivation.):

[Local Control Performance Limitations] $\eta^*_{\ell i}(\delta)$

$$
\begin{aligned}
&\text{Suppose} \quad \delta > \underline{\delta}, \ \ \overline{k}/(1+\delta) < b < \underline{k}/(1-\delta). \\
&\text{Then} \quad \eta^*_{\ell i}(\delta) = \begin{cases} R_\alpha(a^*(\delta)), & a^*(\delta) \geq h_o \\ R_\beta(a^*(\delta)), & a^*(\delta) < h_o \end{cases}
\end{aligned} \tag{75}
$$

Note that $\eta^*_{\ell i}(\delta)$ is an increasing function of δ. Fig. 3 shows a numerical simulation of $\eta^*_g(\delta)$ and $\eta^*_{\ell i}(\delta)$, where $a = 1.2$, $b = 1$, $k_i = 1$, $h_i = 0.6$, $\underline{k}_p = 0.9$, and $\overline{k}_p = 1.1$. The figure shows a trade-off between $\eta^*_g(\delta)$ and $\eta^*_{\ell i}(\delta)$ by using the size δ of the shared model set $\mathcal{M}_\delta$.

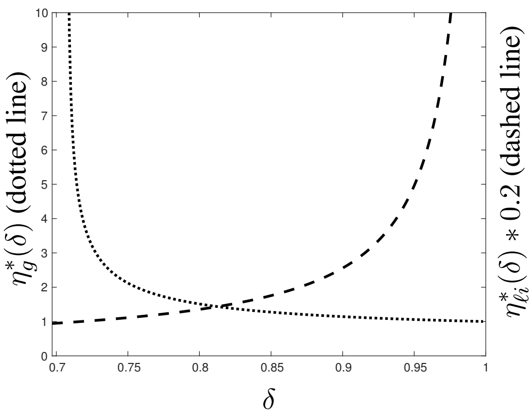

Fig. 3: Trade-off curves between the global control performance $\eta^*_g(\delta)$ (dotted line) and the local control performance $\eta^*_{\ell i}(\delta) * 0.2$ (dashed line)

5 Stochastic analysis on the variations of the local systems

In the previous sections, we discuss trade-off between the global control performance in the upper layer and the local control performance in the lower layer. It is known that the results are derived from the deterministic worst case paradigm. Then, a question arises; is this kind of trade-off avoidable by assuming some appropriate conditions on the systems?

In the previous sections, we deal with the error between a nominal model M_o and the aggregated system $\boldsymbol{M}$ with a hard bound. On the other hand, for example, in the electric networks, it is known that aggregating a bunch of electric suppliers and regarding it as a supplier, the variation of the supplies or the dynamics of the suppliers is averaged and can be ignored when the number of the suppliers is enough large; this is called "averaging effect."

From above consideration, in this section, we introduce conditions local systems should satisfy in order to model the phenomena of the averaging effect and discuss if the trade-off in the previous sections can be avoided when the number of subsystems N is enough large. In concrete, we assume a stochastic property on the perturbations and the model errors on the subsystems in the lower layer and analyze the trade-off in stochastic sense.

At first, we consider that the local controllers C_i are designed to satisfy (35) for a given W_{SL}. As mentioned in Remark 3.3, a setting $q_i \simeq \frac{\beta_i}{n_i}$ attains an arbitrary high local control performance. Next set the nominal model M_o as a center of the group of Φ_i, where the meaning of the center is explained hereafter. On the frequency ω, we consider the following variable transformation:

$$\tilde{\omega} := \frac{e^{\omega} - 1}{e^{\omega} + 1} \tag{76}$$

Note that the domain of ω is transferred from $0 \le \omega \le \infty$ to $0 \le \tilde{\omega} \le 1$. Then, we assume the following on the additive model error $\Delta_i = \Phi_i - M_o$ and that gives the meaning of the center of $\{\Phi_i\}$:

Assumption 5.1 *The real parts or the imaginary parts of the model errors Δ_i and Δ_j, that is, $\mathrm{Re}[\Delta_i(\tilde{\omega})]$ and $\mathrm{Re}[\Delta_j(\tilde{\omega})]$ or $\Im[\Delta_i(\tilde{\omega})]$ and $\Im[\Delta_j(\tilde{\omega})]$, are independent each other for any i and j, and the following is satisfied:*

$$\begin{bmatrix}\mathrm{Re}[\Delta_i(\tilde{\omega})] & \Im[\Delta_i(\tilde{\omega})]\end{bmatrix}^\top = \int \mathrm{d}B_i(\tilde{\omega}), \tag{77}$$

where $B_i(\tilde{\omega}) \in \mathcal{R}^2$ is a Brownian motion on $\tilde{\omega}$ with a probabilistic density

$$\begin{aligned}
&p(\tilde{\omega}_1 - \tilde{\omega}_2, B_i(\tilde{\omega}_1), B_i(\tilde{\omega}_2)) \\
&= (2\pi(\tilde{\omega}_1 - \tilde{\omega}_2))^{-1} (\det \Sigma)^{-\frac{1}{2}} \\
&\quad \times \exp\left(-\frac{1}{2(\tilde{\omega}_1 - \tilde{\omega}_2)} (B_i(\tilde{\omega}_1) - B_i(\tilde{\omega}_2))^\top \Sigma^{-1} \right. \\
&\quad \left. \times (B_i(\tilde{\omega}_1) - B_i(\tilde{\omega}_2)) \right), \\
&\Sigma = \mathrm{diag}(\overline{\delta}^2, \overline{\delta}^2). \qquad (78)
\end{aligned}$$

This assumption introduces a randomness of the change of $\Delta_i(\omega)$ for ω and it is reasonable when the transfer functions of local systems have stochastic variations.

Then, the following holds:

Lemma 5.1 *For a given any probability $\epsilon > 0$, the following holds:*

$$\mathsf{P}\left(\sup_{\omega} |\Delta(\omega)| \geq \frac{\overline{\delta}}{\sqrt{N\epsilon}} \right) \leq \epsilon \qquad (79)$$

Proof. From Assumption 5.1 on Δ_i, $\Delta = \frac{1}{N}\sum_{i=1}^{N} \Delta_i$ satisfies

$$\begin{aligned}
&\begin{bmatrix} \mathrm{Re}[\Delta(\tilde{\omega})] & \Im[\Delta(\tilde{\omega})] \end{bmatrix}^\top = \int \mathrm{d}B(\tilde{\omega}), \qquad (80) \\
&p(\tilde{\omega}_1 - \tilde{\omega}_2, B(\tilde{\omega}_1), B(\tilde{\omega}_2)) \\
&= (2\pi(\tilde{\omega}_1 - \tilde{\omega}_2))^{-1} (\det \Sigma)^{-\frac{1}{2}} \\
&\quad \times \exp\left(-\frac{1}{2(\tilde{\omega}_1 - \tilde{\omega}_2)} (B(\tilde{\omega}_1) - B(\tilde{\omega}_2))^\top \Sigma^{-1} \right. \\
&\quad \left. \times (B(\tilde{\omega}_1) - B(\tilde{\omega}_2)) \right), \\
&\Sigma = \mathrm{diag}\left(\frac{\overline{\delta}^2}{N}, \frac{\overline{\delta}^2}{N} \right). \qquad (81)
\end{aligned}$$

From martingale inequality [6] on Brownian motion $B(\tilde{\omega})$, for any $\lambda > 0$ and $p \geq 1$ we get the following inequality:

$$\mathsf{P}\left(\sum_{0 \leq \tilde{\omega} \leq T} |B(\tilde{\omega})| \geq \lambda \right) \leq \frac{1}{\lambda^p} \mathsf{E}\left(|B(T)|^p \right) \qquad (82)$$

Then, we can get (79) by substituting $T = 1$, $\lambda = \frac{\overline{\delta}}{\sqrt{N\epsilon}}$, and $p = 2$.

Remark 5.1 From Lemma 5.1, it is known that by allowing a risk of probability ϵ, $\|\Delta(\omega)\|_\infty$ has an upper bound $\delta' := \frac{\overline{\delta}}{\sqrt{N\epsilon}}$. When $\epsilon \ll 1$, however the number of local systems N is enough large such that it satisfies $\delta' = \frac{\bar{\delta}}{\sqrt{N\epsilon}} \ll 1$, the bound δ' becomes enough small with a high probability $1-\epsilon$. For example, when $\bar{\delta}^2 = 0.1$, $\epsilon = 0.01$, and $N = 10^4$, the upper bound δ' becomes $\delta' = \frac{\bar{\delta}}{\sqrt{N\epsilon}} \simeq 0.0316$ and it is realized with a probability $1-\epsilon = 0.99$.

From above observation, we set the weight W_U as

$$W_U = \frac{\bar{\delta}}{\sqrt{N\epsilon}} W_{Uo} \tag{83}$$

and consider the following robust control performance problem in the upper layer:

$$\begin{aligned} &\text{find } q \in H_\infty \\ &\text{s.t. } |W_S(1-M_o q)| + |W_U q| \\ &\quad = |W_S(1-M_o q)| + \left| \frac{\bar{\delta}}{\sqrt{N\epsilon}} W_{Uo} q \right| \le 1, \forall \omega \end{aligned} \tag{84}$$

Then, we get the following:

Proposition 5.1 Under Assumption 5.1, a global controller C_g which satisfies (84) attains the robust control performance with a probability $1-\epsilon$.

Remark 5.2 *When N is enough large, the second term of* (84) *is negligible in a stochastic sense and setting $q \simeq M_o^{-1}$ gives an arbitrary high control performance. The local controllers attain arbitrary high local control performances as mentioned before and we can avoid the trade-off between the global control objective and the local control objectives in the deterministic case as discussed in Section 3.2 and 3.3. This result also gives a design strategy for hierarchically decentralized control systems on the size and the variations of the aggregated systems.*

6 Conclusion

In this article, we have proposed a fairly general formulation of hierarchically decentralized control for large-scaled systems by aggregation. We have introduced *Global/Local Shared Model Set* to split the global and local control problems and to clarify the global/local performance trade-off. The effectiveness of the method has been confirmed by a simple example. The future topics include to investigate the analytical formulae of the trade-off by an optimal choice of the nominal model M_o.

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A Proof of Local Control Performance Limitations

The stabilizing controller C_i for P_i is given by

$$C_i = -\frac{q_i}{1 - P_i q_i}, \; q_i \in RH_\infty, \tag{85}$$

and hence we have

$$\Phi_i = \frac{P_i}{1 - P_i C_i} = \frac{k_i}{s + h_i}\left(1 - \frac{k_i}{s + h_i} q_i\right). \tag{86}$$

The constraint (74) of the class of Φ_i gives $b_i = k_i$ and

$$q_i = \frac{a_i - h_i}{k_i}\frac{s + h_i}{s + a_i}. \tag{87}$$

This leads to

$$S = \frac{1}{1 - P_i C_i} = \frac{s + h_i}{s + a_i}. \tag{88}$$

and

$$\frac{\Phi_i - M_o}{M_o} = \frac{(b_i - b)s + (b_i a - b a_i)}{b(s + a_i)} = \frac{(k_i - b)s + (k_i a - b a_i)}{b(s + a_i)}. \tag{89}$$

Consequently, the control performance problem can be represented by

$$\left\| \frac{\eta}{s+1} \frac{s+h_i}{s+a_i} \right\|_\infty < 1 \tag{90}$$

and model error problem is represented by

$$\left\| \frac{(k_i - b)s + (k_i a - b a_i)}{b(s+a_i)} \right\|_\infty < \delta. \tag{91}$$

A necessary condition for (91) is

$$\left| \frac{(k_i - b)s + (k_i a - b a_i)}{b(s+a_i)} \right|_{s=j\infty} = \left| \frac{k_i - b}{b} \right| < \delta, \tag{92}$$

that is,

$$(1-\delta^2)b^2 - 2k_i b + k_i^2 < 0. \tag{93}$$

From the condition (93), we get a necessary condition on b as

$$\frac{k_i}{1+\delta} < b < \frac{k_i}{1-\delta}. \tag{94}$$

A.1 Model error constraint

A.1.1 Case (1)

The pole of the model error system is a_i and the zero is

$$\frac{k_i a - b a_i}{k_i - b}. \tag{95}$$

We first consider the following case:

$$\mathbf{Case(1)}: \quad a_i^2 > \left(\frac{k_i a - b a_i}{k_i - b} \right)^2. \tag{96}$$

In this case, (92) becomes a necessary and sufficient condition for (91) and we get (94). On the other hand, the condition (96) can be rewritten by

$$\frac{1}{(k_i^2 - b)^2} \left\{ (b^2 - (k_i - b)^2) a_i^2 - 2k_i b a a_i + k_i^2 a^2 \right\} < 0. \tag{97}$$

From (94), we can show

$$b^2 - (k_i - b)^2 > 0, \tag{98}$$

that is,

$$0 < k_i < 2b, \tag{99}$$

then, the condition (96) or (97) gives the range of possible a_i as

$$\underline{a}_i^* := \frac{k_i a(b - |k_i - b|)}{b^2 - (k_i - b)^2} < a_i < \frac{k_i a(b + |k_i - b|)}{b^2 - (k_i - b)^2} =: \overline{a}_i^*. \tag{100}$$

Case (1)-1

Moreover, when

$$k_i - b > 0, \tag{101}$$

we get

$$\underline{a}_i^* = \frac{k_i a(2b - k_i)}{b^2 - (k_i - b)^2} = a < a_i < \overline{a}_i^* = \frac{k_i^2 a}{b^2 - (k_i - b)^2} = \frac{k_i a}{2b - k_i}. \tag{102}$$

Case (1)-2

In the other case of

$$k_i - b < 0, \tag{103}$$

we get

$$\underline{a}_i^* = \frac{k_i a}{2b - k_i} < a_i < \overline{a}_i^* = a. \tag{104}$$

A.1.2 Case (2)

Next, consider the case;

$$\mathbf{Case(2)}: \quad a_i^2 < \left(\frac{k_i a - b a_i}{k_i - b}\right)^2. \tag{105}$$

In this case, we get

$$\left\| \frac{(k_i - b)s + (k_i a - b a_i)}{b(s + a_i)} \right\|_\infty = \left| \frac{(k_i - b)s + (k_i a - b a_i)}{b(s + a_i)} \right|_{s=0} = \left| \frac{k_i a - b a_i}{a_i b} \right| < \delta. \tag{106}$$

The condition (106) can be reduced into

$$b^2(1 - \delta^2)a_i^2 - 2k_i b a a_i + k_i^2 a^2 < 0, \tag{107}$$

which provides the possible range of a_i as

$$\underline{a}_i = \frac{abk_i(1 - \delta)}{b^2(1 - \delta^2)} = \frac{ak_i}{b(1 + \delta)} < a_i < \frac{ak_i}{b(1 - \delta)} = \underline{a}_i. \tag{108}$$

On the other hand, similar to the case (1), from the necessary condition (92) or (94), (99) is safisfied and the condition (105) also gives the range of possible a_i as

$$a_i < \underline{a}_i^* = \frac{k_i a(b - |k_i - b|)}{b^2 - (k_i - b)^2} \text{ or } \frac{k_i a(b + |k_i - b|)}{b^2 - (k_i - b)^2} = \overline{a}_i^* < a_i. \tag{109}$$

<u>Case (2)-1</u>

Moreover, when $k_i - b > 0$, we get

$$a_i < \underline{a}_i^* = a \text{ or } \overline{a}_i^* = \frac{k_i a}{2b - k_i} < a_i. \tag{110}$$

<u>Case (2)-2</u>

On the other hand, when$k_i - b < 0$, we get

$$a_i < \underline{a}_i^* = \frac{k_i a}{2b - k_i} \text{ or } \overline{a}_i^* = a < a_i. \tag{111}$$

A.1.3 Summary of model error constraint

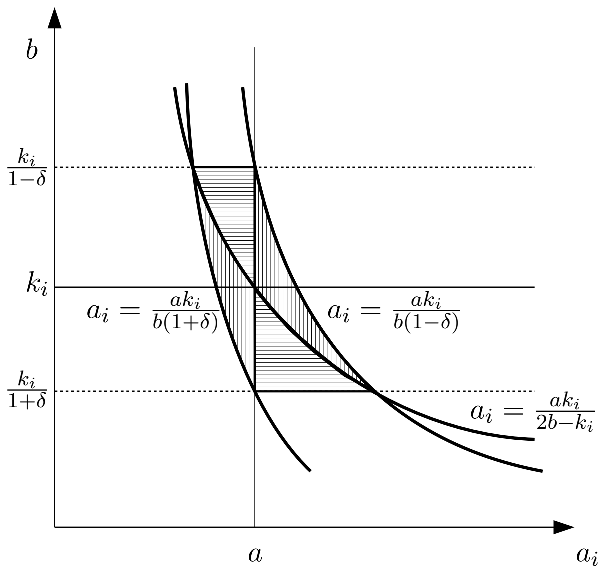

Fig. 4: The allowable region of a_i and b

The combination of the results of case(1)-1, case(1)-2, case(2)-1, and case(2)-2 gives the allowable region on the plain of a_i and b, which is shown as in Fig. 4. The two triangular regions of the horizontal-hatted pattern represent the allowable set of (a_i, b) given in case (1)-1 and case (1)-2. The other two triangular regions of the vertical-hatted pattern represent the allowable set of (a_i, b) given in case (2)-1 and case (2)-2.

Note that the intersection point between $a_i = \frac{ak_i}{2b-k_i}$ and $b = \frac{k_i}{1+\delta}$ is $(a_i, b) = (a\frac{1+\delta}{1-\delta}, \frac{k_i}{1+\delta})$ and it is also the intersection between $a_i = \frac{ak_i}{2b-k_i}$ and $a_i = \frac{ak_i}{b(1-\delta)}$.

Similarly, the intersection point between $a_i = \frac{ak_i}{2b-k_i}$ and $b = \frac{k_i}{1-\delta}$ is $(a_i, b) = (a\frac{1-\delta}{1+\delta}, \frac{k_i}{1-\delta})$ and it is also the intersection between $a_i = \frac{ak_i}{2b-k_i}$ and $a_i = \frac{ak_i}{b(1+\delta)}$. Moreover, at $a_i = a$ on the line $a_i = \frac{ak_i}{b(1-\delta)}$, $b = \frac{k_i}{1-\delta}$ holds and at $a_i = a$ on the line $a_i = \frac{ak_i}{b(1+\delta)}$, $b = \frac{k_i}{1+\delta}$ holds.

As a result, the rectangular region composed of the two triangular regions of the horizontal-hatted pattern and the two triangular regions of the vertical-hatted pattern represents the allowable region of (a_i, b).

A.2 Control performance

Consider the following case:

$$h_i < 1,\ a_i. \tag{112}$$

Define

$$g(X) := \frac{1}{X+1}\frac{X+h_i^2}{X+a_i^2},\ X \in R,\ X \geq 0 \tag{113}$$

and investigate the maximum by considering the following equation:

$$\frac{d}{dX}g(X) = (X+1)^{-2}(X+a_i)^{-2}(-X^2 - 2h_i^2 X + a_i^2 - a_i^2 h_i^2 - h_i^2) = 0. \tag{114}$$

A.2.1 Case (i)

In order to that $g(X)$ has a positive maximizer $X^* > 0$ of $g(X)$, a condition

$$a_i^2 - a_i^2 h_i^2 - h_i^2 > 0, \tag{115}$$

that is,

$$a_i^2 > \frac{h_i^2}{1-h_i^2} \tag{116}$$

is necessary. In this case, such X^* is given by

$$X^*(a_i) = -h_i^2 + \sqrt{h_i^4 + a_i^2 - a_i^2 h_i^2 - h_i^2} > 0. \tag{117}$$

Note that from (112), $X^*(a_i)$ is an increasing function of a_i. Moreover, X_{sup}, which is defined by

$$\left\| \frac{\eta}{s+1}\frac{s+h_i}{s+a_i} \right\|_\infty^2 = \left| \frac{1}{X+1}\frac{X+h_i^2}{X+a_i^2} \right|_{X=X_{\mathrm{sup}}}, \tag{118}$$

is given by

$$X_{\text{sup}} = X^*(a_i). \tag{119}$$

Then, we get

$$\eta^2 = \frac{1}{g(X_{\text{sup}})} = (X_{\text{sup}} + 1)\frac{X_{\text{sup}} + a_i^2}{X_{\text{sup}} + h_i^2} = \frac{(X^*(a_i) + 1)(X^*(a_i) + a_i^2)}{X^*(a_i) + h_i^2} =: \eta^2(a_i). \tag{120}$$

Note that this is an increasing function of a_i in the region $a_i^2 > \frac{h_i^2}{1-h_i^2}$,.

A.2.2 Case (ii)

On the other hand, when

$$a_i^2 - a_i^2 h_i^2 - h_i^2 < 0, \tag{121}$$

that is,

$$a_i^2 < \frac{h_i^2}{1 - h_i^2}, \tag{122}$$

$$X_{\text{sup}} = 0. \tag{123}$$

Then, we get

$$\eta^2 = \frac{1}{g(X_{\text{sup}})} = (X_{\text{sup}} + 1)\frac{X_{\text{sup}} + a_i^2}{X_{\text{sup}} + h_i^2} = \frac{a_i^2}{h_i^2} =: \eta^2(a_i). \tag{124}$$

This is also an increasing function of a_i.

Summary of Case (i) and Case (ii)

$$\eta^2(a_i) = \frac{(X^*(a_i) + 1)(X^*(a_i) + a_i^2)}{X^*(a_i) + h_i^2} \quad \text{for } a_i^2 > \frac{h_i^2}{1 - h_i^2}, \tag{125}$$

$$\eta^2(a_i) = \frac{a_i^2}{h_i^2} \quad \text{for } a_i^2 < \frac{h_i^2}{1 - h_i^2}. \tag{126}$$

From the increasing property of $\eta^2(a_i)$, the largest a_i for each b in their allowable region, which is given in Fig. 4, gives the best local performance. Such choice of a_i, called a_i^*, is given by the intersection between lines $b = b$ and $a_i = \frac{ak_i}{b(1-\delta)}$, which is shown at a black-circled point in Fig. 5. This concludes the statement of the Local Control Performance Limitation.

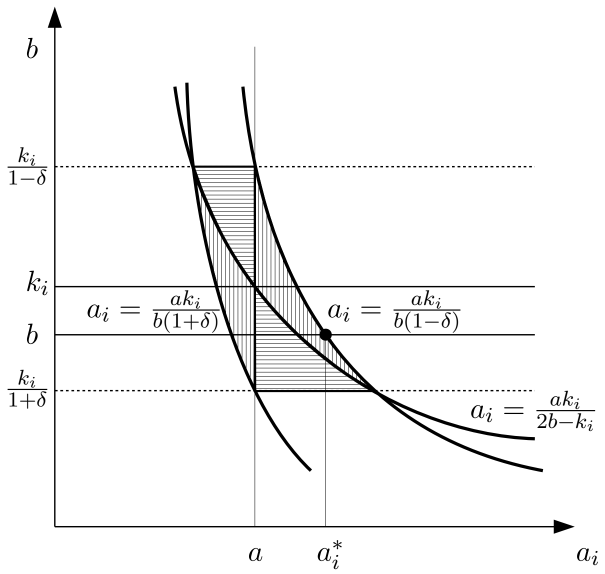

Fig. 5: The allowable region of (a_i, b) and the best choice a_i^* of a_i for a given b