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The Average Distance of Dense Homogeneous Random Graphs

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Abstract

This paper investigates the average distance of two types of homogeneous random graphs (i.e. random graphs where all the vertices are equivalent in the definition) : an Erdős-Rényi graph $G(n,p)$ and a random d-regular graph $G_{n,d}$. It is widely known that $\mathrm{AD}(G(n,p)) = (1+o(1))\,\mathrm{diam}(G(n,p)) = (1+o(1))\log n/\log np$ holds w.h.p. (with high probability) if $n^{-1} < p = n^{-1+o(1)}$, where $\mathrm{AD}(G(n,p))$ and $\mathrm{diam}(G(n,p))$ are the average distance, and respectively, the diameter of $G(n,p)$. A similar result holds for $G_{n,d}$ with $3 \le d = O(1)$; one can easily obtain that $(1-o(1))\log_{d-1} n \le \mathrm{AD}(G_{n,d}) \le \mathrm{diam}(G_{n,d}) = (1+o(1))\log_{d-1} n$ holds w.h.p. from Bollobás and de la Vega (1982).

In this paper, we prove that for $p = (\beta + o(1))\,n^{-1+\alpha}$ where $\alpha \in (0,1)$ and $\beta > 0$ are arbitrary constants, $\mathrm{AD}(G(n,p))$ is asymptotically *concentrated* on $\alpha^{-1} + \exp(-\beta^{1/\alpha})$ if $\alpha \in \mathbb{N}$, and on $\lceil \alpha^{-1} \rceil$ otherwise. Moreover, we prove that the same concentration result holds for $G_{n,d}$ with $d = (\beta + o(1))\,n^{\alpha}$. The result is consistent with an analytical result due to Katzav et al. (2015) in which they did not present a rigorous proof. Our result demonstrates a phase transition of $\mathrm{AD}(G(n,p))$ (and $\mathrm{AD}(G_{n,d})$) between $\alpha^{-1} \in \mathbb{N}$ and $\alpha^{-1} \notin \mathbb{N}$. In particular, an asymptotic gap between the average distance and the diameter can be seen from our result if $\alpha^{-1} \in \mathbb{N}$, whereas such a gap does not appear if $\alpha^{-1} \notin \mathbb{N}$. Furthermore, one can observe that the parameter $p = \beta n^{-1+\alpha}$ with $\alpha^{-1} \in \mathbb{N}$ is on a *critical phase* of the phase transition.

1 Introduction

The average distance is a principal measure of a graph and plays an important role in network analysis. Since typical real-world networks contain "hub"s and reveal power-law degree distribution, the average distance of *inhomogeneous* random graphs has been fairly explored [6, 7, 17, 21, 22]. On the other hand, the average distance of *homogeneous* random graphs has attracted a great deal of attention recently [1, 11, 13, 17, 19, 20]. In particular, regular graphs with low average distance guarantee efficient network topologies in HPC (High Performance Computing) area, where random regular graphs have been considered to perform well in the sense of low latency and fault tolerance [11, 19, 20].

It is widely accepted by random graph theorists that the average distance of "sparse" homogeneous random graphs is with high probability (w.h.p.) the same as the diameter up to a factor $1 + o(1)$ [7, 9]. The following explanation provides an intuitive reasoning. Let G be a sparse homogeneous random graph and assume that G is connected. Consider a breadth first search on G starting from a fixed vertex. Let n_i be the number of vertices we visit in the ith depth for the first time during the search. All the vertices are equivalent because of the homogeneousness; and thus every degree of G tends to take one typical value $\tilde{d}$. Therefore, we obtain $n_i \approx \tilde{d}(\tilde{d}-1)^{i-1} \approx \tilde{d}^i$ (one might be concerned with the duplication of edges, which occurs unlikely because G is sparse). Thus, the average distance of G is likely to be

$$\begin{aligned}\mathrm{AD}(G) &\approx (n-1)^{-1} \sum_{i=1}^{\mathrm{diam}(G)} i \cdot n_i \\ &\approx (n-1)^{-1} \sum_{i=1}^{\mathrm{diam}(G)} i \cdot \tilde{d}^i \\ &\approx \mathrm{diam}(G).\end{aligned}$$

This paper deals with two types of homogeneous random graphs: an Erdős-Rényi graph $G(n,p)$ and a random regular graph $G_{n,d}$. Chung and Lu [7] investigated the average distance of a graph generated by the *expected degree model*, a typical inhomogeneous random graph model that contains the Erdős-Rényi model as a special case. It follows from [7] that the average distance of $G(n,p)$ is $\mathrm{AD}(G(n,p)) = (1+o(1)) \log n / \log np$ w.h.p. if its mean degree np satisfies $1 < np = n^{o(1)}$. According to [9], the diameter of such $G(n,p)$ is $\mathrm{diam}(G(n,p)) = (1+o(1)) \log n/\log np$ w.h.p. Therefore, one can observe that $\mathrm{AD}(G(n,p)) = (1+o(1))\,\mathrm{diam}(G(n,p))$ w.h.p. if $1 < np = n^{o(1)}$.

As for random d-regular graphs, we can evaluate the average distance as follows. Suppose that we have a connected d-regular graph of order n and diameter D with $d \geq 3$. Consider a breadth first search from a fixed vertex. In the ith depth, one visits at most $d(d-1)^{i-1}$ vertices. In other words, the number of vertices at distance exact i from the origin vertex is at most $d(d-1)^{i-1}$ for each $i = 1, \ldots, D$. By summing up over i, we obtain

$$n \leq 1 + \sum_{i=1}^{D} d(d-1)^{i-1}.$$

This upper bound on n is known as the *Moore bound* in graph theory [15], which immediately implies the following lower bound D_0 on the diameter of any d-regular graph of order n.

$$\begin{aligned}D_0 &= \min\left\{ D \in \mathbb{N} \,:\, n \leq 1 + \sum_{i=1}^{D} d(d-1)^{i-1} \right\} \\ &= \left\lceil \log_{d-1} n + \log_{d-1}\left(1 - \frac{2}{d}\left(1 - \frac{1}{n}\right)\right)\right\rceil. \qquad (1)\end{aligned}$$

Similarly, one can derive the following lower bound AD_0 on the average distance of any d-regular graph of order n.

$$\begin{aligned}\mathrm{AD}_0 &= \frac{1}{n-1}\left(\sum_{i=1}^{D_0-1} i \cdot d(d-1)^{i-1} + D_0\left(n - 1 - \sum_{i=1}^{D_0-1} d(d-1)^{i-1}\right)\right) \\ &= D_0 - \frac{d(d-1)^{D_0}}{(n-1)(d-2)^2} + \frac{dD_0}{(n-1)(d-2)} + \frac{d}{(n-1)(d-2)^2}. \qquad (2)\end{aligned}$$

It should be noted that the bounds (1) and (2) are tight (consider the Petersen graph). A simple calculation yields

$$\begin{aligned}D_0 &= \log_{d-1} n + O(1), \\ \mathrm{AD}_0 &= D_0 - O(1) = \log_{d-1} n + O(1)\end{aligned}$$

for any $d = d(n) \geq 3$. According to [5], $\mathrm{diam}(G_{n,d}) = (1+o(1)) \log n / \log(d-1)$ holds w.h.p. for fixed $d \geq 3$ (note that $G_{n,d}$ is not connected w.h.p. if $d \leq 2$). Therefore, we have

$$\frac{\log n}{\log(d-1)} - O(1) = \mathrm{AD}_0 \leq \mathrm{AD}(G_{n,d}) \leq \mathrm{diam}(G_{n,d}) = (1+o(1))\frac{\log n}{\log(d-1)},$$

which yields that $\mathrm{AD}(G_{n,d}) = (1 + o(1)) \log n / \log(d-1)$ holds w.h.p. for fixed $d \geq 3$.

For d such that $d = \omega(1)$ and $d = n^{o(1)}$, $\mathrm{diam}(G_{n,d})$ has not been known explicitly. However, one would derive that

$$\begin{aligned}\mathrm{AD}(G_{n,d}) &= (1 + o(1))\mathrm{diam}(G_{n,d}) \\ &= (1 + o(1))\frac{\log n}{\log d}\end{aligned}$$

holds w.h.p. for degree d with $d = \omega(\log n)$ and $d = n^{o(1)}$ by combining the breadth first search argument mentioned above and the *embedding theorem* due to Dudek et al. [8, 10]. Intuitively speaking, the embedding theorem states the existence of a coupling (i.e. joint distribution) of $G(n,p)$ and $G_{n,d}$ such that $d = (1 + o(1))\, np$ and $G(n,p) \subseteq G_{n,d}$ hold for degree d with $d = \omega(\log n)$ and $d = n^{o(1)}$. As $\mathrm{AD}(G(n,p)) \leq \log n / \log np + o(1)$ w.h.p., so does $G_{n,d}$. The proof of our result on $\mathrm{AD}(G_{n,d})$ also adopts the argument using the embedding theorem. On the other hand, for $d = n^{o(1)}$, the lower bound (2) implies that $\mathrm{AD}(G_{n,d}) \geq \mathrm{AD}_0 = (1 + o(1)) \log n / \log d$ holds w.h.p. Therefore, we have $\mathrm{AD}(G_{n,d}) = (1 + o(1)) \log n / \log d$ for $d = \omega(\log n)$ and $d = n^{o(1)}$.

In summary, the average distance is almost equal to the diameter for homogeneous random graphs of mean degree $n^{o(1)}$. On the other hand, the average distance of "dense" homogeneous random graphs (i.e. ones with mean degree $n^{\Omega(1)}$) is much less understood, though it is theoretically interesting in its own right as we shall present. It follows from Bollobás [3] that $\mathrm{diam}(G(n,p)) = \lfloor \alpha^{-1} \rfloor + 1$ holds w.h.p. for $p = (\beta + o(1))\, n^{-1+\alpha}$. The author [18] proved that $\mathrm{diam}(G_{n,d}) = \lfloor \alpha^{-1} \rfloor + 1$ holds w.h.p. for $d = (\beta + o(1))\, n^{\alpha}$. Therefore, the diameter of dense homogeneous random graphs is asymptotically bounded (to $O(1)$) as well as the average distance, while that of sparse (i.e. mean degree is $n^{o(1)}$) ones are not. In this paper, we prove that for $p = (\beta + o(1))\, n^{-1+\alpha}$ with arbitrary constants $\alpha \in (0, 1)$ and $\beta > 0$, $\mathrm{AD}(G(n,p))$ is asymptotically concentrated on $\alpha^{-1} + \exp(-\beta^{1/\alpha})$ if $\alpha^{-1} \in \mathbb{N}$, and on $\lceil \alpha^{-1} \rceil$ otherwise. Moreover, we prove that the same concentration result holds for a random d-regular graph $G_{n,d}$ of order n and degree $d = np = (\beta + o(1))\, n^{\alpha}$. Actually, a variety of analytical approaches to "compute" the distance distribution of random graphs have been established in the literature of network analysis. These results are in good agreement with numerical experiments, though it usually lacks in mathematical rigor [1, 13, 17]. Katzav et al. [13] and Nitzan et al. [17] presented the analytical results (but without a rigorous proof) of the average distance of our target random graphs, which are consistent with our result. The details will be mentioned in Section 1.3.

One can see from our result that there exists a gap $1 - \exp(-\beta^{1/\alpha})$ between the diameter and the average distance if $\alpha^{-1} \in \mathbb{N}$, while there does not if $\alpha^{-1} \notin \mathbb{N}$. Figure 1 gives a brief explanation for a possible factor of the gap based on the behaviors of the lower bounds D_0 and AD_0.

Fix $\alpha^{-1} \in \mathbb{N}$. The limit of $\beta \to 0$ establishes $\mathrm{AD} \to \alpha^{-1} + 1$, while $\beta \to \infty$ does $\mathrm{AD} \to \alpha^{-1}$. Therefore, the parameter $p = \beta n^{-1+\alpha}$ is on a *critical phase* of the phase transition between $\mathrm{AD}(G(n,p)) = \alpha^{-1}$ and $\mathrm{AD}(G(n,p)) = \alpha^{-1} + 1$ (see Figure 2).

1.1 Formal definition

For a finite set X and a positive integer m with $0 < m \leq |X|$, we use

$$\binom{X}{m} = \{\{x_1, \dots, x_m\} \,:\, |\{x_1, \dots, x_m\}| = m\},$$
$$(X)_m = \left\{(x_1, \dots, x_m) \,:\, \{x_1, \dots, x_m\} \in \binom{X}{m}\right\}.$$

A *graph* $G = (V, E)$ is a pair of finite sets V and $E \subseteq \binom{V}{2}$. We deal with only undirected simple graphs. Each element $v \in V$ is called a *vertex* and $e \in E$ an *edge* of G. For a graph G,

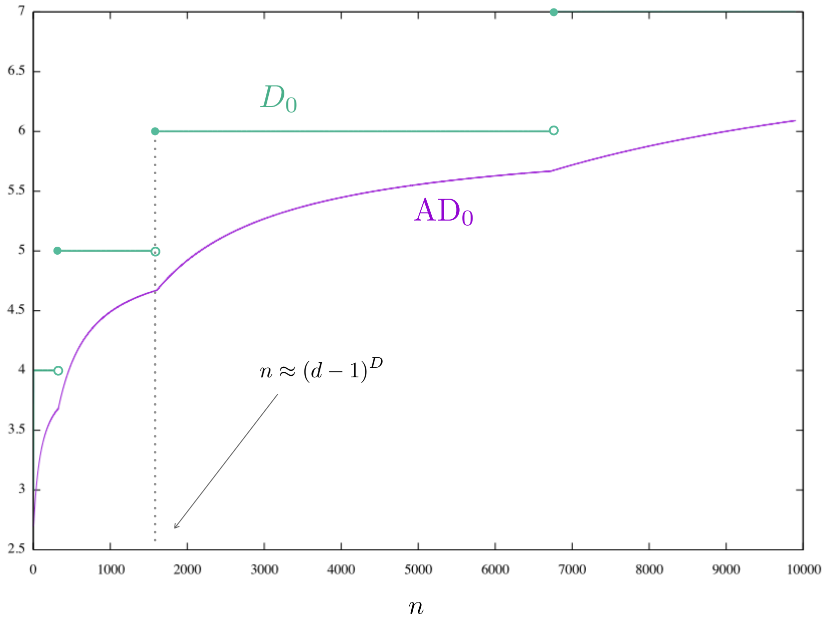


Figure 1: D_0 and AD_0 are plotted with fixed $d = 5$. The horizontal axes is for the order n. The gap $D_0 - \mathrm{AD}_0$ gets increment suddenly at "jumping" points, where D_0 increases by 1. The bound (1) indicates that $n \approx (d-1)^D \iff d \approx n^\alpha$ holds at the "jumping" points, where $D = \alpha^{-1} \in \mathbb{N}$. Hence, the point $\alpha^{-1} \in \mathbb{N}$ captures the "jumping" points; and thus we observe the gap of the diameter and the average distance at $\alpha^{-1} \in \mathbb{N}$ (note that the explanation here only looks at the lower bounds D_0 and AD_0, which lacks in mathematical rigor).

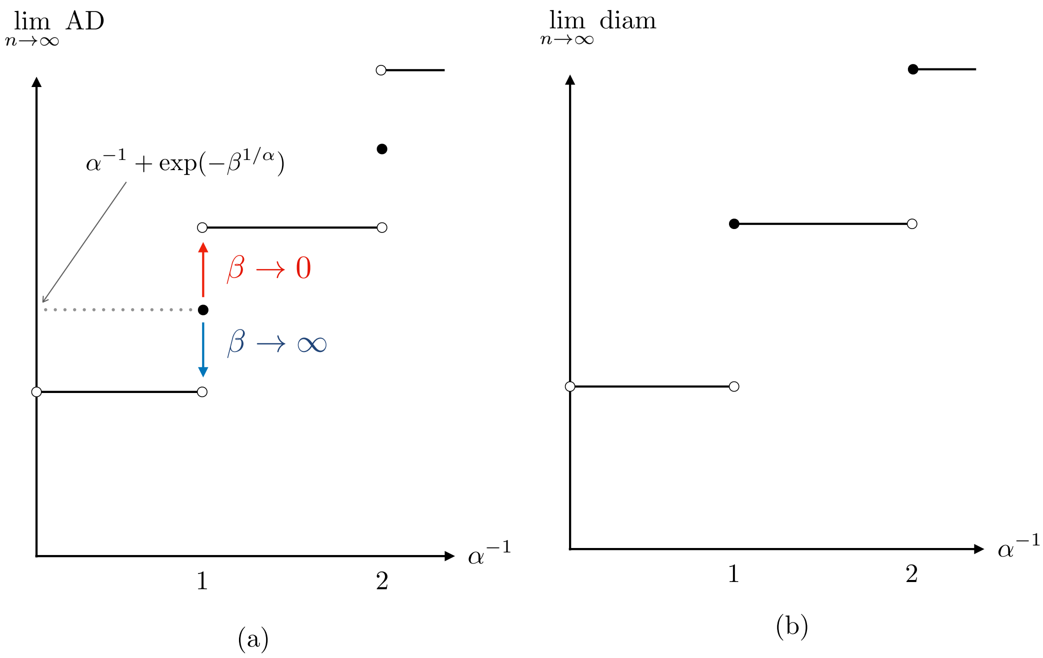


Figure 2: The asymptotic value of (a) $\mathrm{AD}(G(n,p))$ and (b) $\mathrm{diam}(G(n,p))$ for $p = \beta n^{-1+\alpha}$. The plot (a) illustrates the critical phase of the average distance for fixed $\alpha^{-1} \in \mathbb{N}$.

we denote by $V(G)$ and $E(G)$, respectively, the vertex set and the edge set of G. The *order* of a graph is the number of vertices. Throughout the paper, we refer to n as the order of a graph and the vertex set is denoted by $V = \{1, \dots, n\}$. Note that our graphs are *labelled*, that is, all the vertices of a graph are distinguishable. For a graph G, the *degree* of a vertex $v \in V(G)$ is $|\{e \in E(G) : v \in e\}|$. A graph G is *d-regular* if each vertex has degree exact d.

For two graphs G and H, we say G *contains* H if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write "$H \subseteq G$" if G contains G. Two graphs $G \cup H$ and $G \cap H$ are defined as

$$\begin{aligned} G \cup H &= (V(G) \cup V(H),\, E(G) \cup E(H)),\\ G \cap H &= (V(G) \cap V(H),\, E(G) \cap E(H)). \end{aligned}$$

It should be noted that G and H are labelled.

A *path* is a graph $P = (\{v_0, \dots, v_l\}, \{\{v_0, v_1\}, \dots, \{v_{l-1}, v_l\}\})$ where $v_0, \dots, v_l$ are distinct vertices. For such a path P, the vertices v_0 and v_l are called *endpoints.* We refer st-path to a path of endpoints s and t. The *length* of a path is the number of edges. For a graph G and its two distinct vertices s and t, the *distance* $\mathrm{dist}_G(s,t)$ is the minimum among the length of all st-paths contained in G. We define $\mathrm{dist}_G(s,t) = |V(G)|$ if G does not contain an st-path. For a graph $G = (V, E)$ of order n, the *average distance* $\mathrm{AD}(G)$ of G is

$$\mathrm{AD}(G) = \binom{n}{2}^{-1} \sum_{\{s,t\} \in \binom{V}{2}} \mathrm{dist}_G(s,t).$$

Throughout the paper, the *diameter* $\mathrm{diam}(G)$ of G is

$$\mathrm{diam}(G) = \begin{cases} \max_{s \neq t} \mathrm{dist}_G(s,t) & \text{if } G \text{ is connected,}\\ \infty & \text{otherwise.} \end{cases}$$

We refer $\mathrm{dist}(s,t)$ to $\mathrm{dist}_G(s,t)$ if it is clear from the context.

The present definition of $\mathrm{diam}(G)$ above follows from [3, 18]. Our definition of $\mathrm{dist}_G(s,t)$ is due to some technical reason. If we define $\mathrm{dist}_G(s,t) = \infty$ for unreachable vertex pair (s,t), then the expected value of $\mathrm{dist}_{G(n,p)}(s,t)$ and $\mathrm{AD}(G(n,p))$ do not exist (they will be infinity), where our proof turns out to be invalid. Note that our random graphs are so dense that they are connected w.h.p. Therefore, the definition of $\mathrm{AD}(G)$ captures the usual concept of average distance in this paper.

An Erdős-Rényi graph is a graph $G(n,p)$ of order n, where each vertex pair is connected with probability p and independently to all other pairs. A random d-regular graph $G_{n,d}$ is a graph selected uniformly at random from the set of all labelled d-regular graphs of order n (where nd is even). Throughout the paper, $p = p(n) \in [0,1]$ and $d = d(n) \in \mathbb{N}$ can be functions of $n \in \mathbb{N}$.

Let G_n be a random graph of order n (either $G(n,p)$ or $G_{n,d}$ throughout the paper) and $\mathcal{P}$ be a graph property (e.g., being connected, being planar). We say $\mathcal{P}$ holds *with high probability (w.h.p.)* if $\lim_{n\to\infty} \Pr(G_n \text{ satisfies } \mathcal{P}) = 1$.

1.2 Main result

Theorem 1. *Suppose $p = (\beta + o(1))\, n^{-1+\alpha} \in [0,1]$ where $\alpha \in (0,1)$ and $\beta > 0$ are arbitrary constants and let*

$$\mu = \begin{cases} \alpha^{-1} + \exp(-\beta^{1/\alpha}) & \text{if } \alpha^{-1} \in \mathbb{N},\\ \lceil \alpha^{-1} \rceil & \text{otherwise} \end{cases}$$

be a constant.

Then for every $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(|\mathrm{AD}(G(n,p)) - \mu| > \epsilon) = 0.$$

In other words, it holds w.h.p. that

$$\mathrm{AD}(G(n,p)) = \mu + o(1).$$

The following result states that one can replace $G(n,p)$ by $G_{n,d}$ with $d = np$ as follows.

Theorem 2. *Suppose $d = (\beta + o(1))\, n^{\alpha} \in \mathbb{N}$ where $\alpha \in (0,1)$ and $\beta > 0$ are arbitrary constants and let*

$$\mu = \begin{cases} \alpha^{-1} + \exp(-\beta^{1/\alpha}) & \text{if } \alpha^{-1} \in \mathbb{N}, \\ \lceil \alpha^{-1} \rceil & \text{otherwise.} \end{cases}$$

be a constant.

Then for every $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(|\mathrm{AD}(G_{n,d}) - \mu| > \epsilon) = 0.$$

In other words, it holds w.h.p. that

$$\mathrm{AD}(G_{n,d}) = \mu + o(1).$$

As illustrated in Figure 2, our result implies that $\mathrm{AD}(G(n,p))$ is on the critical phase for $p = (\beta + o(1))\, n^{-1+\alpha}$ with $\alpha^{-1} \in \mathbb{N}$. To see this, fix $\alpha \in (0,1)$ be such that $\alpha^{-1} \in \mathbb{N}$. From Theorem 1, it holds w.h.p. that

$$\mathrm{AD}(G(n,p)) \to \begin{cases} \alpha^{-1} + o(1) & \text{if } \beta \to \infty, \\ \alpha^{-1} + 1 + o(1) & \text{if } \beta \to 0, \end{cases}$$

which reveal the critical phase between $\mathrm{AD} = \alpha^{-1}$ and $\mathrm{AD} = \alpha^{-1} + 1$. This observation leads us to the following corollary.

Corollary 3. *Suppose $p = \beta_n \cdot n^{-1+\alpha} \in [0,1]$ where $\alpha \in (0,1)$ is an arbitrary constant and β_n satisfies $|\log(\beta_n)| = o(\log n)$ (i.e. $n^{-o(1)} \le \beta_n \le n^{o(1)}$).*

(i) *If $\alpha^{-1} \notin \mathbb{N}$, it holds w.h.p. that*

$$\mathrm{AD}(G(n,p)) = \lceil \alpha^{-1} \rceil + o(1).$$

(ii) *If $\alpha^{-1} \in \mathbb{N}$, it holds w.h.p. that*

$$\mathrm{AD}(G(n,p)) = \begin{cases} \alpha^{-1} + 1 + o(1) & \text{if } \beta_n \to 0, \\ \alpha^{-1} + \exp(-\beta^{1/\alpha}) + o(1) & \text{if } \beta_n \to \beta, \\ \alpha^{-1} + o(1) & \text{if } \beta_n \to \infty, \end{cases}$$

where $\beta > 0$ is an arbitrary constant.

1.3 Related work

As mentioned earlier, the average distance is the same as the diameter up to a factor $1 + o(1)$ w.h.p. if the homogeneous random graph is so sparse that its mean degree is $n^{o(1)}$. However, as shown in Section 1.2, this is not the case if the random graph is so dense that its mean degree is $n^{\Omega(1)}$. Such random graphs have asymptotically bounded diameter as well as the average distance w.h.p. [3, 18]. In this paper, we consider $G(n, p)$ and $G_{n,d}$ with $np = (\beta + o(1))\, n^{\alpha}$, $d = (\beta + o(1))\, n^{\alpha}$.

Katzav et al. [13] presented an analytical approach to compute the distance distribution of $G(n, p)$ with $p = \beta n^{-1+\alpha}$ approximately. They consider the distance between two random vertices i, j of such $G(n, p)$ and derived that if $\alpha^{-1} \in \mathbb{N}$,

$$\Pr(\mathrm{dist}(i, j) = k) \approx \begin{cases} 1 - \exp(-\beta^{1/\alpha}) & \text{if } k = \alpha^{-1}, \\ \exp(-\beta^{1/\alpha}) & \text{if } k = \alpha^{-1} + 1, \\ 0 & \text{otherwise} \end{cases}$$

holds as $n \to \infty$. Moreover, they also presented the concentration of the average distance, which is in consistent with our result. Note that the results are derived from calculations based on "heuristic" assumptions, though they are in good agreement with the numerical experiment result.

Nitzan et al. [17] investigates the distance distribution of a graph generated by *configuration model*, a common model for inhomogeneous random graphs. The configuration model $C(\mathbf{d})$ with given degree sequence $\mathbf{d} = (d_i)_{i=1}^n$ is a graph generated as follows. We assume that $d_i \in \mathbb{N}$ and $\sum_i d_i$ is even. Consider a finite set U of cardinality $\sum_i d_i$ and let $(\mathcal{P}_i)_{i=1}^n$ be a partition of U such that $|\mathcal{P}_i| = d_i$ for each $i = 1, \ldots, n$ (i.e. $\bigcup_i \mathcal{P}_i = U$ and $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$ for every $i \neq j$). As $\sum_i d_i$ is even, one can generate a uniformly random matching on U, which we denoted by M. The matching M can be regarded to form edges of a graph whose vertex set is $\{1, \ldots, n\}$ (i.e. a pair $ij \in M$ forms an edge $\{x, y\}$ if $i \in \mathcal{P}_x$ and $j \in \mathcal{P}_y$) hold. Let $C(\mathbf{d})$ denote a graph generated by this procedure. Note that $C(\mathbf{d})$ may contain self loops or multiple edges, which hardly affects the distance property such as the diameter. The analytical result due to Nitzan et al. [17] is similar to that in [13]. It can be expected from (46) in [17] that the average distance of $C(\mathbf{d})$ is $\alpha^{-1} + \exp(-\beta^{1/\alpha})$ for $d_i = (\beta + o(1))\, n^{\alpha}$ with $\alpha^{-1} \in \mathbb{N}$, which is consistent with Theorem 2. However, $C(\mathbf{d})$ differs from $G_{n,d}$ because $C(\mathbf{d})$ may contain self loops or multiple edges.

We now explain a brief background of *random regular graph theorem* related to this paper. The configuration model can be applied to analyze random d-regular graphs $G_{n,d}$ by setting $d_i = d$ for each $i = 1, \ldots, n$. For a graph property $\mathcal{P}$, one can see

$$\begin{aligned} \Pr(G_{n,d} \text{ satisfies } \mathcal{P}) &= \Pr(C(\mathbf{d}) \text{ satisfies } \mathcal{P} \mid C(\mathbf{d}) \text{ is simple}) \\ &= \frac{\Pr(C(\mathbf{d}) \text{ satisfies } \mathcal{P} \wedge C(\mathbf{d}) \text{ is simple})}{\Pr(C(\mathbf{d}) \text{ is simple})} \\ &\leq \frac{\Pr(C(\mathbf{d}) \text{ satisfies } \mathcal{P})}{\Pr(C(\mathbf{d}) \text{ is simple})}. \end{aligned}$$

Bollobás [2] proved that, for $3 \leq d = O(1)$,

$$\lim_{n \to \infty} \Pr(C(\mathbf{d}) \text{ is simple}) = 1 - \exp\left(-\frac{(d-1)^2}{4}\right) > 0$$

as $n \to \infty$. Therefore, in order to prove that $G_{n,d}$ does not satisfy $\mathcal{P}$ w.h.p., it suffices to show $\Pr(C(\mathbf{d}) \text{ satisfies } \mathcal{P}) = o(1)$. The asymptotic behavior of the diameter [5], the connectivity [10] and several other properties of $G_{n,d}$ are obtained via the analysis of configuration models for fixed $d \geq 3$. See [16] for details.

As for $G_{n,d}$ with growing degree $d = d(n) = \omega(1)$, it seems difficult to apply the configuration model argument directly because $\Pr(C(\mathbf{d}) \text{ is simple}) = o(1)$. However, it has recently been noticed that $G_{n,d}$ and $G(n,p)$ are "similar" if $d = np = \omega(\log n)$. Haber and Krivelevich [12] proved that $G_{n,d}$ and $G(n,p)$ are "equivalent" in the sense of *first order logic property* for $p = n^{-\alpha}$ and $d = (1+o(1))\, n^{1-\alpha}$ for an irrational $\alpha \in (0,1)$. More precisely, they proved that

$$\lim_{n\to\infty} \Pr(G(n,p) \text{ satisfies } \mathcal{A}) = \lim_{n\to\infty} \Pr(G_{n,d} \text{ satisfies } \mathcal{A})$$

holds where $\mathcal{A}$ is any property that can be captured by a first order logic sentence. Kim and Vu [14] conjectured the existence of a coupling of $G(n,p')$, $G_{n,d}$ and $G(n,p)$ such that $G(n,p') \subseteq G_{n,d} \subseteq G(n,p)$ with $p' = (1-o(1))\, d/n$ and $p = (1+o(1))\, d/n$ for $d = \omega(\log n)$. In other words, the conjecture asserts that $G_{n,d}$ can be "approximated" by $G(n,p)$ with $d = np$ if $d = \omega(\log n)$. Dudek et al. [8] proved the *embedding theorem*, which states the existence of coupling of $G(n,p)$ and $G_{n,d}$ such that $G(n,p) \subseteq G_{n,d}$ for $d = (1+o(1))\, np$, $d = \omega(\log n)$ and $d = o(n)$. By using several results concerning to the similarity, one might analyze $G_{n,d}$ by considering $G(n,p)$ with $p = (1+o(1)\, d/n$. The author [18] investigated the the diameter of $G_{n,d}$ with $d = (\beta + o(1))\, n^\alpha$ by using the embedding theorem [8, 10]. However, to the best of our knowledge, $\mathrm{AD}(G_{n,d})$ and $\mathrm{AD}(G(n,p))$ for $d = (1+o(1))\, np = (\beta+o(1))\, n^\alpha$ is unexplored. Our result adds an evidence to the "similarity" of $G(n,d/n)$ and $G_{n,d}$ for $d = (\beta + o(1))\, n^\alpha$.

1.4 Proof outline

The average distance $\mathrm{AD}(G)$ of a graph $G = (V,E)$ can be rewritten as

$$\begin{aligned} \mathrm{AD}(G) &= \mathop{\mathrm{E}}_{s,t}\left(\mathrm{dist}_G(s,t)\right) \\ &= \sum_{l=1}^{n} \Pr_{s,t}\left(\mathrm{dist}_G(s,t) \geq l\right) \\ &= \binom{n}{2}^{-1} \sum_{l=1}^{n} \sum_{\{s,t\}\in\binom{V}{2}} \mathbb{1}_{[\,\mathrm{dist}_G(s,t)\geq l\,]}(G), \end{aligned} \tag{3}$$

where $\mathbb{1}_{[X]}(G)$ is the indicator defined as

$$\mathbb{1}_{[X]}(G) = \begin{cases} 1 & \text{if } G \text{ satisfies the property } X, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the expectation $\mathrm{E}_{s,t}(\cdot)$ and the probability $\Pr_{s,t}(\cdot)$ are concerned with a random vertex pair $\{s,t\} \in \binom{V}{2}$.

For two constants $\alpha \in (0,1)$ and $\beta > 0$, set

$$\mu = \begin{cases} \lceil \alpha^{-1} \rceil & \text{if } \alpha^{-1} \notin \mathbb{N}, \\ \alpha^{-1} + \exp(-\beta^{1/\alpha}) & \text{if } \alpha^{-1} \in \mathbb{N}. \end{cases}$$

Fix $\epsilon > 0$. Let G be an Erdős-Rényi graph $G(n,p)$ and $\mathrm{AD} = \mathrm{AD}(G(n,p))$ be a random variable. Then, from the Markov inequality, we have

$$\begin{aligned} \Pr(|\mathrm{AD} - \mu| > \epsilon) &= \Pr((\mathrm{AD}-\mu)^2 > \epsilon^2) \\ &\leq \frac{\mathrm{E}(\mathrm{AD}^2) - 2\mu\, \mathrm{E}(\mathrm{AD}) + \mu^2}{\epsilon^2}. \end{aligned}$$

Therefore, it suffices to show $\mathrm{E}(\mathrm{AD}) = \mu + o(1)$ and $\mathrm{E}(\mathrm{AD}^2) = \mu^2 + o(1)$ to prove Theorem 1.

From the expression (3) of AD, we see

$$\mathrm{E}(\mathrm{AD}) = \binom{n}{2}^{-1} \sum_{l=1}^{n} \sum_{\{s,t\} \in \binom{V}{2}} \Pr(\mathrm{dist}(s,t) \geq l),$$

$$\mathrm{E}(\mathrm{AD}^2) = \binom{n}{2}^{-2} \sum_{l,l'=1}^{n} \sum_{\{s,t\},\, \{s',t'\} \in \binom{V}{2}} \Pr(\mathrm{dist}(s,t) \geq l \,\wedge\, \mathrm{dist}(s',t') \geq l').$$

We now go on to the evaluation of the expectations above. The random variable $\mathrm{dist}(s,t) = \mathrm{dist}_{G(n,p)}(s,t)$ denoting the distance is considered, where s and t are fixed vertices. We will show that the random variable w.h.p. satisfies

$$\lceil \alpha^{-1} \rceil \leq \mathrm{dist}(s,t) \leq \lfloor \alpha^{-1} \rfloor + 1.$$

The proof of "$\lceil \alpha^{-1} \rceil \leq \mathrm{dist}(s,t)$" is based on the *first order method*, a typical proof technique known in random graph theory (see, e.g., [4, 10]). The statement "$\mathrm{dist}(s,t) \leq \lfloor \alpha^{-1} \rfloor + 1$" follows from the proof of Theorem 7.1 in [10].

Suppose $\alpha^{-1} \notin \mathbb{N}$. Then, $\mathrm{dist}(s,t) = \lfloor \alpha^{-1} \rfloor + 1 = \lceil \alpha^{-1} \rceil$ holds w.h.p. Actually, the probability $\Pr(\mathrm{dist}(s,t) > \lfloor \alpha^{-1} \rfloor + 1)$ is $O(n^{-3})$, and thus we have

$$\Pr(\mathrm{dist}(s,t) \geq l) = \begin{cases} 1 - o(1) & \text{if } l \leq \lceil \alpha^{-1} \rceil, \\ O(n^{-3}) & \text{otherwise,} \end{cases}$$

which implies

$$\mathrm{E}(\mathrm{AD}) = \lceil \alpha^{-1} \rceil + o(1) = \mu + o(1).$$

Moreover, for fixed $\{s,t\}$, $\{s',t'\} \in \binom{V}{2}$ with $\{s,t\} \cap \{s',t'\} = \emptyset$, it holds that

$$\Pr(\mathrm{dist}(s,t) \geq l \,\wedge\, \mathrm{dist}(s',t') \geq l') = \begin{cases} 1 - o(1) & \text{if } l \leq \lceil \alpha^{-1} \rceil \text{ and } l' \leq \lceil \alpha^{-1} \rceil, \\ O(n^{-3}) & \text{otherwise.} \end{cases}$$

These facts implies that $\mathrm{E}(\mathrm{AD}^2) = \lceil \alpha^{-1} \rceil^2 + o(1) = \mu^2 + o(1)$ holds, which completes the proof of Theorem 1 in the case where $\alpha^{-1} \notin \mathbb{N}$.

Suppose $\alpha^{-1} \in \mathbb{N}$. Then $\mathrm{dist}(s,t) \in \{\alpha^{-1}, \alpha^{-1}+1\}$ holds w.h.p. as $\alpha^{-1} = \lceil \alpha^{-1} \rceil \leq \mathrm{dist}(s,t) \leq \lfloor \alpha^{-1} \rfloor + 1 = \alpha^{-1} + 1$ holds w.h.p. Let X be the number of st-paths of length α^{-1} contained in $G(n,p)$. Then it holds that

$$\Pr(\mathrm{dist}(s,t) = \alpha^{-1} + 1) = \Pr(X = 0) + o(1).$$

An argument similar to the one presented in [5] renders that the random variable X is asymptotically Poisson distributed of mean $\beta^{1/\alpha}$, which yields

$$\Pr(\mathrm{dist}(s,t) = \alpha^{-1} + 1) = \Pr(X = 0) + o(1) = \exp(-\beta^{1/\alpha}) + o(1).$$

Therefore, we have

$$\mathrm{E}(\mathrm{AD}) = \alpha^{-1} + \exp(-\beta^{1/\alpha}) + o(1) = \mu + o(1).$$

Fix two pairs $\{s_1,t_1\}$, $\{s_2,t_2\} \in \binom{V}{2}$ with $\{s_1,t_1\} \cap \{s_2,t_2\} = \emptyset$. Let $X^{(i)}$ be the number of $s_i t_i$-paths and of length α^{-1} contained in $G(n,p)$. Actually, it holds that the joint distribution of $X^{(1)}$ and $X^{(2)}$ tends to be that of two independent and identical Poisson random variables of mean $\beta^{1/\alpha}$. The proof of this fact is done by carrying over Lemma 3.2 in [18], which states that the same result holds for $G_{n,d}$, into $G(n,p)$ with $p = d/n$. The fact implies

$$\Pr(\mathrm{dist}(s,t) \geq l \wedge \mathrm{dist}(s',t') \geq l') = \begin{cases} 1 - o(1) & \text{if } l \leq \alpha^{-1} \text{ and } l' \leq \alpha^{-1}, \\ \exp(-\beta^{1/\alpha}) & \text{if } l < l' = \alpha^{-1} + 1 \text{ (and vice versa)}, \\ \exp(-2\beta^{1/\alpha}) & \text{if } l = l' = \alpha^{-1}, \\ O(n^{-3}) & \text{otherwise,} \end{cases}$$

and thus, we obtain

$$\mathrm{E}(\mathrm{AD}^2) = (\alpha^{-1} + \exp(-\beta^{1/\alpha}))^2 + o(1) = \mu^2 + o(1),$$

which completes the proof of Theorem 1.

Indeed, the argument described above requires $\Pr(\mathrm{dist}(s,t) > \lfloor \alpha^{-1} \rfloor + 1)$ to be so small that the expected value E(AD) converges. For $G(n,p)$, it can be shown that $\Pr(\mathrm{dist}(s,t) > \lfloor \alpha^{-1} \rfloor + 1) = O(n^{-3})$ holds, which the probability is small enough for our proof. On the other hand, it seems to be difficult to prove the same result for $G_{n,d}$. However, we can obtain

$$\mathrm{AD}(G_{n,d}) \geq \mu - o(1)$$

holds w.h.p. by evaluating $\mathrm{E}(\mathrm{AD}(G_{n,d}))$ and $\mathrm{E}(\mathrm{AD}(G_{n,d})^2)$ using (3). Thus, it suffices to show that $\mathrm{AD} \leq \mu + o(1)$ holds w.h.p. To this end, we use the *embedding theorem* due to Dudek et al. [8, 10].

Theorem 4 ([8], Theorem 1)**.** *There is a constant $C > 0$ such that if for some real $\gamma = \gamma(n)$ and positive integer $d = d(n)$,*

$$C\left(\left(\frac{d}{n} + \frac{\log n}{d}\right)^{1/3} + \frac{1}{n}\right) \leq \gamma < 1,$$

and $m = (1-\gamma)\frac{nd}{2}$ is an integer, then there exists a joint distribution of $G(n,m)$ and $G_{n,d}$ with

$$\lim_{n\to\infty} \Pr\left(G(n,m) \subseteq G_{n,d}\right) = 1.$$

We have the following corollary immediately from Theorem 4.

Corollary 5. *Let d and m be as described in Theorem 4 with $d = \omega(\log n)$ and $d = o(n)$. If $\mathrm{AD}(G(n,m)) \leq A$ holds w.h.p. then $\mathrm{AD}(G_{n,d}) \leq A$ holds w.h.p.*

As noted in [8], one can replace $G(n,m)$ by $G(n,p)$ with $p = (1-2\gamma)\frac{d}{n-1}$. For $d = (\beta + o(1))\, n^{\alpha}$ with $\alpha^{-1} \in \mathbb{N}$, one can take γ described in Theorem 4 such that $\gamma = o(1)$ and let $p = (1-2\gamma)\frac{d}{n-1} = (\beta + o(1))\, n^{-1+\alpha}$ be a function. Then we obtain that

$$\mathrm{AD}(G_{n,d}) \leq \mathrm{AD}(G(n,p)) = \alpha^{-1} + \exp(-\beta^{1/\alpha}) + o(1) \tag{4}$$

holds w.h.p.

Intuitively speaking, the embedding theorem states the existance of a coupling of $G_{n,d}$ and $G(n,p)$ such that $d = (1+o(1))\, np$ and $G(n,p) \subseteq G_{n,d}$ hold w.h.p. As $\mathrm{AD}(G(n,p)) \leq \mu + o(1)$ holds w.h.p., so does $G_{n,d}$. This completes the proof of Theorem 2.

2 Average distance of dense Erdős-Rényi graphs

Proof of Theorem 1. Set $p = (\beta + o(1))\, n^{-1+\alpha}$ and

$$\mu = \begin{cases} \alpha^{-1} + \exp(-\beta^{1/\alpha}) & \text{if } \alpha^{-1} \in \mathbb{N}, \\ \lceil \alpha^{-1} \rceil & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$ and $\beta > 0$ are arbitrary constants. If $\mathrm{E}(\mathrm{AD}^2) < \infty$ exists, we have

$$\begin{aligned} \Pr(|\mathrm{AD} - \mu| > \epsilon) &= \Pr((\mathrm{AD} - \mu)^2 > \epsilon^2) \\ &\leq \frac{\mathrm{E}(\mathrm{AD}^2) - 2\mu\, \mathrm{E}(\mathrm{AD}) + \mu^2}{\epsilon^2}. \end{aligned} \tag{5}$$

Remark that the expectation $\mathrm{E}(\cdot)$ and probability $\Pr(\cdot)$ are concerned with $G(n,p)$. From (5), it suffices to show that $\mathrm{E}(\mathrm{AD}) = \mu + o(1)$ and $\mathrm{E}(\mathrm{AD}^2) = \mu^2 + o(1)$ hold.

As mentioned in Section 1.4, we have

$$\begin{aligned} \mathrm{AD}(G) &= \binom{n}{2}^{-1} \sum_{l=1}^{n} \sum_{\{s,t\} \in \binom{V}{2}} \mathbb{1}_{[\,\mathrm{dist}(s,t) \geq l\,]}(G), \\ \mathrm{AD}(G)^2 &= \binom{n}{2}^{-2} \sum_{l,l'=1}^{n} \sum_{\{s,t\},\, \{s',t'\} \in \binom{V}{2}} \mathbb{1}_{[\,\mathrm{dist}(s,t) \geq l \ \wedge \ \mathrm{dist}(s',t') \geq l'\,]}(G), \end{aligned} \tag{6}$$

where

$$\mathbb{1}_{[X]}(G) = \begin{cases} 1 & \text{if } G \text{ satisfies } X, \\ 0 & \text{otherwise.} \end{cases}$$

We use the following lemmas as we shall prove later.

Lemma 6. *Suppose $p = (\beta + o(1))\, n^{-1+\alpha}$ for $\alpha \in (0,1)$ and $\beta > 0$ are arbitrary constants. Let s and t be two fixed distinct vertices. Then it holds w.h.p. that*

$$\mathrm{dist}(s,t) \geq \lceil \alpha^{-1} \rceil.$$

Lemma 7. *Suppose $p = (\beta + o(1))\, n^{-1+\alpha}$ for $\alpha \in (0,1)$ and $\beta > 0$ are arbitrary constants. Let s and t be two fixed distinct vertices. Then, it holds that*

$$\Pr(\mathrm{dist}(s,t) > \lfloor \alpha^{-1} \rfloor + 1) = O(n^{-3}).$$

Lemma 8. *Suppose $p = (\beta + o(1))\, n^{-1+\alpha}$ for $\alpha^{-1} \in \mathbb{N}$ and $\beta > 0$. Let $k \in \{1,2\}$ and $s_1, \ldots, s_k$, $t_1, \ldots, t_k$ be fixed $2k$ distinct vertices. Then, it holds that*

$$\Pr\left(\bigwedge_{i=1}^{k} [\mathrm{dist}(s_i,t_i) \geq \alpha^{-1} + 1] \right) = \exp(-k\beta^{1/\alpha}) + o(1).$$

It should be noted that these results are immediately obtained from existing works [10, 18]. Indeed, Lemma 8 can be extended to every $k \in \mathbb{N}$, though we do not use the fact in this paper.

Rest of the proof of Theorem 1. Assuming Lemma 6 to 8, it is straightforward to see

$$\begin{aligned} \mathrm{E}(\mathrm{AD}) &= \binom{n}{2}^{-1} \sum_{l=1}^{n} \sum_{\{s,t\} \in \binom{V}{2}} \Pr(\mathrm{dist}(s,t) \geq l) \\ &= \sum_{l=1}^{n} \Pr(\mathrm{dist}(1,2) \geq l) \\ &= \mu + o(1). \end{aligned}$$

Here, $V = \{1, \ldots, n\}$ denotes the vertex set. Note that $\Pr(\mathrm{dist}(s,t) \geq l)$ does not depend on the label "s" and "t".

We evaluate $\Pr(\mathrm{dist}(s,t) \geq l \ \wedge \ \mathrm{dist}(s',t') \geq l')$.

If $\max(l,l') > \lfloor \alpha^{-1} \rfloor + 1$, from Lemma 7, we obtain

$$\begin{aligned}\Pr(\mathrm{dist}(s,t) \geq l \ \wedge \ \mathrm{dist}(s',t') \geq l') &\leq \Pr(\mathrm{dist}(s,t) > \lfloor \alpha^{-1} \rfloor + 1) \\ &= O(n^{-3}).\end{aligned}$$

If $\max(l,l') \leq \lceil \alpha^{-1} \rceil$, Lemma 6 implies

$$\begin{aligned}1 &\geq \Pr(\mathrm{dist}(s,t) \geq l \ \wedge \ \mathrm{dist}(s',t') \geq l') \\ &\geq 1 - \Pr(\mathrm{dist}(s,t) < l) - \Pr(\mathrm{dist}(s',t') < l') \\ &= 1 - o(1).\end{aligned}$$

Suppose $\alpha^{-1} \notin \mathbb{N}$. Then $\mu = \lceil \alpha^{-1} \rceil$ and we have

$$\begin{aligned}\mathrm{E}(\mathrm{AD}^2) &= \binom{n}{2}^{-2} \sum_{l,l'=1}^{n} \sum_{\{s,t\},\,\{s',t'\} \in \binom{V}{2}} \Pr(\mathrm{dist}(s,t) \geq l \ \wedge \ \mathrm{dist}(s',t') \geq l') \\ &= \sum_{1 \leq l,l' \leq \lceil \alpha^{-1} \rceil} (1 + o(1)) && \text{(terms correspond to } \max(l,l') \leq \lceil \alpha^{-1} \rceil) \\ &\quad + O(n^{-4} \cdot n^2 \cdot n^4 \cdot n^{-3}) && \text{(terms correspond to } \max(l,l') > \lceil \alpha^{-1} \rceil) \\ &= \mu^2 + o(1).\end{aligned}$$

Suppose $\alpha^{-1} \in \mathbb{N}$. If $l = \alpha^{-1} + 1$ and $l' < \alpha^{-1}$ (or $l' = \alpha^{-1} + 1$ and $l < \alpha^{-1}$), then it holds that

$$\begin{aligned}\Pr(\mathrm{dist}(s,t) \geq l) &\geq \Pr(\mathrm{dist}(s,t) \geq l \ \wedge \ \mathrm{dist}(s',t') \geq l') \\ &\geq 1 - \Pr(\mathrm{dist}(s,t) < l) - \Pr(\mathrm{dist}(s',t') < l') \\ &= \Pr(\mathrm{dist}(s,t) \geq l) - o(1),\end{aligned}$$

and thus, Lemma 8 with letting $k = 1$ implies

$$\begin{aligned}\Pr(\mathrm{dist}(s,t) \geq l \ \wedge \ \mathrm{dist}(s',t') \geq l') &= \Pr(\mathrm{dist}(s,t) \geq l) - o(1) \\ &= \exp(-\beta^{1/\alpha}) + o(1).\end{aligned}$$

If $l = l' = \alpha^{-1} + 1$, from Lemma 8 with letting $k = 2$, we obtain

$$\Pr(\mathrm{dist}(s,t) \geq l \ \wedge \ \mathrm{dist}(s',t') \geq l') = \exp(-2\beta^{1/\alpha}) + o(1).$$

Finally, we have

$$\begin{aligned}\mathrm{E}(\mathrm{AD}^2) &= \binom{n}{2}^{-2} \sum_{l,l'=1}^{n} \sum_{\{s,t\},\,\{s',t'\} \in \binom{V}{2}} \Pr(\mathrm{dist}(s,t) \geq l \ \wedge \ \mathrm{dist}(s',t') \geq l') \\ &= \sum_{1 \leq l,l' \leq \alpha^{-1}} (1 + o(1)) && \text{(terms correspond to } \max(l,l') \leq \alpha^{-1}) \\ &\quad + 2 \sum_{l' \leq \alpha^{-1}} (\exp(-\beta^{1/\alpha}) + o(1)) && \text{(terms correspond to } \max(l,l') = \alpha^{-1} + 1) \\ &\quad + \exp(-2\beta^{1/\alpha}) + o(1) && \text{(a term corresponds to } l = l' = \alpha^{-1} + 1) \\ &\quad + O(n^{-4} \cdot n^2 \cdot n^4 \cdot n^{-3}) && \text{(terms correspond to } \max(l,l') > \alpha^{-1} + 1) \\ &= \mu^2 + o(1),\end{aligned}$$

which completes the proof of Theorem 1. □

2.1 Proof of the Lemmas

Proof of Lemma 6. The proof of Lemma 6 is obtained by just applying the first order method (see, e.g., [10, 4]). Fix two distinct vertices s and t. Let $\mathcal{P}_l$ be the set of all st-paths of length l in a complete graph K_n and $X_l = |\{p \in \mathcal{P}_l : p \subseteq G(n,p)|$ be a random variable. In other words, X_l denotes the number of elements of $\mathcal{P}_l$ contained in $G(n,p)$.

From the Markov inequality, we have

$$\begin{aligned}
\Pr(\mathrm{dist}(s,t) < \lceil \alpha^{-1} \rceil) &= \Pr\left(X_1 + \cdots + X_{\lceil \alpha^{-1} \rceil - 1} > 0\right) \\
&\le \sum_{l=1}^{\lceil \alpha^{-1} \rceil - 1} \mathrm{E}(X_l) \\
&= \sum_{l=1}^{\lceil \alpha^{-1} \rceil - 1} \sum_{P \in \mathcal{P}_l} \Pr(P \subseteq G(n,p)) \\
&\le \sum_{l=1}^{\lceil \alpha^{-1} \rceil - 1} n^{l-1} p^l \\
&\le O(n^{-\alpha}).
\end{aligned}$$

which completes the proof. □

Proof of Lemma 7. Our proof is almost same as a part of the proof of Theorem 7.1 in [10]. We check the validity of the proof for $p = (\beta + o(1))\, n^{-1+\alpha}$. In this paper, we render the proof clear and more rigorous.

Consider a random graph $G = G(n,p)$ where $p = (\beta + o(1))\, n^{-1+\alpha}$. Set $l = \lfloor \alpha^{-1} \rfloor + 1$. We begin with the analysis of the spread process of a breadth first search on $G(n,p)$ from a fixed vertex. For a vertex v and $k \in \mathbb{N}$, define $N_k(v) = \{w \in V : \mathrm{dist}_G(v,w) = k\}$ and let $\mathcal{F}_k^{(v)} = \mathcal{F}_k = \left\{ |N_k(v)| \ge \left(\frac{np}{2}\right)^k \right\}$ be an event. The degree of v by $\deg(v)$ is denoted by $\deg(v)$. The random variable $\Delta = \Delta(G(n,p))$ is denoted by the maximum degree of $G(n,p)$. Let $\mathrm{Bin}(m,q)$ be a binomial distributed random variable with m trials and probability q.

The Markov inequality and the Chernoff-Hoeffding bound lead to

$$\begin{aligned}
\Pr(\Delta \ge 1.1np) &= \Pr\left(\bigvee_{w \in V} [\deg(w) \ge 1.1np] \right) \\
&\le n \Pr(\mathrm{Bin}(n-1,p) \ge 1.1np) \\
&\le n \exp(-\Theta(np)) \\
&= O(n^{-3}).
\end{aligned}$$

Assuming $\Delta \le 1.1np$, we have

$$\begin{aligned}
n - \sum_{i=0}^{k-1} |N_k(v)| &\ge n - \sum_{i=0}^{k-1} (1.1np)^i \\
&= n - o(n) \\
&\ge 0.9n,
\end{aligned}$$

for sufficiently large n because $|N_i(v)| \le (1.1np)^i$.

Also, with the assumption of $\mathcal{F}_{k-1}$, we obtain

$$\begin{aligned}
1 - (1-p)^{|N_{k-1}(v)|} &\ge 1 - \exp\left(-p \left(\frac{np}{2}\right)^{k-1} \right) \\
&\ge \frac{1}{1.8n} \left(\frac{np}{2}\right)^k,
\end{aligned}$$

for sufficiently large n because $|N_k(v)| \geq \left(\frac{np}{2}\right)^{k-1}$.

Therefore, we obtain

$$\begin{aligned}
&\Pr(\overline{\mathcal{F}_k} \mid \mathcal{F}_1, \dots, \mathcal{F}_{k-1}) \\
&\leq \Pr(\overline{\mathcal{F}_k} \mid \mathcal{F}_1, \dots, \mathcal{F}_{k-1},\, \Delta \leq 1.1np) + \Pr(\Delta > 1.1np) \\
&= \Pr\left(\mathrm{Bin}\left(n - \sum_{i=0}^{k-1} |N_k(v)|,\, 1-(1-p)^{|N_k(v)|}\right) < \left(\frac{np}{2}\right)^k \,\middle|\, \mathcal{F}_1, \dots, \mathcal{F}_{k-1},\, \Delta \leq 1.1np\right) + O(n^{-3}) \\
&\leq \Pr\left(\mathrm{Bin}\left(0.9n,\, \frac{1}{1.8n}\left(\frac{np}{2}\right)^k\right) < \left(\frac{np}{2}\right)^k\right) + O(n^{-3}) \\
&\leq \exp\left(-\Theta(np)\right) + O(n^{-3}) \\
&= O(n^{-3})
\end{aligned}$$

for every $k = 1, \dots, \lceil \frac{l}{2} \rceil$. Now, we have

$$\begin{aligned}
\Pr\left(\bigcap_{i=1}^{\lceil l/2 \rceil} \mathcal{F}_i\right) &= \Pr(\mathcal{F}_1) \prod_{i=2}^{\lceil l/2 \rceil} \Pr(\mathcal{F}_i \mid \mathcal{F}_1, \dots, \mathcal{F}_{i-1}) \\
&= 1 - O(n^{-3}). \qquad (7)
\end{aligned}$$

In other words, a breadth first search on $G(n,p)$ from a fixed vertex v likely spreads as $|N_k(v)| \geq \left(\frac{np}{2}\right)^{k-1}$ until $k \leq \lceil \frac{l}{2} \rceil$.

Fix two distinct vertices s and t and let $X = N_{\lfloor l/2 \rfloor}(s)$ and $Y = N_{\lceil l/2 \rceil}(t)$ be the sets of vertices. Define $E(X,Y) = \{e \in E(G(n,p)) : e \cap X \neq \emptyset,\, e \cap Y \neq \emptyset\}$. If $\mathrm{dist}(s,t) > l$, it obviously holds that $E(X,Y) = \emptyset$. Then, we have

$$\begin{aligned}
\Pr\left(E(X,Y) = \emptyset \mid \mathcal{F}_1^{(s)}, \dots, \mathcal{F}_{\lfloor l/2 \rfloor}^{(s)}, \mathcal{F}_1^{(t)}, \dots, \mathcal{F}_{\lceil l/2 \rceil}^{(t)}\right) &\leq (1-p)^{\left(\frac{np}{2}\right)^l} \\
&\leq \exp\left(-p\left(\frac{np}{2}\right)^l\right) \\
&= O(n^{-3}). \qquad (8)
\end{aligned}$$

We now look at the probability $\Pr(\mathrm{dist}(s,t) > l)$. From (7) and (8), we obtain

$$\begin{aligned}
\Pr(\mathrm{dist}(s,t) > l) &\leq \Pr(E(X,Y) = \emptyset) \\
&\leq \Pr\left(E(X,Y) = \emptyset \mid \mathcal{F}_1^{(s)}, \dots, \mathcal{F}_{\lfloor l/2 \rfloor}^{(s)}, \mathcal{F}_1^{(t)}, \dots, \mathcal{F}_{\lceil l/2 \rceil}^{(t)}\right) \\
&\qquad + \Pr\left(\overline{\mathcal{F}_1^{(s)} \cap \dots \cap \mathcal{F}_{\lfloor l/2 \rfloor}^{(s)} \cap \mathcal{F}_1^{(t)} \cap \dots \cap \mathcal{F}_{\lceil l/2 \rceil}^{(t)}}\right) \\
&\leq O(n^{-3}) + 2 - 2\Pr\left(\bigcap_{i=1}^{\lceil l/2 \rceil} \mathcal{F}_i\right) \\
&= O(n^{-3}).
\end{aligned}$$

□

Proof of Lemma 8. For $k \in \{1,2\}$, fix $2k$ distinct vertices $s_1, \dots, s_k, t_1, \dots, t_k$. For each $i = 1, \dots, k$, let $\mathcal{P}^{(i)}$ be the set of all $s_i t_i$-paths of length $\alpha^{-1} + 1$ contained in a complete graph K_n. We denote by $X^{(i)}$ the number of paths of $\mathcal{P}^{(i)}$ contained in $G(n,p)$, that is, $X^{(i)} := |\{q \in \mathcal{P}^{(i)} : q \subseteq G(n,p)|$.

Lemma 6 and Lemma 7 yields that $\mathrm{dist}(s_i, t_i) \in \{\alpha^{-1}, \alpha^{-1} + 1\}$ holds w.h.p. Thus, we have

$$\Pr(\mathrm{dist}(s_i, t_i) \geq \alpha^{-1} + 1) = \Pr(X^{(i)} = 0) + o(1).$$

With the notation $(x)_r = x(x-1)\cdots(x-r+1)$, we will show later that

$$\mathrm{E}\left(\prod_{i=1}^{k}(X^{(i)})_{r_i}\right) = (\beta^{1/\alpha})^{\sum_{i=1}^{k} r_i} + o(1) \tag{9}$$

holds for every $k \in \{1,2\}$ and every fixed nonnegative integers $r_1, \ldots, r_k$. On the assumption of (9), the Poisson approximation theorem (see, e.g., [4, 10, 16]) implies that the random variables $X^{(1)}$ and $X^{(2)}$ are asymptotically independent Poisson distributed with means $\beta^{1/\alpha}$. Therefore we have

$$\begin{aligned}\Pr\left(\bigwedge_{i=1}^{k}[\mathrm{dist}(s_i,t_i) \geq \alpha^{-1}+1]\right) &= \Pr\left(\bigwedge_{i=1}^{k}\left[X^{(i)}=0\right]\right) + o(1)\\ &= \exp(-k\beta^{1/\alpha}) + o(1),\end{aligned}$$

which completes the proof of Lemma 8.

Actually, the author [18] deals with $G_{n,d}$ and proved the following lemma which is the same as (9) and derived the same result as Lemma 8 for general $k \in \mathbb{N}$.

Lemma 9 (Lemma 3.2, [18]). *Suppose $d = (\beta + o(1))\, n^{\alpha}$ with two constants $\alpha \in (0,1)$ and $\beta > 0$ where $\alpha^{-1} \in \mathbb{N}$. For $k \in \mathbb{N}$, fix $2k$ vertices $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ with $S \cap T = \emptyset$ and let $X^{(i)}$ denote the number of $s_i t_i$-paths of length α^{-1}. Then, it holds that*

$$\mathrm{E}\left(\prod_{i=1}^{k}(X^{(i)})_{r_i}\right) = (\beta^{1/\alpha})^{\sum_{i=1}^{k} r_i} + o(1),$$

where the expectation E *is taken over random regular graphs $G_{n,d}$.*

His proof for Lemma 9 above works for our goal (9) by replacing $G_{n,d}$ by $G(n, d/n)$. This paper presents a brief version of the proof, since it suffices to see the special case $k \in \{1,2\}$ here.

Setting $r_2 = 0$, we can assume $k = 2$ without loss of generality and do so. As each $X^{(i)}$ can be rewritten as the sum of indicators $\mathbb{1}_{[\,q \subseteq G(n,p)\,]}$ over $q \in \mathcal{P}^{(i)}$, we have

$$\mathrm{E}\left(\prod_{i=1}^{2}(X^{(i)})_{r_i}\right) = \sum_{\substack{\mathbf{P}=(P_1,\ldots,P_{r_1})\in(\mathcal{P}^{(1)})_{r_1}\\ \mathbf{Q}=(Q_1,\ldots,Q_{r_2})\in(\mathcal{P}^{(2)})_{r_2}}} \Pr\left(U(\mathbf{P},\mathbf{Q}) \subseteq G(n,p)\right),$$

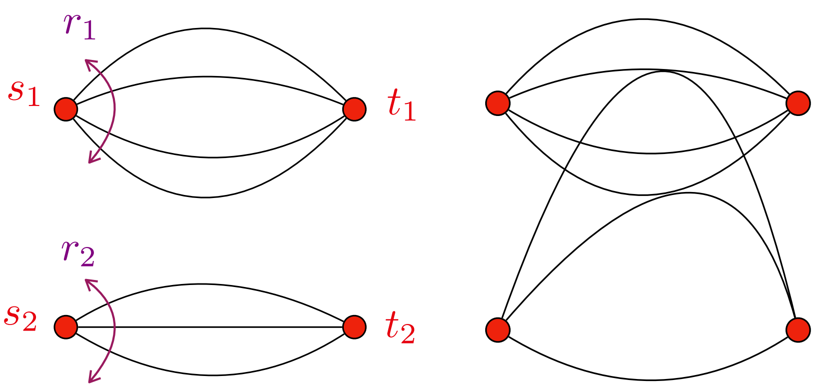
where

$$U(\mathbf{P},\mathbf{Q}) = \bigcup_{i=1}^{r_1}\bigcup_{j=1}^{r_2}(P_i \cup Q_j)$$

for two tuples of paths $\mathbf{P} = (P_1, \ldots, P_{r_1})$ and $\mathbf{Q} = (Q_1, \ldots, Q_{r_2})$. In other words, $U(\mathbf{P},\mathbf{Q})$ denotes a graph represented by the union of r_1 paths $\mathbf{P}$ connecting s_1 and t_1, and r_2 ones $\mathbf{Q}$ connecting s_2 and t_2.

For a fixed graph H, let

$$\begin{aligned}N_H &:= \left|\left\{(\mathbf{P},\mathbf{Q}) \in (\mathcal{P}^{(1)})_{r_1} \times (\mathcal{P}^{(2)})_{r_2} \;:\; U(\mathbf{P},\mathbf{Q}) = H\right\}\right|\\ &= \begin{cases} O(n^{|V(H)|-4}) & \text{if } \exists\, \mathbf{P},\mathbf{Q} \text{ such that } H = U(\mathbf{P},\mathbf{Q})\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$



(a) (b)

Figure 3: Candidates for H, which is the union of paths of length α^{-1} and having label only on red vertices. (a) illustrates the union of "disjoint paths" and (b) illustrates the union of "crossing paths".

be the number of ways for representing H as the union of s_1t_1-paths and s_2t_2-paths. Then it is straightforward to see

$$\mathrm{E}\left(\prod_{i=1}^{2}(X^{(i)})_{r_i}\right) = \sum_H N_H \cdot p^{|E(H)|}.$$

The summation above is over every graph H having label *only on the endpoints* (i.e. s_1, s_2, t_1 and t_2, as shown in Figure 3).

If $|V(H)| = (r_1 + r_2)(\alpha^{-1} - 1) + 4$ (intuitively speaking, such H is represented as the union of "disjoint paths", as shown in Figure 3(a)), it holds that $N_H = (1 + o(1))n^{(r_1+r_2)(\alpha^{-1}-1)}$ and $|E(H)| = (r_1 + r_2)\alpha^{-1}$. Thus, we have

$$N_H \cdot p^{|E(H)|} = (1 + o(1)) = (1 + o(1))\,(\beta^{1/\alpha})^{r_1+r_2}.$$

Otherwise (in this case, the graph H is represented as the union of "crossing paths", as shown in Figure 3(b)), we have

$$\begin{aligned} N_H \cdot p^{e_H} &= O(n^{|V(H)|} \cdot n^{-|E(H)|+\alpha|E(H)|}) \\ &= o(1), \end{aligned}$$

which follows from Lemma 3.4 in [18].

Therefore, we obtain

$$\mathrm{E}\left(\prod_{i=1}^{2}(X^{(i)})_{r_i}\right) = \sum_H N_H \cdot p^{e_H} = (\beta^{1/\alpha})^{r_1+r_2} + o(1).$$

Note that the number of H with $N_H > 0$ is $O(1)$ because such an H has order at most $2(\alpha^{-1} + 1)$. □

3 Average distance of dense random regular graphs

3.1 Lower bound from the Moore bound argument

The Moore bound implies the lower bound (1) of the diameter of any d-regular graph of order n. For $d = (\beta + o(1))\, n^{\alpha}$, it is straightforward see

$$\lim_{n\to\infty} D_0 = \begin{cases} \lfloor \alpha^{-1} \rfloor + 1 & \text{either } \alpha^{-1} \notin \mathbb{N}, \text{ or } \alpha^{-1} \in \mathbb{N} \text{ and } \beta < 1, \\ \alpha^{-1} & \alpha^{-1} \in \mathbb{N} \text{ and } \beta > 1, \\ \text{depends on the term } o(1) & \text{otherwise,} \end{cases} \tag{10}$$

as we shall show in Section A.1.

Similarly, we have the lower bound (2) of the average distance of any d-regular graph of order n. For $d = (\beta + o(1))\, n^{\alpha}$, a simple calculation shows

$$\lim_{n\to\infty} \mathrm{AD}_0 = \begin{cases} \lfloor \alpha^{-1} \rfloor + 1 & \text{if } \alpha^{-1} \notin \mathbb{N}, \\ \alpha^{-1} & \text{if } \alpha^{-1} \in \mathbb{N} \text{ and } \beta > 1, \\ \alpha^{-1} - \beta^{1/\alpha} + 1 & \text{if } \alpha^{-1} \in \mathbb{N} \text{ and } \beta < 1, \\ \text{depends on the term } o(1) & \text{otherwise,} \end{cases} \tag{11}$$

as we shall show in Section A.2.

From (11), we have the following result that gives a proof of Theorem 2 in the case where $\alpha^{-1} \notin \mathbb{N}$.

Proposition 10. *Set $d = (\beta + o(1))\, n^{\alpha}$, where $\alpha \in (0, 1)$ and $\beta > 0$ are arbitrary constants. If $\alpha^{-1} \notin \mathbb{N}$,*

$$\mathrm{AD}(G_{n,d}) = \lceil \alpha^{-1} \rceil - o(1).$$

holds w.h.p.

Proof. Suppose $\alpha^{-1} \notin \mathbb{N}$. It is obvious that

$$\lceil \alpha^{-1} \rceil - o(1) = \lfloor \alpha^{-1} \rfloor + 1 - o(1) = \mathrm{AD}_0(G_{n,d}) \le \mathrm{AD}(G_{n,d}) \le \mathrm{diam}(G_{n,d}).$$

Proposition 10 follows from the fact that $\mathrm{diam}(G_{n,d}) = \lfloor \alpha^{-1} \rfloor + 1 = \lceil \alpha^{-1} \rceil$ holds w.h.p. shown in [18]. Note that $\mathrm{AD}_0(G) \le \mathrm{AD}(G)$ still holds for a disconnected graph G. □

3.2 Upper bound from the embedding theorem

The previous section indicates that the lower bound AD_0 is asymptotically the same as $\mathrm{diam}(G_{n,d})$ for $d = (\beta + o(1))\, n^{\alpha}$ with $\alpha^{-1} \notin \mathbb{N}$. However, If $\alpha^{-1} \in \mathbb{N}$, a gap between $\mathrm{diam}(G_{n,d}) = \lfloor \alpha^{-1} \rfloor + 1$ and AD_0 exists. Theorem 4 (the embedding theorem) enables us to obtain a sharper upper bound on AD, as we introduced in Section 1.4. That is, we have $\mathrm{AD} \le \alpha^{-1} + \exp(-\beta^{1/\alpha})$ by combining Theorem 1 and 4. Still, there exists a gap between AD_0 and the sharp upper bound. We present a sharp lower bound for the $\mathrm{AD}(G_{n,d})$ in the next section.

3.3 Sharp lower bound

Suppose $d = (\beta + o(1))\, n^{\alpha}$ where $\alpha \in (0, 1)$ and $\beta > 0$ are arbitrary constants such that $\alpha^{-1} \in \mathbb{N}$. The following lemma implies the sharp lower bound on $\mathrm{AD}(G_{n,d})$.

Lemma 11. *Suppose $d = (\beta + o(1))\, n^{\alpha}$ where $\alpha \in (0,1)$ and $\beta > 0$ are arbitrary constants with $\alpha^{-1} \in \mathbb{N}$. Set $\mu = \alpha^{-1} + \exp(-\beta^{1/\alpha})$. Then for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \Pr(\mathrm{AD}(G_{n,d}) \le \mu - \epsilon) = 0.$$

In other words, it holds w.h.p. that

$$\mathrm{AD}(G_{n,d}) \ge \mu - o(1).$$

Proof. The average distance $\mathrm{AD}(G_{n,d})$ satisfies

$$\begin{aligned} \mathrm{AD}(G_{n,d}) &= \sum_{l=1}^{n} \Pr_{s,t}\left(\mathrm{dist}(s,t) \ge l\right) \\ &\ge \sum_{l=1}^{\alpha^{-1}+1} \Pr_{s,t}\left(\mathrm{dist}(s,t) \ge l\right). \end{aligned}$$

Here, the probability $\Pr_{s,t}(\cdot)$ is concerned with a random vertex pair $\{s,t\} \in \binom{V}{2}$.

Let $p_l = p_l(G_{n,d}) = \Pr_{s,t}\left(\mathrm{dist}(s,t) \ge l\right)$ be random variable on $G_{n,d}$. The random variables p_l can be rewritten as

$$\begin{aligned} p_l &= \binom{n}{2}^{-1} \sum_{\{s,t\} \in \binom{V}{2}} \mathbb{1}_{[\,\mathrm{dist}(s,t) \ge l\,]}, \\ p_l^2 &= \binom{n}{2}^{-2} \sum_{\{s,t\},\, \{s',t'\} \in \binom{V}{2}} \mathbb{1}_{[\,\mathrm{dist}(s,t) \ge l \ \wedge\ \mathrm{dist}(s',t') \ge l\,]}. \end{aligned}$$

Hence, as $n \to \infty$, we obtain

$$\begin{aligned} \mathrm{E}(p_l) &= \binom{n}{2}^{-1} \sum_{\{s,t\} \in \binom{V}{2}} \Pr(\mathrm{dist}(s,t) \ge l) \\ &= \Pr(\mathrm{dist}(1,2) \ge l) \\ &\to \begin{cases} 1 & \text{if } 1 \le l \le \alpha^{-1}, \\ \exp(-\beta^{1/\alpha}) & \text{if } l = \alpha^{-1} + 1, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}(p_l^2) &= \binom{n}{2}^{-2} \sum_{\{s,t\},\, \{s',t'\} \in \binom{V}{2}} \Pr(\mathrm{dist}(s,t) \ge l \ \wedge\ \mathrm{dist}(s',t') \ge l) \\ &= \binom{n}{2}^{-2} \left(O(n^3) + \sum_{\{s,t\} \cap \{s',t'\} = \emptyset} \Pr(\mathrm{dist}(s,t) \ge l \ \wedge\ \mathrm{dist}(s',t') \ge l) \right) \\ &= \Pr(\mathrm{dist}(1,2) \ge l \ \wedge\ \mathrm{dist}(3,4) \ge l) + o(1) \\ &\to \begin{cases} 1 & \text{if } 1 \le l \le \alpha^{-1}, \\ \exp(-2\beta^{1/\alpha}) & \text{if } l = \alpha^{-1} + 1. \end{cases} \end{aligned}$$

Remark that the expectations $\mathrm{E}(\cdot)$ above are concerned with $G_{n,d}$. Here, we have used the fact

$$\Pr(\mathrm{dist}(1,2) \ge \alpha^{-1} + 1 \ \wedge\ \mathrm{dist}(3,4) \ge \alpha^{-1} + 1) = \exp(-2\beta^{1/\alpha}) + o(1),$$

which follows from the Poisson approximation theorem (see, e.g., [4, 10, 16]) and Lemma 9 with letting $k = 2$. We also note that each term in the summation (e.g., "$\Pr(\mathrm{dist}(s,t) \geq l)$") does not depend on the label of the vertices.

Set

$$\mu_l = \begin{cases} 1 & \text{if } 1 \leq l \leq \alpha^{-1}, \\ \exp(-\beta^{1/\alpha}) & \text{if } l = \alpha^{-1} + 1, \end{cases}$$

for $l = 1, \ldots, \alpha^{-1} + 1$, and $\mu = \alpha^{-1} + \exp(-\beta^{1/\alpha})$. For every fixed $\epsilon > 0$, we have

$$\begin{aligned} \Pr(|p_l - \mu_l| > \epsilon) &= \Pr((p_l - \mu_l)^2 > \epsilon^2) \\ &\leq \frac{\mathrm{E}(p_l^2) - 2\mu_l \,\mathrm{E}(p_l) + \mu_l^2}{\epsilon^2} \\ &= o(1), \end{aligned}$$

and thus

$$\begin{aligned} \Pr\left(\left| \sum_{l=1}^{\alpha^{-1}+1} p_l - \mu \right| > \epsilon \right) &\leq \Pr\left(\exists l,\ |p_l - \mu_l| > \epsilon/(\alpha^{-1}+1)\right) \\ &\leq \sum_{l=1}^{\alpha^{-1}+1} o(1) \\ &= o(1). \end{aligned}$$

Therefore, it holds w.h.p. that

$$\begin{aligned} \mathrm{AD}(G_{n,d}) &\geq \sum_{l=1}^{\alpha^{-1}+1} p_l \\ &\geq \mu - o(1). \end{aligned}$$

□

Proof of Theorem 2. If $\alpha^{-1} \notin \mathbb{N}$, apply Proposition 10. Otherwise, combine (4) and Lemma 11. □

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A Analysis of the asymptotic behavior of the Moore bound

A.1 Diameter

The asymptotic behavior (10) of D_0 is proved in this section. Set $d = (\beta + o(1))\, n^{\alpha}$. We shall look at D_0 defined as (1). A simple calculation implies

$$\begin{aligned}\log(d-1) &= \log d + \log\left(1 - \frac{1}{d}\right) \\ &= \log(\beta + o(1)) + \alpha \log n - O\left(d^{-1}\right) \\ &= \alpha \log n \left(\frac{\log(\beta + o(1)) - O\left(n^{-\alpha}\right)}{\alpha \log n} + 1\right),\end{aligned}$$

and

$$\begin{aligned}\log_{d-1}\left(1 - \frac{2}{d}\left(1 - \frac{1}{n}\right)\right) &= -O\left(\frac{1}{d \log d}\right) \\ &= -O\left(\frac{1}{n^{\alpha} \log n}\right).\end{aligned}$$

Therefore, substitution of $d = (\beta + o(1))\, n^{\alpha}$ into (1) yields

$$\begin{aligned}D_0 &= \left\lceil \alpha^{-1}\left(1 + \frac{\log(\beta + o(1)) - O(n^{-\alpha})}{\alpha \log n}\right)^{-1} - O\left(\frac{1}{n^{\alpha} \log n}\right)\right\rceil \\ &= \left\lceil \alpha^{-1} - \frac{\epsilon(n)}{1 + \epsilon(n)} - O\left(\frac{1}{n^{\alpha} \log n}\right)\right\rceil,\end{aligned}$$

where $\epsilon(n) = \frac{\log(\beta+o(1)) - O(n^{-\alpha})}{\alpha \log n}$.

Suppose $\alpha^{-1} \notin \mathbb{N}$ and let $\delta = \lceil \alpha^{-1} \rceil - \alpha^{-1} \in (0,1)$ be a constant. Since $\epsilon(n) = o(1)$, we have

$$D_0 = \left\lceil \lceil \alpha^{-1} \rceil - \delta \pm o(1) \right\rceil \to \lceil \alpha^{-1} \rceil = \lfloor \alpha^{-1} \rfloor + 1.$$

Here, "$\pm$" means that either "+" or "−" (whether "+" or not does not matter because the term $o(1)$ above is much less than the constant $\delta > 0$).

If $\alpha^{-1} \in \mathbb{N}$ and $\beta > 1$ then $\log(\beta + o(1)) > 0$ holds, which implies

$$D_0 = \left\lceil \alpha^{-1} - O\left(\frac{1}{\log n}\right) \pm o\left(\frac{1}{\log n}\right)\right\rceil \to \alpha^{-1}.$$

Similarly, if $\alpha^{-1} \in \mathbb{N}$ and $\beta < 1$, we have

$$D_0 = \left\lceil \alpha^{-1} + O\left(\frac{1}{\log n}\right) \pm o\left(\frac{1}{\log n}\right)\right\rceil \to \alpha^{-1} + 1.$$

If $\alpha^{-1} \in \mathbb{N}$ and $\beta = 1$, then D_0 depends on the term $o(1)$ in $d = (\beta + o(1))\, n^{\alpha}$. □

A.2 Average distance

We show (11). Set $d = (\beta + o(1))\, n^{\alpha}$ with two arbitrary constants $\alpha \in (0,1)$ and $\beta > 0$. Since $D_0 \le \lfloor \alpha^{-1} \rfloor + 1 = O(1)$ holds, the equality (2) yields

$$\mathrm{AD}_0 = D_0 - \frac{d^{D_0 - 1}}{n} + o(1). \tag{12}$$

If $\alpha^{-1} \notin \mathbb{N}$, it holds that $D_0 = \lfloor \alpha^{-1} \rfloor + 1$ and $\lfloor \alpha^{-1} \rfloor < \alpha^{-1}$; thus we have

$$\mathrm{AD}_0 = D_0 - \frac{d^{1/\alpha}}{n} \cdot \frac{1}{d^{1/\alpha - \lfloor 1/\alpha \rfloor}} + o(1) = D_0 + o(1).$$

If $\alpha^{-1} \in \mathbb{N}$ and $D_0 = \alpha^{-1}$, substitution of $D_0 = \alpha^{-1}$ into (12) shows

$$\mathrm{AD}_0 = D_0 - \frac{d^{1/\alpha - 1}}{n} + o(1) = D_0 + o(1).$$

Similarly, if $\alpha^{-1} \in \mathbb{N}$ and $D_0 = \alpha^{-1} + 1$, we obtain

$$\mathrm{AD}_0 = D_0 - \frac{d^{1/\alpha}}{n} + o(1) = \alpha^{-1} + 1 - \beta^{1/\alpha} + o(1).$$

□